On the pricing of defaultable bonds using the framework of barrier options

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Abstract

In the framework of the structural approach of bond pricing, we extend the Fujita-Ishizaka model by considering more realistic payoffs. The payoff to the bondholder at time of default, provided that default occurs prior to maturity, depends on the firm value at time of default. We also find the new measure with the advantage to calculate the value of bond and its financial interpretation. In addition, we present some numerical exmaples.

1 Introduction

The models for the quantitative analysis of credit risk are usually classified into two groups : the reduced-form models and the structural models. The present article is concerned with the latter method.

The structural-model approach is based on the evolution of the firm value and uses the option pricing theory to determine the value of the firm's bond. The approach was initiated by Merton [10] and Black & Scholes [3]. In Merton's model, the default occurs if, at the maturity of the bond, the firm value is less than the amount of the firm's debt. Note that the default in his model could occur, by definition, only at the maturity.

Black & Cox [2] proposed a more realistic model by allowing premature default. In their model, the default occurs when the firm value crosses some non-constant barrier. In other words, the default time is defined by the first passage time of the firm value to some barrier, corresponding to knock-out options. Later, Longstaff & Schwartz [9] and Cathcart & El-Jahel [4] extended the Black-Cox model by considering stochastic interest rate. Those models are called the first passage time models.

The structural approach captures so-called safety covenants in the indenture provisions. A safety covenant allows the bondholders to force bankruptcy if certain conditions are met. In the first passage time models, the condition for default is that the firm value falls to default boundary. The first passage time models facilitate the modeling of safety covenants, but such safety covenants are often too strict for the company. Recently Fujita & Ishizaka [7] proposed bonds with more relaxed safety covenants by giving some generalizations of the first passage

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time models. In their model, the first passage time of the firm value to some barrier is only considered to be the "caution time," and the default occurs if some conditions are satisfied after the caution time. This corresponds to Edokko options proposed by Fujita & Miura [8]. The various conditions for the default considered in the Fujita-Ishizaka model give more flexibility to the structural-model approach.

However, the Fujita-Ishizaka model has some drawbacks. When the default occurs prior to maturity, the firm's value at the default time should be invested in riskless bonds up to maturity, but the payoffs to the bondholder in the Fujita-Ishizaka model are not made that way.

The purpose of the present article is to consider models with more realistic payoffs and to value corporate bonds accordingly. The conditions for default considered here are the same as (part of) Black & Cox [2] and Fujita & Ishizaka [7]. But, contrary to Fujita-Ishizaka model, we consider that the payoff to the bondholder at time of default, provided that default occurs prior to maturity, depends on the firm value at time of default. We also find the new measure with the advantage to calculate the value of bond and its financial interpretation. This paper is organized as follows. In Section 2 we propose our basic model and explain the difference from the Fujita-Ishizaka model. In Section 3, we consider four conditions for default and price the corporate bonds for each condition. In Section 4, some numerical examples of our models are presented. Section 5 gives a summary.

2 Our basic model

Suppose that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{F})$. We assume that trading occurs continuously in a frictionless market with no tax and no transaction cost, and risk free rate r is a constant.

Consider a firm that issues a single zero coupon bond with face value L and maturity T. The firm value is modeled by the geometric Brownian motion:

$$\begin{aligned} dV_t &= rV_t dt + \sigma V_t dW_t, \quad t \geq 0, \\ V_0 &= v, \end{aligned}$$

where σ and v are constants and W is a standard Brownian motion. We assume that $v \geq L$ Note that $\mathbf{P}$ is already the martingale measure, that is, the discounted firm value $e^{-rt}V_t$ is a martingale under $\mathbf{P}$. This assumption is not restrictive after a suitable measure change.

The stopping time τ_A is defined as the first passage time of V to the level A:

$$\tau_A := \inf\{t > 0 | V_t = A\}, \quad (\inf \emptyset = \infty)$$

where A is a positive constant satisfying $A < L$. We call τ_A the "caution time." Moreover, we specify the "default time" depending on τ_A and denote by $g(\tau_A)$. Note that $g(\tau_A) \geq \tau_A$. We say that the default occurs if and only if $g(\tau_A) \leq T$. Four examples of $g(\tau_A)$ are considered in the next section, corresponding to four different safety covenants of the corporate bond. As soon as the default occurs, the firm is forced into restructuring or bankruptcy in which case the bondholder take over the firm.

The payoff to the bondholder at maturity is modeled as follows.

1. If the default does not occur and if the firm value at maturity is greater than L, then the payoff is L, where L is a positive constant smaller than v.

2. If the default does not occur but if the firm value is less than L, then the payoff is $\beta_1 V_T$, where β_1 is a positive constant satisfying $0 \leq \beta_1 \leq 1$. (This case is essentially a default, but in this paper we refer to the next case as "default".)

3. If the default occurs, then the payoff is a constant fraction of the firm value at default time invested in riskless bonds up to maturity. That is, the payoff is $e^{r(T-g(\tau_A))}\beta_2 V_{g(\tau_A)}$, where β_2 is a positive constant satisfying $0 \leq \beta_2 \leq 1$.

Then, the payoff in our model is

$$X(T) = L\mathbf{1}_{\{g(\tau_A)>T, V_T \geq L\}} + \beta_1 V_T \mathbf{1}_{\{g(\tau_A)>T, V_T<L\}} + e^{r(T-g(\tau_A))}\beta_2 V_{g(\tau_A)}\mathbf{1}_{\{g(\tau_A)\leq T\}}. \tag{1}$$

Note that the third term is different from that of Fujita & Ishizaka [7].

Under this setting, the price of corporate bond at time 0, $D(0,T)$, is derived as follows(see Baxter & Rennie [1], for example):

$$D(0,T) = \mathbf{E}[e^{-rT}X(T)]. \tag{2}$$

3 Four examples with pricing

In the sequel, we will sometimes use the new measure $\mathbf{Q}$ with its Radon-Nikodym density defined by

$$\frac{d\mathbf{Q}}{d\mathbf{P}} := \frac{e^{-rT}V_T}{v}.$$

Girsanov's theorem implies that $W_t^{\mathbf{Q}} := W_t - \sigma t$ is a Brownian motion under $\mathbf{Q}$. Note the following points about $\mathbf{Q}$. First, this measure-change from $\mathbf{P}$ to $\mathbf{Q}$ can simplify the calculation. Second, $\mathbf{Q}$ is the matingale measure, with V_t as a numéraire.

Example 3.1

The default time is defined as

$$g(\tau_A) := \tau_A.$$

This is nothing but the first-passage-time model of Black & Cox [2], except for the default boundary.

In this case, the payoff to the bondholder at maturity is

$$X^{(1)}(T) = L\mathbf{1}_{\{\tau_A>T, V_T \geq L\}} + \beta_1 V_T \mathbf{1}_{\{\tau_A>T, V_T<L\}} + e^{r(T-\tau_A)}\beta_2 A\mathbf{1}_{\{\tau_A \leq T\}}. \tag{3}$$

Then, the price of corporate bond at time 0 is given by

$$\begin{aligned} D^{(1)}(0,T) &= \mathbf{E}[e^{-rT}L\mathbf{1}_{\{\tau_A>T, V_T\geq L\}}] + \mathbf{E}[e^{-rT}\beta_1 V_T\mathbf{1}_{\{\tau_A>T, V_T<L\}}] \\ &\quad + \mathbf{E}[e^{-r\tau_A}\beta_2 A\mathbf{1}_{\{\tau_A\leq T\}}] \\ &=: I_1^{(1)} + I_2^{(1)} + I_3^{(1)}, \end{aligned}$$

where

$$
\begin{aligned}
I_1^{(1)} &= \mathbf{E}[e^{-rT} L \mathbf{1}_{\{\tau_A > T, V_T \geq L\}}] \\
&= e^{-rT} L \cdot \mathbf{P}\{\tau_A > T, V_T \geq L\} \\
&= e^{-rT} L \left\{ \Phi\left(\frac{\log \frac{v}{L} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - (\frac{A}{v})^{\frac{2r}{\sigma^2}-1} \Phi\left(\frac{\log \frac{A^2}{vL} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right\}, \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
I_2^{(1)} &= \mathbf{E}[e^{-rT} \beta_1 V_T \mathbf{1}_{\{\tau_A > T, V_T < L\}}] \\
&= \beta_1 v \mathbf{E}\left[\frac{e^{-rT} V_T}{v} \mathbf{1}_{\{\tau_A > T, V_T < L\}} \right] \\
&= \beta_1 v \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau_A > T, V_T < L\}}] \\
&= \beta_1 v \cdot \mathbf{Q}\{\tau_A > T, V_T < L\} \\
&= \beta_1 v \left[\Phi\left(\frac{\log \frac{L}{v} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - \Phi\left(\frac{\log \frac{A}{v} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right. \\
&\quad \left. - (\frac{A}{v})^{\frac{2r}{\sigma^2}+1} \left\{ \Phi\left(\frac{\log \frac{vL}{A^2} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - \Phi\left(\frac{\log \frac{v}{A} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right\} \right], \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
I_3^{(1)} &= \mathbf{E}[e^{-r\tau_A} \beta_2 A \mathbf{1}_{\{\tau_A \leq T\}}] \\
&= \beta_2 v \cdot \mathbf{Q}\{\tau_A \leq T\} \\
&= \beta_2 v \left\{ \Phi\left(\frac{\log \frac{A}{v} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) + (\frac{A}{v})^{\frac{2r}{\sigma^2}+1} \Phi\left(\frac{\log \frac{A}{v} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right\}, \qquad (6)
\end{aligned}
$$

and $\Phi(\cdot)$ denotes the standard normal distribution function for a standardized normal variable.
Remark. There are errors in the equations (6) and (7) of Fujita & Ishizaka [7]. They must be replaced by (4) and (5), respectively, of the present paper.

Example 3.2

The default time is defined as

$$g(\tau_A) := \inf\{t \geq \tau_A | \int_{\tau_A}^{t} \mathbf{1}_{(-\infty,A]}(V_u) du > \alpha T\}, \quad (\inf \emptyset = \infty)$$

where α is a positive constant satisfying $0 < \alpha < 1$. Under this setting, default does not occur if the occupation time of the firm value below A is relatively short. This setting corresponds to the cumulative Parisian option of Chesney, Jeanblanc-Picqué & Yor [5]. This option is a down-and-out option knocked out if the occupation time of the underlying asset below A exceeds αT.

In this case, the payoff to the bondholder at maturity is

$$
\begin{aligned}
X^{(2)}(T) &= L \mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u) du \leq \alpha T, V_T \geq L\}} \\
&\quad + \beta_1 V_T \mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u) du \leq \alpha T, V_T < L\}} \\
&\quad + e^{r(T - g(\tau_A))} \beta_2 V_{g(\tau_A)} \mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u) du > \alpha T\}}. \qquad (7)
\end{aligned}
$$

Then, the price of corporate bond at time 0 is given by

$$\begin{aligned}
D^{(2)}(0,T) &= \mathbf{E}[e^{-rT}L\mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du\le\alpha T, V_T\ge L\}}] \\
&\quad + \mathbf{E}[e^{-rT}\beta_1 V_T \mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du\le\alpha T, V_T< L\}}] \\
&\quad + \mathbf{E}[e^{-rg(\tau_A)}\beta_2 V_{g(\tau_A)} \mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du>\alpha T\}}] \\
&=: I_1^{(2)} + I_2^{(2)} + I_3^{(2)},
\end{aligned}$$

where

$$\begin{aligned}
I_1^{(2)} &= \mathbf{E}[e^{-rT}L\mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du\le\alpha T, V_T\ge L\}}] \\
&= e^{-rT}L\cdot\mathbf{P}\left\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du\le\alpha T, V_T\ge L\right\} \\
&= e^{-rT}L\int_0^{T(1-\alpha)}du\int_{\frac{1}{\sigma}\log\frac{L}{A}}^{+\infty}da\int_0^{\alpha T}db\,\exp\{\frac{r-\frac{1}{2}\sigma^2}{\sigma}a-\frac{(r-\frac{1}{2}\sigma^2)^2(T-u)}{2\sigma^2}\} \\
&\quad\times h_{(W_{T-u},\int_0^{T-u}\mathbf{1}_{(-\infty,0]}(W_s)ds)}(a,b)f_{\tau_A}(u) \\
&\quad + e^{-rT}L\int_{T(1-\alpha)}^T du\,\Phi\left(\frac{\log\frac{A}{L}+(r-\frac{1}{2}\sigma^2)(T-u)}{\sigma\sqrt{T-u}}\right)f_{\tau_A}(u)+I_1^{(1)}, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
I_2^{(2)} &= \mathbf{E}[e^{-rT}\beta_1 V_T\mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du\le\alpha T, V_T< L\}}] \\
&= \beta_1 v\cdot\mathbf{Q}\left\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du\le\alpha T, V_T< L\right\} \\
&= \beta_1 v\int_0^{T(1-\alpha)}du\int_{-\infty}^{\frac{1}{\sigma}\log\frac{L}{A}}da\int_0^{\alpha T}db\,\exp\{\frac{r+\frac{1}{2}\sigma^2}{\sigma}a-\frac{(r+\frac{1}{2}\sigma^2)^2(T-u)}{2\sigma^2}\} \\
&\quad\times h_{(W_{T-u},\int_0^{T-u}\mathbf{1}_{(-\infty,0]}(W_s)ds)}(a,b)f'_{\tau_A}(u) \\
&\quad + \beta_1 v\int_{T(1-\alpha)}^T du\,\Phi\left(\frac{\log\frac{L}{A}-(r+\frac{1}{2}\sigma^2)(T-u)}{\sigma\sqrt{T-u}}\right)f'_{\tau_A}(u)+I_2^{(1)}, \qquad (9)
\end{aligned}$$

$$\begin{aligned}
I_3^{(2)} &= \mathbf{E}[e^{-rg(\tau_A)}\beta_2 V_{g(\tau_A)}\mathbf{1}_{\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du>\alpha T\}}] \\
&= \beta_2 v\cdot\mathbf{Q}\left\{\int_0^T \mathbf{1}_{(-\infty,A]}(V_u)du>\alpha T\right\} \\
&= \beta_2 v\int_0^{T(1-\alpha)}du\int_{-\infty}^{+\infty}da\int_{\alpha T}^{T-u}db\,\exp\{\frac{r+\frac{1}{2}\sigma^2}{\sigma}a-\frac{(r+\frac{1}{2}\sigma^2)^2(T-u)}{2\sigma^2}\} \\
&\quad\times h_{(W_{T-u},\int_0^{T-u}\mathbf{1}_{(-\infty,0]}(W_s)ds)}(a,b)f'_{\tau_A}(u), \qquad (10)
\end{aligned}$$

$$f_{\tau_A}(u) = \frac{\log\frac{v}{A}}{\sqrt{2\pi\sigma^2u^3}}\exp\{-\frac{(\log\frac{A}{v}-(r-\frac{1}{2}\sigma^2)u)^2}{2\sigma^2u}\}, \qquad (11)$$

$$h_{(W_t,\int_0^t \mathbf{1}_{(-\infty,0]}(W_s)ds)}(a,b) = \begin{cases} \dfrac{a}{2\pi}\displaystyle\int_b^t ds \frac{1}{\sqrt{s^3(t-s)^3}} \exp\{-\frac{a^2}{2(t-s)}\} & \text{for } a>0 \\ \dfrac{-a}{2\pi}\displaystyle\int_0^b ds \frac{1}{\sqrt{s^3(t-s)^3}} \exp\{-\frac{a^2}{2s}\} & \text{for } a<0. \end{cases} \tag{12}$$

Example 3.3

The default time is defined as

$$g(\tau_A) := \inf\{t \geq \tau_A | \int_{\tau_A}^t \mathbf{1}_{(-\infty,A]}(V_u)du > \alpha(T-\tau_A)\}.$$

This setting is very similar to Example 3.2, except that the default time depends on the caution time. That is, in this setting, it is less probable that default occurs, as a period between the caution time and maturity is longer. This condition for default is the same as Section IV.1 of Fujita & Ishizaka [7], corresponding to the cumulative Parisian Edokko option of Fujita & Miura [8]. This option is also a down-and-out option knocked out if, after the caution time, the occupation time of the underlying asset below A exceeds $\alpha(T-\tau_A)$.

In this case, the payoff to the bondholder at maturity is

$$\begin{aligned} X^{(3)}(T) \quad = \quad & L\mathbf{1}_{\{\{\tau_A\leq T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du\leq\alpha(T-\tau_A), V_T\geq L\}\cup\{\tau_A>T, V_T\geq L\}\}} \\ & + \beta_1 V_T \mathbf{1}_{\{\{\tau_A\leq T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du\leq\alpha(T-\tau_A), V_T< L\}\cup\{\tau_A>T, V_T< L\}\}} \\ & + e^{r(T-\tau_A)}\beta_2 V_{g(\tau_A)} \mathbf{1}_{\{\tau_A<T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du>\alpha(T-\tau_A)\}}. \end{aligned} \tag{13}$$

Then, the price of corporate bond at time 0 is given by

$$\begin{aligned} D^{(3)}(0,T) \quad = \quad & \mathbf{E}[e^{-rT}L\mathbf{1}_{\{\{\tau_A\leq T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du\leq\alpha(T-\tau_A), V_T\geq L\}\cup\{\tau_A>T, V_T\geq L\}\}}] \\ & + \mathbf{E}[e^{-rT}\beta_1 V_T \mathbf{1}_{\{\{\tau_A\leq T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du\leq\alpha(T-\tau_A), V_T< L\}\cup\{\tau_A>T, V_T< L\}\}}] \\ & + \mathbf{E}[e^{-rg(\tau_A)}\beta_2 V_{g(\tau_A)} \mathbf{1}_{\{\tau_A<T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du>\alpha(T-\tau_A)\}}] \\ =: \quad & I_1^{(3)} + I_2^{(3)} + I_3^{(3)}, \end{aligned}$$

where

$$\begin{aligned} I_1^{(3)} \quad = \quad & \mathbf{E}[e^{-rT}L\mathbf{1}_{\{\{\tau_A\leq T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du\leq\alpha(T-\tau_A), V_T\geq L\}\cup\{\tau_A>T, V_T\geq L\}\}}] \\ = \quad & e^{-rT}L\cdot\mathbf{P}\left\{\{\tau_A\leq T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du\leq\alpha(T-\tau_A), V_T\geq L\}\cup\{\tau_A>T, V_T\geq L\}\right\} \\ = \quad & e^{-rT}L\int_0^T du \int_{\frac{1}{\sigma}\log\frac{L}{A}}^{+\infty} da \int_0^{\alpha(T-u)} db \, \exp\{\frac{r-\frac{1}{2}\sigma^2}{\sigma}a - \frac{(r-\frac{1}{2}\sigma^2)^2(T-u)}{2\sigma^2}\} \\ & \times h_{(W_{T-u},\int_0^{T-u}\mathbf{1}_{(-\infty,0]}(W_s)ds)}(a,b) f_{\tau_A}(u) + I_1^{(1)}, \end{aligned} \tag{14}$$

$$I_2^{(3)} \quad = \quad \mathbf{E}[e^{-rT}\beta_1 V_T \mathbf{1}_{\{\{\tau_A\leq T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du\leq\alpha(T-\tau_A), V_T< L\}\cup\{\tau_A>T, V_T< L\}\}}]$$

$$
\begin{aligned}
&= \beta_1 v \cdot \mathbf{Q}\left\{\{\tau_A \le T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du \le \alpha(T-\tau_A), V_T < L\} \cup \{\tau_A > T, V_T < L\}\right\} \\
&= \beta_1 v \int_0^T du \int_{-\infty}^{\frac{1}{\sigma}\log\frac{L}{A}} da \int_0^{\alpha(T-u)} db \, \exp\{\frac{r+\frac{1}{2}\sigma^2}{\sigma}a - \frac{(r+\frac{1}{2}\sigma^2)^2(T-u)}{2\sigma^2}\} \\
&\quad \times h_{(W_{T-u}, \int_0^{T-u}\mathbf{1}_{(-\infty,0]}(W_s)ds)}(a,b) f'_{\tau_A}(u) + I_2^{(1)}, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
I_3^{(3)} &= \mathbf{E}[e^{-rg(\tau_A)}\beta_2 V_{g(\tau_A)}\mathbf{1}_{\{\tau_A<T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du > \alpha(T-\tau_A)\}}] \\
&= \beta_2 v \cdot \mathbf{Q}\left\{\tau_A < T, \int_{\tau_A}^T \mathbf{1}_{(-\infty,A]}(V_u)du > \alpha(T-\tau_A)\right\} \\
&= \beta_2 v \int_0^T du \int_{-\infty}^{+\infty} da \int_{\alpha(T-u)}^{T-u} db \, \exp\{\frac{r+\frac{1}{2}\sigma^2}{\sigma}a - \frac{(r+\frac{1}{2}\sigma^2)^2(T-u)}{2\sigma^2}\} \\
&\quad \times h_{(W_{T-u}, \int_0^{T-u}\mathbf{1}_{(-\infty,0]}(W_s)ds)}(a,b) f'_{\tau_A}(u). \qquad (16)
\end{aligned}
$$

Example 3.4

The default time is defined as

$$
g(\tau_A) := \begin{cases} (1-\alpha)\tau_A + \alpha T & \text{if } \tau'_B > (1-\alpha)\tau_A + \alpha T \\ \infty & \text{if } \tau'_B \le (1-\alpha)\tau_A + \alpha T, \end{cases}
$$

where

$$
\tau'_B := \inf\{t \ge \tau_A | V_t = B\},
$$

with B a positive constant greater than A. Under this setting, default does not occur if the firm value recovers level B within a given peiod depending on the caution time. This condition for default is the same as in Section IV.2 of Fujita & Ishizaka [7], corresponding to the simple Parisian like Edokko option of Fujita & Miura [8]. This option is also a down-and-out option knocked out if, after the caution time, it takes more than $\alpha(T-\tau_A)$ for the underlying asset to return to another bar B.

In this case, the payoff to the bondholder at maturity is

$$
\begin{aligned}
X^{(4)}(T) &= L\mathbf{1}_{\{\{\tau_A \le T, \tau'_B \le (1-\alpha)\tau_A + \alpha T, V_T \ge L\} \cup \{\tau_A > T, V_T \ge L\}\}} \\
&\quad + \beta_1 V_T \mathbf{1}_{\{\{\tau_A \le T, \tau'_B \le (1-\alpha)\tau_A + \alpha T, V_T < L\} \cup \{\tau_A > T, V_T < L\}\}} \\
&\quad + e^{r(1-\alpha)(T-\tau_A)}\beta_2 V_{(1-\alpha)\tau_A + \alpha T}\mathbf{1}_{\{\tau_A \le T, \tau'_B > (1-\alpha)\tau_A + \alpha T\}}. \qquad (17)
\end{aligned}
$$

Then, the price of corporate bond at time 0 is given by

$$
\begin{aligned}
D^{(4)}(0,T) &= \mathbf{E}[e^{-rT}L\mathbf{1}_{\{\{\tau_A \le T, \tau'_B \le (1-\alpha)\tau_A + \alpha T, V_T \ge L\} \cup \{\tau_A > T, V_T \ge L\}\}}] \\
&\quad + \mathbf{E}[e^{-rT}\beta_1 V_T \mathbf{1}_{\{\{\tau_A \le T, \tau'_B \le (1-\alpha)\tau_A + \alpha T, V_T < L\} \cup \{\tau_A > T, V_T < L\}\}}] \\
&\quad + \mathbf{E}[e^{-r\{(1-\alpha)\tau_A + \alpha T\}}\beta_2 V_{(1-\alpha)\tau_A + \alpha T}\mathbf{1}_{\{\tau_A \le T, \tau'_B > (1-\alpha)\tau_A + \alpha T\}}] \\
&=: I_1^{(4)} + I_2^{(4)} + I_3^{(4)},
\end{aligned}
$$

where

$$
\begin{aligned}
I_1^{(4)} &= \mathbf{E}[e^{-rT}L\mathbf{1}_{\{\{\tau_A\le T,\tau'_B\le(1-\alpha)\tau_A+\alpha T,V_T\ge L\}\cup\{\tau_A>T,V_T\ge L\}\}}] \\
&= e^{-rT}L\cdot\mathbf{P}\left\{\{\tau_A\le T,\tau'_B\le(1-\alpha)\tau_A+\alpha T,V_T\ge L\}\cup\{\tau_A>T,V_T\ge L\}\right\} \\
&= e^{-rT}L\int_0^T du\int_u^{(1-\alpha)u+\alpha T} ds\,\Phi\left(\frac{\log\frac{B}{L}+(r-\frac{1}{2}\sigma^2)(T-s)}{\sigma\sqrt{T-s}}\right)m_{\tau'_B}(s-u)f_{\tau_A}(u) \\
&\quad +I_1^{(1)}, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
I_2^{(4)} &= \mathbf{E}[e^{-rT}\beta_1V_T\mathbf{1}_{\{\{\tau_A\le T,\tau'_B\le(1-\alpha)\tau_A+\alpha T,V_T<L\}\cup\{\tau_A>T,V_T<L\}\}}] \\
&= \beta_1 v\cdot\mathbf{Q}\left\{\{\tau_A\le T,\tau'_B\le(1-\alpha)\tau_A+\alpha T,V_T<L\}\cup\{\tau_A>T,V_T<L\}\right\} \\
&= \beta_1 v\int_0^T du\int_u^{(1-\alpha)u+\alpha T} ds\,\Phi\left(\frac{\log\frac{L}{B}-(r+\frac{1}{2}\sigma^2)(T-s)}{\sigma\sqrt{T-s}}\right)m'_{\tau'_B}(s-u)f'_{\tau_A}(u) \\
&\quad +I_2^{(1)}, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
I_3^{(4)} &= \mathbf{E}[e^{-r\{(1-\alpha)\tau_A+\alpha T\}}\beta_2V_{(1-\alpha)\tau_A+\alpha T}\mathbf{1}_{\{\tau_A\le T,\tau'_B>(1-\alpha)\tau_A+\alpha T\}}] \\
&= \beta_2 v\cdot\mathbf{Q}\{\tau_A\le T,\tau'_B>(1-\alpha)\tau_A+\alpha T\} \\
&= \beta_2 v\int_0^T du\left\{\Phi\left(\frac{\log\frac{B}{A}-(r+\frac{1}{2}\sigma^2)\alpha(T-u)}{\sigma\sqrt{\alpha(T-u)}}\right)\right. \\
&\quad \left.-(\frac{B}{A})^{\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\log\frac{A}{B}-(r+\frac{1}{2}\sigma^2)\alpha(T-u)}{\sigma\sqrt{\alpha(T-u)}}\right)\right\}f'_{\tau_A}(u), \qquad (20)
\end{aligned}
$$

$$
f'_{\tau_A}(u) = \frac{\log\frac{v}{A}}{\sqrt{2\pi\sigma^2u^3}}\exp\{-\frac{(\log\frac{A}{v}-(r+\frac{1}{2}\sigma^2)u)^2}{2\sigma^2u}\}, \qquad (21)
$$

$$
m_{\tau'_B}(s-u) = \frac{\log\frac{B}{A}}{\sqrt{2\pi\sigma^2(s-u)^3}}\exp\{-\frac{(\log\frac{B}{A}-(r-\frac{1}{2}\sigma^2)(s-u))^2}{2\sigma^2(s-u)}\}, \qquad (22)
$$

$$
m'_{\tau'_B}(s-u) = \frac{\log\frac{B}{A}}{\sqrt{2\pi\sigma^2(s-u)^3}}\exp\{-\frac{(\log\frac{B}{A}-(r+\frac{1}{2}\sigma^2)(s-u))^2}{2\sigma^2(s-u)}\}. \qquad (23)
$$

4 Numerical Examples

In this section, some numerical examples of the models proposed above are presented. In paticular, the effect of level A and α related to default condition on each bond price is investigated. We use quasi-Monte Carlo simulation with Sobol' sequence. It aims at improving the performance of Monte Carlo simulation with regard to the convergence order and is often used for finance applications (see Press, Teukolsky & Vetterling [11] for details).

Figure 1 describes the effect of level A on the bond prices. The parameters have been chosen: $v = 120, r = 0.03, \sigma = 0.2, T = 5, L = 100, B = 90, \alpha = 0.1, \beta_1 = \beta_2 = 1$. As Figure 1 shows, greater A increases $D^{(1)}$. This result seems strange because it becomes more probable for greater A that "default" occurs. But differently stated, level A(or the default boundary) also

Figure 1: **The effect of A on the bond prices** Parameters:$v = 120, r = 0.03, \sigma = 0.2, T = 5, L = 100, B = 90, \alpha = 0.1, \beta_1 = \beta_2 = 1$

assures the bondholder of more amount than A, which is related to the third term denoted by $I_3^{(1)}$. Thus, when the coefficient β_2 is chosen on some small level, for example $\beta_2 = 0.7$, greater A decreases $D^{(1)}$. Contrary to $D^{(1)}$, greater A decreases $D^{(2)}$ and $D^{(3)}$. These payoffs to the bondholders at the default time are always less than A. For $D^{(4)}$, the additional parameter B affects its price differently. As A is lower, each bond price converges to 80.12 which is derived from the framework of Merton [10] with the same parameters.

Figure 2: **The effect of α on the bond prices** Parameters:$v = 120, r = 0.03, \sigma = 0.2, T = 5, L = 100, A = 80, B = 90, \beta_1 = \beta_2 = 1$

Figure 2 describes the effect of α on the bond prices. The parameters have been chosen: $v = 120, r = 0.03, \sigma = 0.2, T = 5, L = 100, A = 80, B = 90, \beta_1 = \beta_2 = 1$. Greater α, which implies a relaxation of default condition, decreases each bond price. Each bond price converges to $D^{(1)}$ with the same parameters, as α is closer to 0.

5 Summary

In this paper, we considered a structural model for corporate default. In our model, if the default occurs before maturity, then the payoff to the bondholder is a constant fraction of the firm value at default time invested in riskless bonds up to maturity. The payoff is thus more realistic than in the existing models. We also valued corporate bonds according to four examples within our framework. The measure-change technique considerably simplifies the calculation and the new measure has a financial interpretation. The prices for some examples are in closed form. In numerical examples, the effect of A and α related to default condition on each bond has been presented.

Empirical studies remain to compare our models with existing ones. Also some generalizations of the model are left for future studies, such as models with caution time different from the first passage time to some constant barrier, and models with stochastic interest rate.

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Figure 1

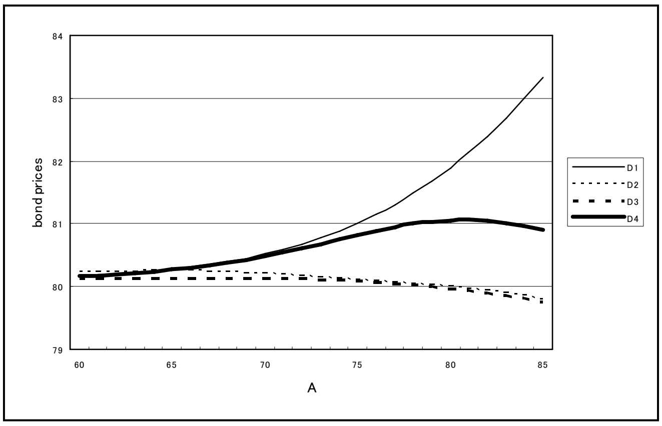

Figure 2