Numerical computation of Lyapunov exponents related to attractors in a free boundary problem

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1. Introduction

The Free boundary problems are boundary value problems defined on domains whose boundaries are unknown and must be determined as the solutions. Due to nonlinearity they easily involve chaotic phenomena. Free boundary problems often arise from the practical situations, so investigation of chaotic phenomena is very important.

The investigation is carried out via analysis of bifurcation and attractors[10]. In the previous work bifurcation phenomena in a free boundary problem related to natural convection were analyzed numerically[13]. Attractors in free boundary porblems were analyzed theoretically[1]. However, concrete analysis was not carried out, because attractors were considered in the infinite-dimensional space.

Attractors of the ODE system are very important. This is because they are useful for concrete analysis[4, 5, 7]. For autonomous ODE systems numerical computation of Lyapunov exponents is easily carried out. If there exist positive Lyapunov exponents, chaotic phenomena exist. However, it is difficult to derive the autonomous ODE system which approximates the PDE system describing a free boundary problems.

In the paper a method for numerical computation of attractors in free boundary problems and Lyapunov exponents is presented. To see the procedure of the method a free boundary problem with some parameters is considered. It is of the type of a two-phase Stefan problem. The method consists of SCM(Spectral Collocation Method) [3], the fixed domain method[12] and transformation from the nonautonomous system into the autonomous one[2]. For one-dimensional free boudary problems it facilitates derivation of ODE systems.

2. Test problem

We consider the following one-dimensional free boundary problem with some parameters.

Problem 1. For parameters $|\alpha^{\pm}|,\ |\beta|,\ |s_0| < 1,\ \ 0 \le r \le 1,\ \ q$ and $\omega^{\pm}$, find $u^{\pm}(x,t)$ and $s(t)$ such that

$$u_t^+(x,t) = u_{xx}^+(x,t) + g^+(x,t), \qquad 0 < t, \qquad -1 < x < s(t), \tag{2.1}$$
$$u^+(-1,t) = h^+(t), \qquad 0 \le t, \tag{2.2}$$
$$u^+(s(t),t) = 0, \qquad 0 \le t, \tag{2.3}$$
$$u^+(x,0) = f^+(x), \qquad -1 < x < s_0, \tag{2.4}$$
$$u_t^-(x,t) = u_{xx}^-(x,t) + g^-(x,t), \qquad 0 < t, \qquad s(t) < x < 1, \tag{2.5}$$
$$u^-(1,t) = h^-(t), \qquad 0 \le t, \tag{2.6}$$
$$u^-(s(t),t) = 0, \qquad 0 \le t, \tag{2.7}$$
$$u^-(x,0) = f^-(x), \qquad s_0 < x < 1, \tag{2.8}$$
$$\frac{d}{dt}s(t) = -k^+(t)\ u_x^+(s(t),t) + k^-(t)\ u_x^-(s(t),t), \qquad 0 < t, \tag{2.9}$$
$$s(0) = s_0 \tag{2.10}$$

where

$$k^{\pm}(t) = r + (1-r)\,\frac{1}{2}\,\frac{1 \pm \beta \sin t}{\pm 1 + \alpha^{\pm} \sin t}\,\beta \cos t, \tag{2.11}$$
$$h^{\pm}(t) = \pm 1 + \alpha^{\pm} \sin(\omega^{\pm} t), \tag{2.12}$$
$$g^{\pm}(x,t) = q\left(\pm\frac{(\beta-\alpha^{\pm})\cos t}{(1 \pm \beta \sin t)^2}(x - \beta \sin t) \pm \frac{\pm 1 + \alpha^{\pm} \sin t}{1 \pm \beta \sin t}\beta \cos t\right), \tag{2.13}$$
$$f^+(x) = (x - s_0)\left(a(x+1) - \frac{1}{s_0+1}\right), \tag{2.14}$$
$$f^-(x) = (x - s_0)\left(b(x-1) + \frac{1}{s_0-1}\right). \tag{2.15}$$

Parameters $a,\ b$ should be determined such that $f^+(x) \ge 0,\ f^-(x) \le 0$.

Remark 1. *For $a = b = s_0 = r = 0,\ \ \omega^{\pm} = 1$ and $q = 1$, there are exact solutions as* *follows:*

$$s(t) = s_p(t) \equiv \beta \sin t, \tag{2.16}$$

$$u^{\pm}(x,t) = \frac{\mp h^{\pm}(t)}{1 \pm s_p(t)}(x - s_p(t)) = \mp \frac{\pm 1 + \alpha^{\pm} \sin t}{1 \pm \beta \sin t}(x - \beta \sin t). \tag{2.17}$$

3. Our method

In analysis of chaotic phenomena attractors play a very important role. Attractors in the PDE system are not so useful for concrete analysis. Therefore, derivation of the ODE system which approximates the PDE system becomes important. However, for free boundary problems it is not easy.

In this section a method for derivation of the ODE system is presented. It consists of the fixed domain method, the spectral (collocation) method and transformation from the nonautonomous system into the autonomous one. To see its procedure it is applied to Problem 1.

3.1. Spectral collocation method

The spectral methods are superior in accuracy[3]. In particular, SCM(Spectral Collocation Method) is preferable to nonlinear problems.

In the paper, SCM using Chebyshev Polynomials and Chebyshev-Gauss-Lobatto case's collocation points are used. In SCM it is easy to increase the order of the approximation by increasing the number of collocation points. This feature is quite remarkable and different from other discretization methods. The application of SCM is similar to that of FDM. So, it is easily applied to nonliear problems.

3.2. Fixed domain method

SCM can not be applied directly to a free boundary problem due to the unknown shape of the domain. To avoid this difficulty, we use the fixed domain method[6, 12]. Mapping functions are introduced for mapping the unknown domain to the fixed rectangular domain.

We use the following mapping function (variable transformation) : $(x,t) \to (\xi, \tilde{t})$ such that

$$t = t(\tilde{t}) = \tilde{t}, \qquad 0 \le t, \tag{3.18}$$

$$x = x(\xi, \tilde{t}) = \begin{cases} \dfrac{\tilde{s}(\tilde{t}) + 1}{2}(\xi + 1) - 1, & 0 \le t, \ -1 \le x \le s(t), \\ \dfrac{1 - \tilde{s}(\tilde{t})}{2}(\xi - 1) + 1, & 0 \le t, \ s(t) \le x \le 1. \end{cases} \tag{3.19}$$

Using these mapping functions, we define

$$\tilde{s}(\tilde{t}) = s(t(\tilde{t})), \tag{3.20}$$

$$\tilde{u}^{+}(\xi, \tilde{t}) = u^{+}(x(\xi, \tilde{t}), t(\tilde{t})), \tag{3.21}$$

$$\tilde{u}^{-}(\xi, \tilde{t}) = u^{-}(x(\xi, \tilde{t}), t(\tilde{t})). \tag{3.22}$$

Then, Problem 1 is transformed into the following fixed boundary problem.

Problem 2. Find $\tilde{u}^{\pm}(\xi,\tilde{t})$ and $\tilde{s}(\tilde{t})$ such that

$$
\begin{aligned}
\tilde{u}^{+}_{\tilde{t}}(\xi,\tilde{t}) &= -k^{+}(\tilde{t})\frac{2(\xi+1)}{\left(\tilde{s}(\tilde{t})+1\right)^2}\tilde{u}^{+}_{\xi}(1,\tilde{t})\tilde{u}^{+}_{\xi}(\xi,\tilde{t}) \\
&\quad - k^{-}(\tilde{t})\frac{2(\xi+1)}{\tilde{s}(\tilde{t})^2-1}\tilde{u}^{-}_{\xi}(-1,\tilde{t})\tilde{u}^{+}_{\xi}(\xi,\tilde{t}) + \frac{4}{\left(\tilde{s}(\tilde{t})+1\right)^2}\tilde{u}^{+}_{\xi\xi}(\xi,\tilde{t}) \\
&\quad + q\left\{\frac{(\beta-\alpha^{+})\cos\tilde{t}}{(1+\beta\sin\tilde{t})^2}\left(\frac{\tilde{s}(\tilde{t})+1}{2}(\xi+1)-1-\beta\sin\tilde{t}\right)\right. \\
&\quad \left. + \frac{(1+\alpha^{+}\sin\tilde{t})\beta\cos\tilde{t}}{1+\beta\sin\tilde{t}}\right\}, \qquad 0<\tilde{t}, \qquad -1<\xi<1, \qquad (3.23)\\
\tilde{u}^{+}(-1,\tilde{t}) &= 1+\alpha^{+}\sin(\omega^{+}\tilde{t}), \qquad 0\le\tilde{t}, \qquad (3.24)\\
\tilde{u}^{+}(1,\tilde{t}) &= 0, \qquad 0\le\tilde{t}, \qquad (3.25)\\
\tilde{u}^{+}(\xi,0) &= \frac{1}{4}(\xi-1)\{a(s_0+1)^2(\xi+1)-2\}, \qquad -1<\xi<1, \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}^{-}_{\tilde{t}}(\xi,\tilde{t}) &= -k^{+}(\tilde{t})\frac{2(\xi-1)}{\tilde{s}(\tilde{t})^2-1}\tilde{u}^{+}_{\xi}(1,\tilde{t})\tilde{u}^{-}_{\xi}(\xi,\tilde{t}) \\
&\quad - k^{-}(\tilde{t})\frac{2(\xi-1)}{\left(\tilde{s}(\tilde{t})-1\right)^2}\tilde{u}^{-}_{\xi}(-1,\tilde{t})\tilde{u}^{-}_{\xi}(\xi,\tilde{t}) + \frac{4}{\left(\tilde{s}(\tilde{t})-1\right)^2}\tilde{u}^{-}_{\xi\xi}(\xi,\tilde{t}) \\
&\quad + q\left\{-\frac{(\beta-\alpha^{-})\cos\tilde{t}}{(1-\beta\sin\tilde{t})^2}\left(\frac{1-\tilde{s}(\tilde{t})}{2}(\xi-1)+1-\beta\sin\tilde{t}\right)\right. \\
&\quad \left. + \frac{(1-\alpha^{-}\sin\tilde{t})\beta\cos\tilde{t}}{1-\beta\sin\tilde{t}}\right\}, \qquad 0<\tilde{t}, \qquad -1<\xi<1, \qquad (3.27)\\
\tilde{u}^{-}(-1,\tilde{t}) &= 0, \qquad 0\le\tilde{t}, \qquad (3.28)\\
\tilde{u}^{-}(1,\tilde{t}) &= -1+\alpha^{-}\sin(\omega^{-}\tilde{t}), \qquad 0\le\tilde{t}, \qquad (3.29)\\
\tilde{u}^{-}(\xi,0) &= \frac{1}{4}(\xi+1)\{b(s_0-1)^2(\xi-1)-2\}, \qquad -1<\xi<1, \qquad (3.30)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\tilde{t}}\tilde{s}(\tilde{t}) &= -k^{+}(\tilde{t})\frac{2}{\tilde{s}(\tilde{t})+1}\tilde{u}^{+}_{\xi}(1,\tilde{t}) - k^{-}(\tilde{t})\frac{2}{\tilde{s}(\tilde{t})-1}\tilde{u}^{-}_{\xi}(-1,\tilde{t}), \quad 0<\tilde{t}, \qquad (3.31)\\
\tilde{s}(0) &= s_0. \qquad (3.32)
\end{aligned}
$$

3.3. ODE system

Numerical computation of attractors in the ODE system can be carried out by the applicaiton of SCM in space and time to Problem 2[6, 8]. However, this procedure is not proper for numerical computation of Lyapunov exponents which are computed for the

ODE system. The ODE system is very important not only in numerical computation of Lyapunov exponents but also in theoretical analysis. For its derivation SCM not in time but in space is first applied.

By applying SCM in space with the following Chebyshev-Gauss-Lobatto points:

$$\xi_i = \cos\frac{i\pi}{N_x}, \qquad i = 0, 1, \cdots, N_x \tag{3.33}$$

to Problem 2, we obtain the following ODE system : Problem 3. For simplicity we substitute the symbol t for the symbol $\tilde{t}$.

Problem 3. Find $\tilde{u}_i^{\pm}(t)$, $i = 1, 2, \cdots, N_x - 1$ and $\tilde{s}(t)$ such that

$$\begin{aligned}
\frac{d}{dt}\tilde{u}_i^+(t) = &-k^+(t)\frac{2(\xi_i+1)}{\big(\tilde{s}(t)+1\big)^2}\left(\sum_{k=1}^{N_x-1}(D_x)_{0,k}\ \tilde{u}_k^+(t) + (D_x)_{0,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\right) \\
&\left(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\ \tilde{u}_k^+(t) + (D_x)_{i,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\right) \\
&- k^-(t)\frac{2(\xi_i+1)}{\tilde{s}(t)^2-1}\left(\sum_{k=1}^{N_x-1}(D_x)_{N_x,k}\ \tilde{u}_k^-(t) + (D_x)_{N_x,0}\big(\alpha^-\sin(\omega^-t)-1\big)\right) \\
&\left(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\ \tilde{u}_k^+(t) + (D_x)_{i,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\right) \\
&+ \frac{4}{\big(\tilde{s}(t)+1\big)^2}\left(\sum_{k=1}^{N_x-1}(D_{xx})_{i,k}\ \tilde{u}_k^+(t) + (D_{xx})_{i,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\right) \\
&+ q\left\{\frac{(\beta-\alpha^+)\cos t}{(1+\beta\sin t)^2}\left(\frac{\tilde{s}(t)+1}{2}(\xi_i+1) - 1 - \beta\sin t\right)\right. \\
&\left. + \frac{(1+\alpha^+\sin t)\beta\cos t}{1+\beta\sin t}\right\}, \qquad\qquad 0 < t,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\frac{d}{dt}\tilde{u}_i^-(t) = &-k^+(t)\frac{2(\xi_i-1)}{\tilde{s}(t)^2-1}\left(\sum_{k=1}^{N_x-1}(D_x)_{0,k}\ \tilde{u}_k^+(t) + (D_x)_{0,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\right) \\
&\left(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\ \tilde{u}_k^-(t) + (D_x)_{i,0}\big(\alpha^-\sin(\omega^-t)-1\big)\right) \\
&- k^-(t)\frac{2(\xi_i-1)}{\big(\tilde{s}(t)-1\big)^2}\left(\sum_{k=1}^{N_x-1}(D_x)_{N_x,k}\ \tilde{u}_k^-(t) + (D_x)_{N_x,0}\big(\alpha^-\sin(\omega^-t)-1\big)\right) \\
&\left(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\ \tilde{u}_k^-(t) + (D_x)_{i,0}\big(\alpha^-\sin(\omega^-t)-1\big)\right) \\
&+ \frac{4}{\big(\tilde{s}(t)-1\big)^2}\left(\sum_{k=1}^{N_x-1}(D_{xx})_{i,k}\ \tilde{u}_k^-(t) + (D_{xx})_{i,0}\big(\alpha^-\sin(\omega^-t)-1\big)\right) \\
&+ q\left\{-\frac{(\beta-\alpha^-)\cos t}{(1-\beta\sin t)^2}\left(\frac{1-\tilde{s}(t)}{2}(\xi_i-1) + 1 - \beta\sin t\right)\right.
\end{aligned}$$

$$+\frac{(1-\alpha^- \sin t)\beta\cos t}{1-\beta\sin t}\Bigg\}, \qquad 0<t, \tag{3.35}$$

$$\frac{d}{dt}\tilde{s}(t) = -k^+(t)\frac{2}{\tilde{s}(t)+1}\left(\sum_{k=1}^{N_x-1}(D_x)_{0,k}\ \tilde{u}_k^+(t) + (D_x)_{0,N_x}\big(\alpha^+\sin(\omega^+ t)+1\big)\right),$$

$$-k^-(t)\frac{2}{\tilde{s}(t)-1}\left(\sum_{k=1}^{N_x-1}(D_x)_{N_x,k}\ \tilde{u}_k^-(t) + (D_x)_{N_x,0}\big(\alpha^-\sin(\omega^- t)-1\big)\right), \quad 0<t, \tag{3.36}$$

$$\tilde{u}_i^+(0) = \left(\frac{a}{4}(s_0+1)^2(\xi_i+1)-\frac{1}{2}\right)(\xi_i-1), \tag{3.37}$$

$$\tilde{u}_i^-(0) = \left(\frac{b}{4}(s_0-1)^2(\xi_i-1)-\frac{1}{2}\right)(\xi_i+1), \tag{3.38}$$

$$\tilde{s}(0) = s_0. \tag{3.39}$$

Of course, it is easy to change N_x. This means original attractors of the PDE system can be approximated arbitrarily by the method. This feature of the method is very important from the theoretical view point. For $N_x = 2$ the ODE system becomes as follows.

Problem 4. Find $\tilde{u}_1^\pm(t)$ and $\tilde{s}(t)$ such that

$$\begin{aligned}
\frac{d}{dt}\tilde{u}_1^+(t) &= -\frac{k^+(t)}{2\big(\tilde{s}(t)+1\big)^2}\big(4\tilde{u}_1^+(t)-\alpha^+\sin(\omega^+ t)-1\big)\big(\alpha^+\sin(\omega^+ t)+1\big)\\
&\quad+\frac{k^-(t)}{2\big(\tilde{s}(t)^2-1\big)}\big(4\tilde{u}_1^-(t)-\alpha^-\sin(\omega^- t)+1\big)\big(\alpha^+\sin(\omega^+ t)+1\big)\\
&\quad-\frac{4}{\big(\tilde{s}(t)+1\big)^2}\big(2\tilde{u}_1^+(t)-\alpha^+\sin(\omega^+ t)-1\big)\\
&\quad+q\left\{\frac{(\beta-\alpha^+)\cos t}{(1+\beta\sin t)^2}\left(\frac{\tilde{s}(t)-1}{2}-\beta\sin t\right)\right.\\
&\quad\left.+\frac{(1+\alpha^+\sin t)\beta\cos t}{1+\beta\sin t}\right\}, \qquad 0<t,
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
\frac{d}{dt}\tilde{u}_1^-(t) &= -\frac{k^+(t)}{2\big(\tilde{s}(t)^2-1\big)}\big(4\tilde{u}_1^+(t)-\alpha^+\sin(\omega^+ t)-1\big)\big(\alpha^-\sin(\omega^- t)-1\big)\\
&\quad+\frac{k^-(t)}{2\big(\tilde{s}(t)-1\big)^2}\big(4\tilde{u}_1^-(t)-\alpha^-\sin(\omega^- t)+1\big)\big(\alpha^-\sin(\omega^- t)-1\big)\\
&\quad-\frac{4}{\big(\tilde{s}(t)-1\big)^2}\big(2\tilde{u}_1^-(t)-\alpha^-\sin(\omega^- t)+1\big)\\
&\quad+q\left\{-\frac{(\beta-\alpha^-)\cos t}{(1-\beta\sin t)^2}\left(\frac{\tilde{s}(t)+1}{2}-\beta\sin t\right)\right.
\end{aligned}$$

$$+\frac{(1-\alpha^- \sin t)\beta\cos t}{1-\beta\sin t}\Bigg\}, \qquad 0<t, \tag{3.41}$$

$$\frac{d}{dt}\tilde{s}(t) = \frac{k^+(t)}{\tilde{s}(t)+1}\big(4\tilde{u}_1^+(t)-\alpha^+\sin(\omega^+ t)-1\big)$$

$$-\frac{k^-(t)}{\tilde{s}(t)-1}\big(4\tilde{u}_1^-(t)-\alpha^-\sin(\omega^- t)+1\big), \qquad 0<t, \tag{3.42}$$

$$\tilde{u}_1^+(0) = \frac{1}{2}-\frac{a}{4}(s_0+1)^2, \tag{3.43}$$

$$\tilde{u}_1^-(0) = -\frac{1}{2}-\frac{b}{4}(s_0-1)^2, \tag{3.44}$$

$$\tilde{s}(0) = s_0, \qquad 0<t. \tag{3.45}$$

3.4. Transformation into the autonomous system

The ODE systems in Problems 3, 4 are not autonomous. So, transformation into the autonomous system is necessary for numerical computation of Lyapunov exponents. It can be done by introducing a new parameter θ[2]. Problem 4 is transformed into the following autonomous system.

Problem 5. Find $\tilde{u}_1^{\pm}(t), \tilde{s}(t)$ and $\theta(t)$ such that

$$\frac{d}{dt}\tilde{u}_1^+(t) = -\frac{k^+(t)}{2\big(\tilde{s}(t)+1\big)^2}\big(4\tilde{u}_1^+(t)-\alpha^+\sin\{\omega^+\theta(t)\}-1\big)\big(\alpha^+\sin\{\omega^+\theta(t)\}+1\big)$$

$$+\frac{k^-(t)}{2\big(\tilde{s}(t)^2-1\big)}\big(4\tilde{u}_1^-(t)-\alpha^-\sin\{\omega^-\theta(t)\}+1\big)\big(\alpha^+\sin\{\omega^+\theta(t)\}+1\big)$$

$$-\frac{4}{\big(\tilde{s}(t)+1\big)^2}\big(2\tilde{u}_1^+(t)-\alpha^+\sin\{\omega^+\theta(t)\}-1\big)$$

$$+q\left\{\frac{(\beta-\alpha^+)\cos\{\theta(t)\}}{(1+\beta\sin\{\theta(t)\})^2}\left(\frac{\tilde{s}(t)-1}{2}-\beta\sin\{\theta(t)\}\right)\right.$$

$$\left.+\frac{(1+\alpha^+\sin\{\theta(t)\})\beta\cos\{\theta(t)\}}{1+\beta\sin\{\theta(t)\}}\right\}, \qquad 0<t, \tag{3.46}$$

$$\frac{d}{dt}\tilde{u}_1^-(t) = -\frac{k^+(t)}{2\big(\tilde{s}(t)^2-1\big)}\big(4\tilde{u}_1^+(t)-\alpha^+\sin\{\omega^+\theta(t)\}-1\big)\big(\alpha^-\sin\{\omega^-\theta(t)\}-1\big)$$

$$+\frac{k^-(t)}{2\big(\tilde{s}(t)-1\big)^2}\big(4\tilde{u}_1^-(t)-\alpha^-\sin\{\omega^-\theta(t)\}+1\big)\big(\alpha^-\sin\{\omega^-\theta(t)\}-1\big)$$

$$-\frac{4}{\big(\tilde{s}(t)-1\big)^2}\big(2\tilde{u}_1^-(t)-\alpha^-\sin\{\omega^-\theta(t)\}+1\big)$$

$$+q\left\{-\frac{(\beta-\alpha^-)\cos\{\theta(t)\}}{(1-\beta\sin\{\theta(t)\})^2}\left(\frac{\tilde{s}(t)+1}{2}-\beta\sin\{\theta(t)\}\right)\right.$$

$$+\frac{(1-\alpha^- \sin\{\theta(t)\})\beta\cos\{\theta(t)\}}{1-\beta\sin\{\theta(t)\}}\Bigg\},\qquad 0 < t, \tag{3.47}$$

$$\frac{d}{dt}\tilde{s}(t) = \frac{k^+(t)}{\tilde{s}(t)+1}\big(4\tilde{u}_1^+(t) - \alpha^+ \sin\{\omega^+\theta(t)\} - 1\big)$$
$$-\frac{k^-(t)}{\tilde{s}(t)-1}\big(4\tilde{u}_1^-(t) - \alpha^- \sin\{\omega^-\theta(t)\} + 1\big),\qquad 0 < t, \tag{3.48}$$

$$\frac{d}{dt}\theta(t) = 1,\qquad 0 < t, \tag{3.49}$$

$$\tilde{u}_1^+(0) = \frac{1}{2} - \frac{a}{4}(s_0+1)^2, \tag{3.50}$$

$$\tilde{u}_1^-(0) = -\frac{1}{2} - \frac{b}{4}(s_0-1)^2, \tag{3.51}$$

$$\tilde{s}(0) = s_0,\qquad 0 < t. \tag{3.52}$$

Of course, this procedure is applicable to the general system : Problem 3.

Then SCM in time[6, 11] is applied for computing Lyapunov exponents.

4. Numerical results

In this section, numerical results are shown. We performed numerical simulation for $N_x = 2,\ q = 0,\ r = 1,\ \alpha = \beta = 0.5$ and $\omega^+ = 1$. For time integration we used SCM with 11 Chebyshev-Gauss-Lobatto collocation points in the interval $\Delta t = 0.1$[6, 11].

Figs. 1 - 4 show attractors in the solution space (the three-dimensional space) and Lyapunov exponents. Attractors are computed from Problem 4. Lyapunov exponents are computed from both Problem 5 and its linearized problem[9].

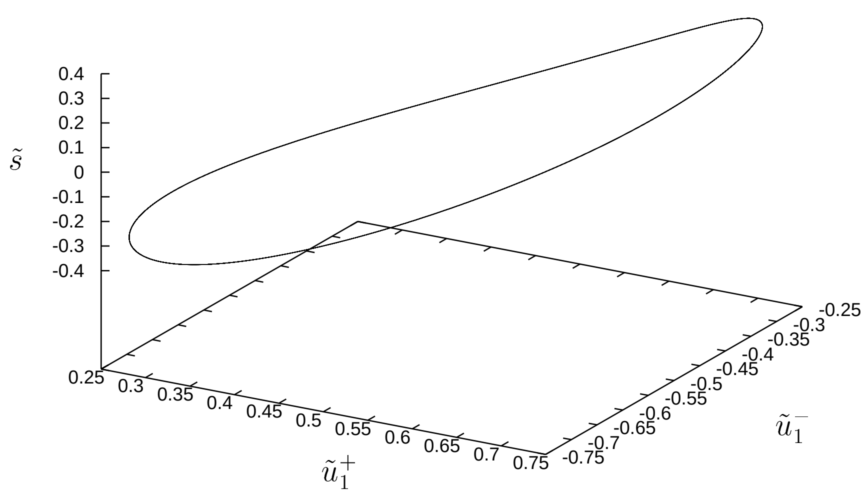

Fig. 1. Attractor in Problem 4 for $\omega^- = 1$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.360,\ \lambda_2 = -6.710,\ \lambda_3 = -19.10,\ \lambda_4 = 0.000$.

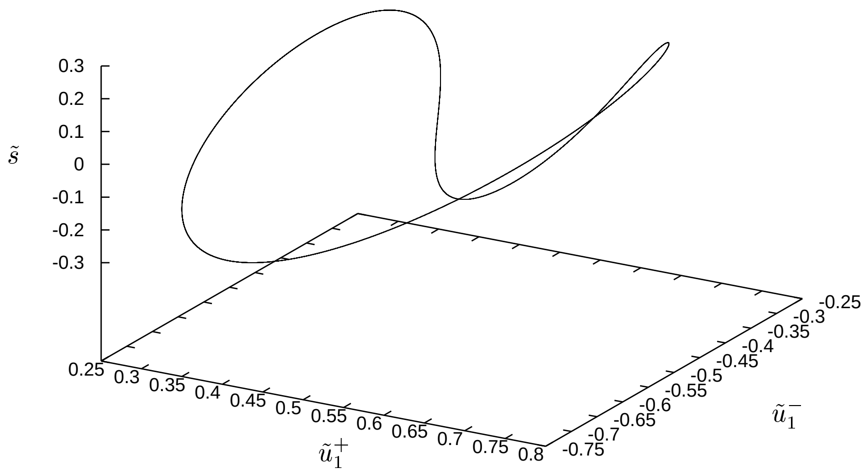

Fig. 2. Attractor in Problem 4 for $\omega^- = 2$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.275,\ \lambda_2 = -7.487,\ \lambda_3 = -14.71,\ \lambda_4 = 0.000.$

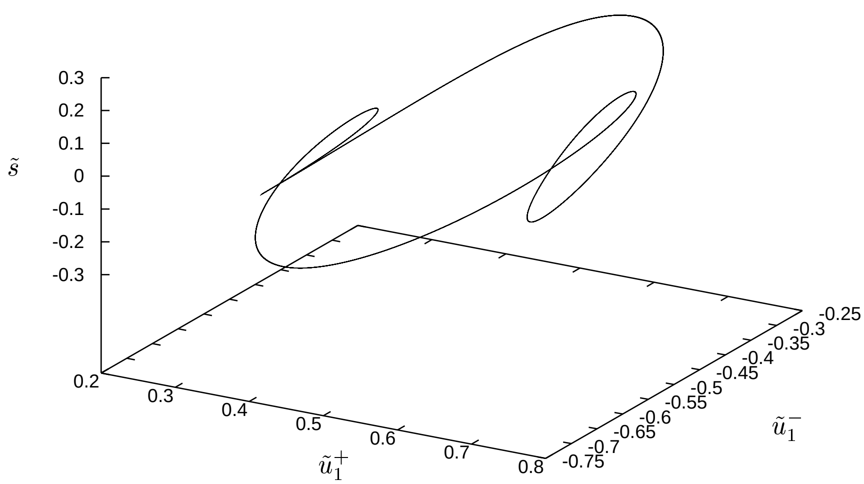

Fig. 3. Attractor in Problem 4 for $\omega^- = 3$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.284,\ \lambda_2 = -7.264,\ \lambda_3 = -15.10,\ \lambda_4 = 0.000.$

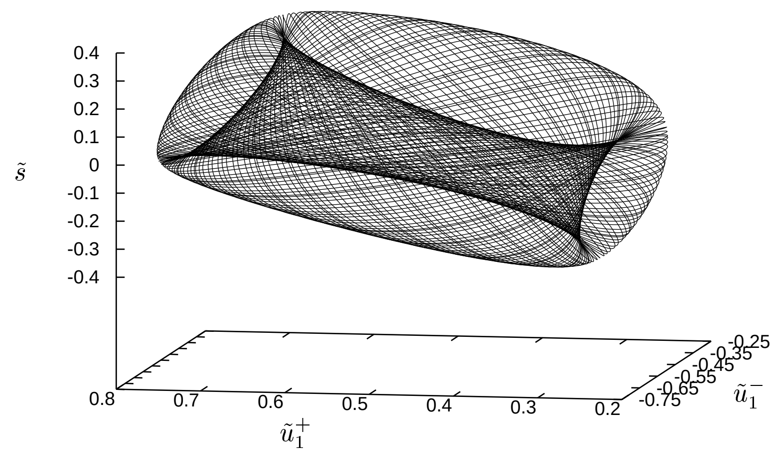

Fig. 4. Attractor in Problem 4 for $\omega^- = \sqrt{2}$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.289$, $\lambda_2 = -7.228$, $\lambda_3 = -15.75$, $\lambda_4 = 0.000$.

For parameters investigated above attractors are not strange. So, there are no positive Lyapunov exponents.

5. Conclusion

In the paper a method for numerical computation of attractors in free boundary problems and their Lyapunov exponents is presented. The method consists of SCM(Spectral Collocation Method), the fixed domain method and transformation from the nonautonomous system into the autonomous system. To see the procedure of the method it is applied to a free boundary problem with some parameters which is of the type of a two-phase Stefan problem. Various attractors are found numerically and Lyapunov exponents are computed.

For one-dimensional free boudary problems the method facilitate the derivation of ODE systems which approximate PDE systems describing free boundary problems. SCM is used in the method, so original attractors of the PDE system can be approximated arbitrarily. This means the method plays a very important role in theoretical analysis.

Our next goal is to find strange attractors (i.e. positive Lyapunov exponents) in free boundary problems by using our method.

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