A complete-market generalization of the Black-Scholes model

Koichiro TAKAOKA[*]

Hitotsubashi University

Abstract

The author proposes a new single-stock generalization of the Black-Scholes model. The stock price process is Markovian, the volatility is time-varying, and the market is complete. We also consider the option pricing based on our model and a connection with the equilibrium theory.

Keywords. Black-Scholes model, complete market models, equilibrium price, option pricing, volatility

1 Introduction

To uniquely determine the arbitrage-free price of derivative securities such as options, we need to model the price behavior of the underlying asset. The celebrated Black-Scholes model [1] formulates the stock price process as

$$S_t \; := \; S_0 \exp\left(\sigma W_t \, + \, \mu t\right)$$

and the riskless bond price as

$$B_t \; := \; e^{rt},$$

where W is a one-dimensional standard Brownian motion, and σ, μ, and r are constants. The model has good mathematical tractability, but it is often pointed out that the following assumptions of the model do not fit the real-world data:

- the volatility process is constant in time;

- for every fixed time $t > 0$, the logarithmic rate of return $\frac{1}{t} \log\left(\frac{S_t}{S_0}\right)$ is normally distributed.

More realistic continuous-time models have also been proposed, such as the CEV model [2], the Hull-White model [6] (including the Heston model [3]), and some jump diffusion models (e.g. Merton [10]). The books of Hull [5] and Musiela

[*]Graduate School of Commerce and Management, Hitotsubashi University, Kunitachi-City, Tokyo 186-8601, Japan. Email: takaoka@math.hit-u.ac.jp

& Rutkowski [11] provide good references for those generalizations. For many of those generalized models – among the above cited papers, all but the CEV model – the market is *incomplete* and thus the absence of arbitrage is not a sufficient criterion for giving the unique price of the derivative securities.

In the present article, we propose another simple single-stock generalization of the Black-Scholes model based on the author's working paper [13]. Our models have the following properties:

- the stock price process is Markovian;

- the volatility is time-varying;

- the market is complete, *i.e.* there exists a unique risk neutral measure;

- the right and the left tails of the distribution of rate of return are asymmetric.

This paper is organized as follows. In Section 2 we give our first model and some examples. Some properties of our model are explained in Section 3, and option pricing is discussed in Section 4. Section 5 gives an equilibrium characterization of our model. In Section 6 we give our second model, the reciprocal of the first one. The appendix is a technical note on the applicability of the stochastic Fubini theorem for the proof of Propositions 3.1 and 3.3.

Notation. Throughout this paper, $\mathbb{R}_{++}$ is defined as the open interval $(0, \infty)$.

2 Our first model and some examples

We model the stock price process S and the riskless bond price B as

$$
\begin{aligned}
S_t &:= S_0 \; e^{rt} \int_0^\infty \exp\left\{ \sigma(W_t + Ct) - \frac{\sigma^2}{2}t \right\} \lambda(d\sigma), \\
B_t &:= e^{rt}.
\end{aligned}
$$

Here

- W is a one-dimensional standard Brownian motion on a filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P \right)$, where the filtration $(\mathscr{F}_t)_{t\geq 0}$ is generated by W ;

- λ is a deterministic measure on $\left(\mathbb{R}_{++}, \mathscr{B}(\mathbb{R}_{++}) \right)$ such that

$$\lambda(\mathbb{R}_{++}) = 1 \quad \text{and} \quad \int_0^\infty \sigma\, \lambda(d\sigma) < \infty\,;$$

- r and C are constants.

The stochastic process S is thus a weighted average of geometric Brownain motions. Note that the average is taken over σ and not over t. The assumption $\int_0^\infty \sigma\, \lambda(d\sigma) < \infty$ is a sufficient condition for the applicability of the stochastic Fubini theorem for the proof of Propositions 3.1 and 3.3: see the Appendix for details.

The measure λ determines the stock price dynamics. We give several examples.

Example 2.1. If the measure λ is concentrated on one point, then S is a constant-volatility geometric Brownian motion, *i.e.* the Black-Scholes stock price model.

Example 2.2. The recent paper of Ishimura & Sakaguchi [7] has pointed out that, for the case where λ has mass $\frac{1}{2}$ on each of the two points $\bar\sigma \pm \epsilon$, the process becomes

$$S_t = S_0\, e^{rt}\, \exp\left\{ \bar\sigma\,(W_t + Ct) - \tfrac{\bar\sigma^2 + \epsilon^2}{2}\, t \right\}\, \cosh\left\{ \epsilon\,(W_t + Ct) - \epsilon\,\bar\sigma\, t \right\}.$$

This particular model is thoroghly investigated in their paper.

Example 2.3. Suppose λ is uniformly distributed over the interval $[a, b]$. Then

$$\begin{aligned}
S_t = {}& S_0\, e^{rt}\, \frac{1}{b-a} \sqrt{\frac{2\pi}{t}}\, \exp\left(\frac{(W_t + Ct)^2}{2t} \right) \\
& \cdot \left\{ \Phi\!\left(b\sqrt{t} - \frac{W_t + Ct}{\sqrt{t}} \right) - \Phi\!\left(a\sqrt{t} - \frac{W_t + Ct}{\sqrt{t}} \right) \right\},
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function.

Example 2.4. Suppose λ has the following density w.r.t. the Lebesgue measure:

$$\frac{d\lambda(\sigma)}{d\sigma} = \frac{1}{Z}\, \exp\left(-\tfrac{a}{2}\sigma^2 + b\,\sigma \right).$$

Here $a > 0$ and $b \in \mathbb{R}$ are both constants, and $Z := \frac{1}{\sqrt{a}} f\!\left(\frac{b}{\sqrt{a}} \right)$ is the normalizing constant where the deterministic function $f : \mathbb{R} \to \mathbb{R}_{++}$ is defined by

$$f(x) := \sqrt{2\pi}\, \exp(\tfrac{x^2}{2})\, \Phi(x).$$

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Then

$$S_t = S_0\ e^{rt} \cdot \frac{1}{\frac{1}{\sqrt{a}} f\left(\frac{b}{\sqrt{a}}\right)} \cdot \frac{1}{\sqrt{a+t}} f\left(\frac{(W_t + Ct) + b}{\sqrt{a+t}}\right).$$

Example 2.5. Suppose that the density of λ is

$$b\ \exp\left\{-b\left(\sigma - \ell\right)\right\} \mathbf{1}_{\{\sigma > \ell\}},$$

where $b > 0$ and $\ell \geq 0$ are both constants. Then

$$\begin{aligned}
S_t &= S_0\ e^{rt}\ \exp\left\{\ell\left(W_t + Ct\right) - \frac{\ell^2}{2} t\right\} \\
&\quad \cdot \frac{b}{\sqrt{t}} f\left(\frac{(W_t + Ct) - \ell t - b}{\sqrt{t}}\right)
\end{aligned}$$

with the function $f(\cdot)$ defined in the previous example.

3 Some properties of the stochastic process S

Proposition 3.1   *There exisits a unique probability measure Q under which the discounted stock price process*

$$\tilde{S}_t := \frac{S_t}{B_t}$$

is a martingale.

Proof.   Since

$$\tilde{S}_t = S_0 \int_0^\infty \exp\left\{\sigma(W_t + Ct) - \frac{\sigma^2}{2} t\right\} \lambda(d\sigma),$$

we see from the Itô formula and the stochastic Fubini theorem (see the Appendix) that

$$d\tilde{S}_t = \left[S_0 \int_0^\infty \sigma \exp\left\{\sigma(W_t + Ct) - \frac{\sigma^2}{2} t\right\} \lambda(d\sigma)\right] \left(dW_t + C\,dt\right).$$

The process $\tilde{S}$ is thus a martingale only under the measure Q with Radon-Nikodym density

$$\frac{dQ}{dP}\Big|_{\mathscr{F}_t} = \exp\left(-CW_t - \frac{C^2}{2} t\right). \qquad \square$$

Remark .   Let $W_t^Q := W_t + Ct$, a Q-Brownian motion. We then have

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \frac{\int_0^\infty \sigma \exp\left(\sigma W_t^Q - \frac{\sigma^2}{2} t\right) \lambda(d\sigma)}{\int_0^\infty \exp\left(\sigma W_t^Q - \frac{\sigma^2}{2} t\right) \lambda(d\sigma)}\, dW_t^Q.$$

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It follows that the volatility at time t, *i.e.* the coefficient of dW_t^Q in the above expression, equals

$$\int_0^\infty \sigma \, \lambda_t(d\sigma),$$

where the measure λ_t is defined by

$$\lambda_t(d\sigma) \; := \; \frac{\exp\left(\sigma W_t^Q - \frac{\sigma^2}{2}t\right)\lambda(d\sigma)}{\int_0^\infty \exp\left(\sigma W_t^Q - \frac{\sigma^2}{2}t\right)\lambda(d\sigma)}.$$

(Note that $\lambda_0 = \lambda$.) The volatility process is therefore time-varying unless the measure λ is concentrated on one point.

Proposition 3.2 *The process S is Markovian both under P and under Q.*

Proof. We define the deterministic function

$$g(x,t) \; := \; S_0 \; e^{rt} \int_0^\infty \exp\left\{\sigma(x+Ct) - \frac{\sigma^2}{2}t\right\}\lambda(d\sigma);$$

then we see that $S_t = g(W_t, t)$. The function g is strictly increasing w.r.t the first argument, so, if we know the value of S_t at time t, then we also see the value of W_t and the future dynamics of W as well. It follows that S is Markovian under P. For the measure Q our assertion can be proved similarly with the function

$$\tilde{g}(x,t) \; := \; S_0 \; e^{rt} \int_0^\infty \exp\left\{\sigma x - \frac{\sigma^2}{2}t\right\}\lambda(d\sigma). \qquad \square$$

The next proposition shows a relationship between the moments of the measure λ and the variation of a "single path" of S, which is relevant to the calibration of λ.

Proposition 3.3 *Let $n \in \mathbb{N}$ and suppose $\int_0^\infty \sigma^n \, \lambda(d\sigma) < \infty$. Define recursively*

$$I_t^{(0)} \; := \; \frac{S_t}{S_0 \, e^{rt}},$$

$$I_t^{(k)} \; := \; \sqrt{\frac{d\langle I_t^{(k-1)}\rangle}{dt}} \qquad for \quad k = 1, 2, \ldots, n,$$

where $\langle \cdot \rangle$ denotes the quadratic variation. Then we have that, for $k = 0, 1, \ldots, n$,

$$I_t^{(k)} \; = \; \int_0^\infty \sigma^k \exp\left(\sigma W_t^Q - \frac{\sigma^2}{2}t\right)\lambda(d\sigma)$$

and

$$\int_0^\infty \sigma^k \, \lambda(d\sigma) \; = \; I_0^{(k)}.$$

Proof. The Itô formula together with the stochastic Fubini theorem proves the assertion. See Appendix for details. □

Remark 3.4 For every fixed time $t > 0$, it is not hard to show the following properties.

- Let M be the supremum of the support of the measure λ. If $M < \infty$ then the right tail of the distribution of the logarithmic rate of return $\frac{1}{t} \log\left(\frac{S_t}{S_0}\right)$ is roughly the same as that of the normal distributions with variance $\frac{M^2}{t}$. If $M = \infty$ then the right tail is heavier than the normal distributions.

- Let m be the infimum of the support of λ. If $m > 0$ then the left tail of the distribution of $\frac{1}{t} \log\left(\frac{S_t}{S_0}\right)$ is roughly the same as that of the normal distributions with variance $\frac{m^2}{t}$. If $m = 0$ then the left tail is thinner than the normal distributions.

- Unless the measure λ is concentrated on one point, the tail distribution of $\frac{1}{t} \log\left(\frac{S_t}{S_0}\right)$ is strictly heavier on the right than on the left.

For the model of Example 2.4, it holds that

$$f(x) \quad \sim \quad \sqrt{2\pi} \, \exp\left(\frac{x^2}{2}\right) \qquad (x \to \infty),$$

$$f(x) \quad \sim \quad \frac{1}{|x|} \qquad\qquad (x \to -\infty),$$

and we can thus recover the above assertion that the distribution of $\frac{1}{t} \log\left(\frac{S_t}{S_0}\right)$ has a heavier right tail than the normal distributions and a thinner left tail.

For our second model of Section 6, things are completely opposite for the right and the left tails.

4 Option pricing based on our model

Consider the European call option of S with maturity $T > 0$ and strike price $K > 0$. Assume that no dividend is paid to the stockholders until time T.

Proposition 4.1 *The call option price at time $0 \leq t < T$ is*

$$E^Q\left[\frac{(S_T - K)^+}{e^{r(T-t)}} \,\Big|\, \mathscr{F}_t \right]$$

$$= \quad S_t \int_0^\infty \Phi\left(-\frac{\hat{x}_t}{\sqrt{T-t}} + \sigma\sqrt{T-t} \right) \lambda_t(d\sigma) \; - \; K\, e^{-r(T-t)}\, \Phi\left(-\frac{\hat{x}_t}{\sqrt{T-t}} \right),$$

where the measure λ_t is defined in the remark after Proposition 3.1 and $\hat{x}_t = \hat{x}_t(W_t^Q)$ is the unique solution of the equation for x :

$$S_t\, e^{r(T-t)} \int_0^\infty \exp\left\{ \sigma\, x - \frac{\sigma^2}{2}(T-t) \right\} \lambda_t(d\sigma) \; = \; K.$$

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Remark. We can also show that, for the hedging portfolio, the amount of stocks to hold at time t is

$$\int_0^\infty \Phi\left(-\frac{\hat{x}_t}{\sqrt{T-t}} + \sigma\sqrt{T-t} \right) \lambda_t(d\sigma),$$

a property similar to the Black-Scholes case.

Proof of Proposition. Since S is a weighted average of constant-volatility geometric Brownain motions, it is possible to use Jamshidian's trick [8], originally for coupon-bearing bonds, to prove our assertion. In more detail, we proceed as follows. Since

$$S_T \;=\; S_t \; e^{r(T-t)} \int_0^\infty \exp\left\{ \sigma\,(W_T^Q - W_t^Q) - \frac{\sigma^2}{2}(T-t) \right\} \lambda_t(d\sigma) \qquad \text{a.s.,}$$

it follows that

$$S_T \,>\, K \iff W_T^Q - W_t^Q \,>\, \hat{x}_t$$

almost surely conditioned by $\mathscr{F}_t$, and

$$E^Q\Big[\frac{(S_T - K)^+}{e^{r(T-t)}} \Big| \mathscr{F}_t \Big]$$

$$= \; e^{-r(T-t)}\, E^Q\big[S_T\, \mathbf{1}_{\{S_T > K\}} \,\big|\, \mathscr{F}_t \big] - K\, e^{-r(T-t)}\, Q\big(S_T > K \,\big|\, \mathscr{F}_t \big)$$

$$= \; S_t \int_0^\infty E^Q\Big[\exp\left\{ \sigma\,(W_T^Q - W_t^Q) - \frac{\sigma^2}{2}(T-t) \right\} \mathbf{1}_{\{W_T^Q - W_t^Q > \hat{x}_t\}} \,\Big|\, \mathscr{F}_t \Big] \lambda_t(d\sigma)$$
$$\quad - K\, e^{-r(T-t)}\, Q\big(W_T^Q - W_t^Q > \hat{x}_t \,\big|\, \mathscr{F}_t \big)$$

$$= \; S_t \int_0^\infty \Phi\left(-\frac{\hat{x}_t}{\sqrt{T-t}} + \sigma\sqrt{T-t} \right) \lambda_t(d\sigma) \; - \; K\, e^{-r(T-t)}\, \Phi\left(-\frac{\hat{x}_t}{\sqrt{T-t}} \right).$$

5   An equilibrium characterization

In this section, we will give an equilibrium formulation of stock price processes and will discuss how our stock price model S is characterized among the equilibria. Suppose that there are two kinds of securities in the market, namely the stock and the riskless bond. The time horizon T is set to be finite.

The assumptions for our equilibrium formulation are as follows.

- The initial stock price is already given.

- Only the initial endowments are considered. No endowment is given at time $t > 0$.

- No consumption is considered. Each agent is interested only in her final wealth realized by her trades together with her initial endowment. The utility functions are heterogeneous and are either power utilities or logarithmic utilities.

- For simplicity, we also assume that

  - no dividend is paid to the stockholders;
  - the interest rate is zero.

Definition 5.1 Let $(A, \mathscr{A}, \nu)$ be a measure space, where $\{a\} \in \mathscr{A}$ for all $a \in A$. Consider the two measurable functions

$$
\begin{aligned}
V &: \quad A \to \mathbb{R}_{++}, \\
p &: \quad A \to (-\infty, 1)
\end{aligned}
$$

and a constant $s_0 > 0$. We assume that

$$
\int_A V(a)\, \nu(da) \; < \; \infty \quad \text{and} \quad \int_A \frac{1}{1 - p(a)}\, V(a)\, \nu(da) \; < \; \infty.
$$

The economic interpretation is as follows. The set of all agents in the market is formulated by the measure space $(A, \mathscr{A}, \nu)$, which is used for the theory of large economies, c.f. Hildenbrand [4]. $V(a)$ is interpreted as the value of the securities endowed initially to agent a, and $p(a)$ is the exponent of the agent's power utility, with the convention that the utility is set to be logarithmic if $p(a) = 0$. (See Definition 5.3 for deatils.) Also, s_0 is interpreted as the initial total market value of the stock. We can interpret $\frac{1}{1-p(a)}$ as the risk tolerance for agent a, so the second assumption of Definition 5.1 says that the risk tolerance of the entire market is finite.

In the following definitions, S_t is the total market value process of the stock, and $\theta_t^*(a)$ represents the weight of agent a's portfolio invested in the stock at time t. Moreover, $V_t\big(a, \theta^*(a)\big)$ is the value process for agent a.

Definition 5.2 Let $\big(\, \Omega,\, \mathscr{F},\, (\mathscr{F}_t)_{t \in [0,T]},\, P \,\big)$ be a filtered probability space, satisfying the usual conditions, carrying a one-dimensional standard Brownian motion $(W_t)_{t \in [0,T]}$. Assume that the filtration $(\mathscr{F}_t)_{t \in [0,T]}$ is generated by W. An $\mathbb{R}_{++}$-valued semimartingale $(S_t)_{t \in [0,T]}$ is said to be of class $\mathscr{S}$ if

$$
\begin{cases}
\dfrac{dS_t}{S_t} &= \quad \sigma(S_t, t)\, dW_t \;+\; \mu(S_t, t)\, dt, \\[2mm]
S_0 &= \quad s_0,
\end{cases}
$$

where the two functions

$$
\begin{aligned}
\sigma &: \quad \mathbb{R}_{++} \times [0, T] \to \mathbb{R}_{++}, \\
\mu &: \quad \mathbb{R}_{++} \times [0, T] \to \mathbb{R}
\end{aligned}
$$

are deterministic and measurable. For notational simplicity, we denote $\sigma(S_t, t)$ and $\mu(S_t, t)$ by σ_t and μ_t, respectively.

Definition 5.3 The pair $\left(S, \{ \theta^*(a) \}_{a \in A} \right)$ of a stochastic process $S \in \mathscr{S}$ and a family of predictable processes $\theta^*(a) = (\theta_t^*(a))_{t \in (0,T]}$ is said to be an *equilibrium* if the following two conditions are satisfied.

1. For almost every $a \in A$, the predictable process $(\theta_t^*(a))_{t \in (0,T]}$ maximises the expected utility

$$
\begin{cases}
\frac{1}{p(a)} E\left[V_T(a, \theta)^{p(a)} \right] & \text{if } p(a) \neq 0, \\[2mm]
E\left[\log V_T(a, \theta) \right] & \text{if } p(a) = 0,
\end{cases}
$$

where $V_t(a, \theta)$ is an $\mathbb{R}_{++}$-valued semimartingale defined by the SDE

$$
\begin{cases}
\dfrac{dV_t(a, \theta)}{V_t(a, \theta)} = \theta_t \dfrac{dS_t}{S_t}, \\[3mm]
V_0(a, \theta) = V(a).
\end{cases}
$$

The maximum is taken over all predictable processes θ such that

$$
\int_0^T \theta_t^2 \sigma_t^2 + |\theta_t \mu_t| \, dt < \infty \quad a.s.
$$

2. The market clearing condition:

$$
\int_A V_t\left(a, \theta^*(a)\right) \theta_t^*(a) \, d\nu(a) = S_t, \qquad t > 0, \quad a.s.
$$

For many of these settings there is more than one equilibrium. Among the equilibria, our stock price model is characterized as follows.

Proposition 5.4 *Suppose that we have the following two additional assumptions:*

1. *The market price of risk is constant, i.e., there exists a constant $C > 0$ such that*

$$
\frac{\mu_t}{\sigma_t} = C \qquad t \geq 0, \quad a.s.
$$

2. *Initially the total amount of the riskless bonds in the entire market is zero, i.e.*

$$
\int_A V(a) \, \nu(da) = s_0.
$$

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Then there is a uniquie equilibrium, for which the stock price process S becomes our stock price model with the measure λ being the distribution of $\frac{C}{1-p(\cdot)}$ w.r.t. the measure

$$\frac{V(a)\,\nu(da)}{\int_A V(a)\,\nu(da)}$$

Remark. For our equilibrium of the proposition, The second assumption of 5.1 corresponds to the assumption $\int_0^\infty \sigma\,\lambda(d\sigma) < \infty$ of Section 2. The measure λ_t, defined in the remark after Proposition 3.1, is the distribution of $\frac{C}{1-p(\cdot)}$ w.r.t. the measure

$$\frac{V_t\big(a,\theta^*(a)\big)\,\nu(da)}{\int_A V_t\big(a,\theta^*(a)\big)\,\nu(da)}.$$

Therefore the stock price volatility at time t, *i.e.* the mean of σ with weight $\lambda_t(d\sigma)$, equals the market mean of the risk tolerance at time t.

Proof of Proposition. A standard discussion of the stochastic optimization problems (c.f. Theorem 3.7.6 of Karatzas & Shreve [9]) gives

$$\theta_t^*(a) = \frac{C}{\sigma_t}\frac{1}{1-p(a)}$$

and

$$V_t\big(a,\theta^*(a)\big) = V(a)\exp\left\{\frac{C}{1-p(a)}(W_t+Ct)-\frac{1}{2}\left(\frac{C}{1-p(a)}\right)^2 t\right\}.$$

Furthermore, since

$$d\left(\int_A V_t\big(a,\theta^*(a)\big)\,\nu(da)\right)$$

$$= \frac{\int_A V_t\big(a,\theta^*(a)\big)\,\theta^*(a)\,\nu(da)}{S_t}\,dS_t$$

$$\text{(by the definition of } V_t\big(a,\theta^*(a)\big)\text{)}$$

$$= dS_t \quad \text{(by the market clearing condition),}$$

it follows from our second additional assumption that

$$S_t = \int_A V_t\big(a,\theta^*(a)\big)\,\nu(da)$$

$$= \int_A V(a)\exp\left\{\frac{C}{1-p(a)}(W_t+Ct)-\frac{1}{2}\left(\frac{C}{1-p(a)}\right)^2 t\right\}\nu(da). \qquad \square$$

Remark. In the working paper [13], the author tries to give a kind of "game-theoretic" formulation of the stock price processes, where even small investors are not purely price takers: each of them is able to give infinitesimal but direct impact for the determination of the price at the next moment.

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6 Our second model

In addtion to our model of Section 2, we are also able to define

$$X_t := X_0 \ e^{rt} \ \frac{1}{\int_0^\infty \exp\left\{ \sigma(W_t + Ct) - \frac{\sigma^2}{2}t \right\} \lambda(d\sigma)}$$

as an alternative stock price model. If the measure λ is concentrated on one point, then the process X becomes a constant-volatility geometric Brownian motion, *i.e.* the Black-Scholes model.

All the properties of Section 3 have their counterparts for the process X. For example, the counterpart for Remark 3.4 is as follows. As we will see, things are opposite for the right and the left tails compared with the stochastic process S of Section 3.

- Let M be the supremum of the support of the measure λ. If $M < \infty$ then the *left* tail of the distribution of the logarithmic rate of return $\frac{1}{t} \log\left(\frac{X_t}{X_0}\right)$ is roughly the same as that of the normal distributions with variance $\frac{M^2}{t}$. If $M = \infty$ then the *left* tail is heavier than the normal distributions.

- Let m be the infimum of the support of λ. If $m > 0$ then the *right* tail of the distribution of $\frac{1}{t} \log\left(\frac{X_t}{X_0}\right)$ is roughly the same as that of the normal distributions with variance $\frac{m^2}{t}$. If $m = 0$ then the *right* tail is thinner than the normal distributions.

- Unless the measure λ is concentrated on one point, the tail distribution of $\frac{1}{t} \log\left(\frac{X_t}{X_0}\right)$ is strictly heavier on the *left* than on the *right*.

The option pricing for X is not so complicated, because a call option for X can be considered as a put option for $\frac{1}{X}$. In more detail, we do as follows. The risk neutral measure, with X chosen as the numeraire, is the same as the measure Q of Sections 3 and 4. The price of the call option at time t is therefore

$$
\begin{aligned}
& X_t \, E^Q\left[\left. \frac{(X_T - K)^+}{X_T} \right| \mathscr{F}_t \right] \\
= \ & X_t \, E^Q\left[\left. (1 - \tfrac{K}{X_T})^+ \right| \mathscr{F}_t \right] \\
= \ & E^Q\left[\left. \left\{ X_t - K \, e^{-r(T-t)} \int_0^\infty \exp\left(\sigma \, (W_T^Q - W_t^Q) - \frac{\sigma^2}{2}(T-t) \right) \lambda_t(d\sigma) \right\}^+ \right| \mathscr{F}_t \right] \\
= \ & X_t \, \Phi\left(\frac{\check{x}_t}{\sqrt{T-t}} \right) - K \, e^{-r(T-t)} \int_0^\infty \Phi\left(\frac{\check{x}_t}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) \lambda_t(d\sigma),
\end{aligned}
$$

where $\check{x}_t$ is the unique solution of the equation for x :

$$X_t \ e^{r(T-t)} \ \frac{1}{\int_0^\infty \exp\left\{ \sigma \, x - \frac{\sigma^2}{2}(T-t) \right\} \lambda_t(d\sigma)} = K.$$

7 Appendix: Applicability of the stochastic Fubini theorem

For a rigorous proof of Propositions 3.1 and 3.3 we need to apply the stochastic Fubini theorem. What we want to show is the following proposition. In the statement, set $\eta := \lambda$ for Proposition 3.1 and $\eta(d\sigma) := \sigma^{k-1}\lambda(d\sigma)$ for Proposition 3.3.

Proposition 7.1 *Let η be a finite measure on $\big(\mathbb{R}_{++}, \mathscr{B}(\mathbb{R}_{++})\big)$ such that*

$$\int_0^\infty \sigma\,\eta(d\sigma) < \infty.$$

Then we have

$$\int_0^\infty \exp\left(\sigma W_t^Q - \frac{\sigma^2}{2}t\right)\eta(d\sigma)$$
$$= \eta(\mathbb{R}_{++}) + \int_0^t \left\{\int_0^\infty \sigma\,\exp\left(\sigma W_u^Q - \frac{\sigma^2}{2}u\right)\eta(d\sigma)\right\}dW_u^Q, \qquad t \geq 0, \quad a.s.,$$

where, as defined before, $W_t^Q := W_t + Ct$.

Proof. We use a version of the stochastic Fubini theorem (Theorem IV.46 of Protter [12]) and reduce the problem to showing that

$$\int_0^t du \int_0^\infty \eta(d\sigma)\,\sigma^2\,\exp\left\{2\sigma W_u^Q - \sigma^2 u\right\} < \infty$$

for all $t > 0$, a.s. Since

$$\max_{\sigma \geq 0}\,\sigma\,\exp\left\{2\sigma\,W_u^Q - \sigma^2\,u\right\}$$
$$= \frac{W_u^Q + \sqrt{(W_u^Q)^2 + 2u}}{2u}\,\exp\left\{\frac{(W_u^Q)^2 + W_u^Q\sqrt{(W_u^Q)^2 + 2u}}{2u} - \frac{1}{2}\right\},$$

it suffices to show that

$$\int_{0+} \frac{W_u^Q + \sqrt{(W_u^Q)^2 + 2u}}{2u}\,\exp\left\{\frac{(W_u^Q)^2 + W_u^Q\sqrt{(W_u^Q)^2 + 2u}}{2u} - \frac{1}{2}\right\}du < \infty$$

almost surely. The last inequality holds by the law of the iterated logarithm.
$\square$

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