Optimal Bond Portfolio for Investors with Long Time Horizons

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We study the optimal bond portfolio for an investor with long time horizon using Japanese interest rate data. A simple onefactor term structure model is used for our numerical example. The optimal portfolio is computed using the technique of stochastic flows and Monte Carlo simulation. The hedging portfolio is not negligible and the mean variance portfolio is very sensitive to parameter values. The optimal portfolio is highly leveraged for a typical parameter value. The investor holds a zero-coupon bond because of the lower bound restriction on investor's wealth. The lower bound constraint may make the optimal portfolio more realistic.

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We study the optimal bond portfolio for an investor with long time horizon using Japanese interest rate data. A simple onefactor term structure model is used for our numerical example. The optimal portfolio is computed using the technique of stochastic flows and Monte Carlo simulation. The hedging portfolio is not negligible and the mean variance portfolio is very sensitive to parameter values. The optimal portfolio is highly leveraged for a typical parameter value. The investor holds a zero-coupon bond because of the lower bound restriction on investor's wealth. The lower bound constraint may make the optimal portfolio more realistic.

1 Introduction

Long-term bonds are one of the most popular securities that are traded for long time. Many articles about pricing of bonds and fixed-income securities are published since the 1960's. Models of the term structure of interest rates have been very successful not only in academic field but also in the application of financial models to everyday business problems. However, it has not been extensively studied about the optimal bond portfolios for longterm investors using sophisticated term structure models such as Vasicek [27], Cox, Ingersoll, and Ross [11], Heath, Jarrow, and Morton [16].

This gap in the literature may be due to the difficulty in obtaining tractable solutions of dynamic portfolio problem. Optimal portfolio and consumption choice in multi-period or in continuous-time settings were investigated by Samuelson [26] and Merton [21] [22]. By assuming a model with constant coefficients and solving the relevant Hamilton-Jacobi-Bellman equation, Merton [21] produces solutions when the utility function is a member of the family of utility functions that is called the Hyperbolic Absolute Risk Aversion (HARA) family. If coefficients are not assumed to be constant, in other words, the investment opportunity set is time varying, explicit solutions for portfolio weights are available only in the special cases, for example, where investors have log-utility.

The difficulty in solving the optimal portfolio problem is particularly unfortunate, because we cannot take advantages of sophisticated term structure models even though there are considerable evidence of time-varying investment opportunities in fixed-income security markets. Recently a number of authors such as Balduzzi and Lynch [1], Barberis [2], Brandt [3], and Brennan, Schwartz, and Lagnando [5], Campbell and Viceira [7], Lynch [20], and Xia [28] have used numerical methods to solve particular long run portfolio choice problems. Kim and Omberg [18] and Liu [19] obtain exact analytical solutions for a range of continuous-time problems with predictability. All of these papers point out the importance of "hedging portfolio" that is a demand for the asset as a vehicle to hedge against "unfavorable" shifts in the investment opportunity set. These papers generally concentrate on the choice between cash and equities rather than the demand for long-term bonds.

Campbell and Viceira [8] study inter-temporal portfolio choice in an environment with random real interest rates, using an approximation technique developed in their earlier papers (Campbell [6]). They calibrate their model to historical data on the US term structure of interest rates, and report optimal portfolios for investors. Investor's preferences are of the form suggested by Epstein and Zin [14]; the investor has constant relative risk aversion and constant inter-temporal elasticity of substitution in consumption. In order to use their approximation results, investor's time horizon is assumed to be infinite.

In this paper, we compute the optimal bond portfolio using Monte Carlo simulation approach. As discussed above, to compute the optimal consumption and portfolio rule is in general difficult when the investment opportunity set is time varying.¹ In order to compute the optimal consumption-portfolio rule, we use Monte Carlo simulation approach in this paper. Our control problem is converted into the martingale formulation by standard arguments, since markets are complete from the investor's point of view. The technique of stochastic flows is then applied to facilitate the computation. (See, for example, Nualart [23] and Protter [25].) It is possible that the result could be derived as a corollary of the general result of Ocone and Karatzas [24] and Detemple, Garcia, and Rindisbacher [12], in which the Malliavin calculus is applied. However any such relation is not transparent, and the stochastic flow is more directly related to the derivatives of the value function.²

¹Brennan, Schwartz, and Lagnando [5] and Xia [28] for example solve non-linear partial differential equation which is associated with the Hamilton-Jacobi-Bellman (HJB) equation. Campbell and Viceira [7] use log-linear approximations to solve the investor's multi-period discrete-time problem. Kim and Omberg [18] and Liu [19] obtain exact analytical solutions for the value function of the problem. Brandt [3] uses the investor's Euler equations to estimate the investor's portfolio allocation to stocks.

²The Malliavin calculus can also be applied to compute option Greeks. See, for example,

Investors are assumed to have power-utility function that is defined on running consumption and wealth at a finite time-horizon. We can thus explicitly study how terminal horizons of investors affect their optimal portfolio choice. For numerical example, we assume that there is no running consumption in order to reduce computational cost. Our numerical procedure however can be easily applied to the case where investors have more complicated utility functions and there are running consumption. In fact, in the later part of this paper, we also study the case where investors have power utility function that is defined only for wealth level that exceeds a certain minimum wealth level at a terminal horizon.

We consider a simple one-factor model of the term structure of interest rates (Brennan and Schwartz [4]) and calibrate the model to historical data on the Japanese term structure of interest rates. The data includes the period of Japanese zero-interest rate policy in late 90's. Nominal short rates that are close to zero in our sample period cause some difficulties in parameter estimations and simulations. Although we decide to use a simple one-factor model so that numerical computation is easily done, the model may not be the best choice from empirical point of view. Some affine-term-structure models that have square root terms in its volatility coefficients, such as Cox, Ingersoll, and Ross model, might be problematic in our example, because the existence of stochastic flow is not guaranteed by standard conditions. Furthermore, to simulate random sample path could be difficult because of the square root term when nominal short rates are close to zero as in our sample period.

The remainder of this paper is structured as follows. Section 2 formally describes the model. Properties of the optimal portfolio are discussed in Section 3. Section 4 discusses the parameter estimation and the calibration of the term structure model. Section 5 studies numerical solutions of the optimal portfolio problem. Section 6 concludes.

2 Model

Let *B* be one dimensional standard Brownian motion in \mathbb{R} , restricted to some time interval $[0,\overline{T}]$, on a given probability space (Ω, \mathcal{F}, P) . We also fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq \overline{T}\}$ of *B*. We take as given an adapted short-rate process *r* with $\int_0^{\overline{T}} |r_t| dt < \infty$. That is, r_t is the continually compounding interest rate on riskless securities at time *t*. There is a zero-

Fournié, Lasry, Lebuchoux, Lions, and Touzi [15]. The relation between the Malliavin calculus and the stochastic flow is also discussed in Colwell, Elliott, and Kopp [9].

coupon bond maturing at some future time $\tau \leq \overline{T}$. By definition, the bond pays no dividends before time τ , and offers a fixed lump-sum payment at time τ that we can take without loss of generality to be 1 unit of account.

We suppose that there is an equivalent martingale measure Q and that there is no arbitrage opportunity. Let \hat{B} be the standard Brownian motion in \mathbb{R} under Q. Let $\Lambda_{t,\tau}$ be the price of the zero-coupon bond at time t maturing at time $\tau, t < \tau < \overline{T}$. It is well known that $\Lambda_{t,\tau}$ is given by

$$\Lambda_{t,\tau} = E_t^Q \left[\exp\left(\int_t^\tau -r_u du\right) \right],$$

where $E^{Q}[\cdot]$ is the expectation under Q. We also assume that there is a smooth function of two variables g that satisfies

$$\Lambda_{t,\tau} = g(r_t, t).$$

No arbitrage argument implies that g satisfies the partial differential equation

$$g_t(r,t) + \hat{\mu}_r g_r(r,t) + \frac{1}{2} \hat{\sigma}_r^2 g_{rr}(r,t) - rg(r,t) = 0,$$

$$g(r,\tau) = 1,$$

where $\hat{\mu}_r$ and $\hat{\sigma}_r$ are drift and volatility parameters of the short rate under an equivalent martingale measure Q. It follows from Ito's lemma that

$$d\Lambda_{t,\tau} = \left(r(t) + \theta(t)\sigma_r \frac{g_r(r(t),t)}{g(r(t),t)}\right)g(r(t),t)\Lambda_{t,\tau}dt + \sigma_r(r(t),t)\left(\frac{g_r(r(t),t)}{g(r(t),t)}\right)\Lambda_{t,\tau}dB_t$$

$$\equiv \mu_{\Lambda}(t,\tau)\Lambda_{t,\tau}dt + \sigma_{\Lambda}(t,\tau)\Lambda_{t,\tau}dB(t).$$
(1)

In this paper, we consider one factor term-structure models, by which we mean models of the short rate r given by an stochastic differential equation of the form

$$dr_t = \mu_r(r_t, t)dt + \sigma_r(r_t, t)dB_t, \qquad (2)$$

where $\mu_r : \mathbb{R}_+ \times [0, \overline{T}] \to \mathbb{R}$ and $\sigma_r : \mathbb{R}_+ \times [0, \overline{T}] \to \mathbb{R}$ are assumed to satisfy regularity conditions that guarantee a unique strong solution for SDE (2).

Let θ be one dimensional, progressively measurable market price of risk process, which satisfies

$$\int_0^{\overline{T}} |\theta(t)| dt < +\infty, \quad P\text{-a.s.}$$

and

$$\theta(t) = \frac{\mu_{\Lambda}(t,\tau) - r_t}{\sigma_{\Lambda}(t,\tau)}.$$

The positive localmartingale ξ is then defined by

$$\xi(t) := \exp\left\{-\int_0^t \theta(s) dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right\},\,$$

which is in fact martingale. It follows from Girsanov's Theorem that Q satisfies

$$\frac{dQ}{dP} = \xi(\overline{T}).$$

Under the probability measure Q,

$$\hat{B}(t) = B(t) + \int_0^t \theta(t)dt$$
(3)

is $\{\mathcal{F}_t\}$ -standard Brownian motion.

Under the equivalent martingale measure Q, the short rate process satisfies the stochastic differential equation

$$dr(t) = \hat{\mu}_r(t)dt + \hat{\sigma}_r(t)d\hat{B}(t), \qquad (4)$$

which means that $\hat{\mu}_r(t) + \theta_t \hat{\sigma}_r(t) = \mu_r(r_t, t)$ and $\hat{\sigma}_r(t) = \sigma_r(r_t, t)$. Therefore θ satisfies

$$\theta_t = \frac{\mu_r(r_t, t) - \hat{\mu}_r(t)}{\sigma_r(r_t, t)}.$$

The state price deflator π is defined by

$$\pi_t = \left(\exp\left[-\int_0^t r_s ds \right] \right) \xi_t.$$
 (5)

It follows from Ito's lemma that ξ and π satisfy

$$d\xi_t = -\xi_t \theta_t dB_t \quad \text{and} \quad d\pi_t = -\pi_t (r_t dt + \theta_t dB_t) \tag{6}$$

with $\xi_0 = 1$ and $\pi_0 = 1$.

We consider investors who invest their wealth into a riskless asset and a zero-coupon bond with maturity τ . Her investment horizon is T, where $0 < T < \tau < \overline{T}$. Utility is defined over the space D of terminal wealth Z, which is an \mathcal{F}_T -measurable non-negative random variable. Specifically, $U: D \to \mathbb{R}$ is defined by

$$U(Z) = E\left[u_T(Z)\right],\tag{7}$$

where $u_T : \mathbb{R}_+ \to \mathbb{R}$ is the power utility function

$$u_T(Z) = \frac{(Z - \bar{Z})^{\alpha}}{\alpha},\tag{8}$$

with $\alpha < 1$, $\alpha \neq 0$, and $\overline{Z} \ge 0$. If we set $\overline{Z} = 0$, the utility function is a usual power utility function. If we set $\overline{Z} > 0$, there is the lower bound below which terminal wealth is not permitted to fall.

Let φ be a fraction of total wealth held in a zero-coupon bond maturing at time τ . Given an adapted process φ , the wealth process W^{φ} satisfies

$$dW_t^{\varphi} = (W_t^{\varphi}\varphi_t(\mu_{\Lambda}(t) - r_t) + r_t W_t^{\varphi}) dt + W_t^{\varphi}\varphi_t \sigma_{\Lambda}(t) dB_t$$

= $\mu_W(t) dt + \sigma_W(t) dB_t,$ (9)

with $W^{\varphi}(0) = w_0$. We say that φ is budget feasible, denoted by $\varphi \in \Gamma(w_0, r_0)$, if $W^{\varphi}(T) \in D$ and φ is a trading strategy satisfying $W^{\varphi}(t) \ge 0$, $0 \le t \le T$, *P*-almost surely, and $W^{\varphi}(0) = w_0$. Investor's problem is defined by, for each initial wealth w_0 and initial short rate r_0 ,

$$V(w_0, r_0) = \sup_{\varphi \in \Gamma(w_0, r_0)} E\left[u_T(W^{\varphi}(T))\right]$$

=
$$\sup_{\varphi \in \Gamma(w_0, r_0)} E\left[\frac{(W^{\varphi}(T) - \bar{Z})^{\alpha}}{\alpha}\right].$$
 (10)

3 Optimal Portfolio Strategy

In this section, we study theoretical properties of the optimal portfolio strategy to problem (10). We first consider a power-utility function with no lower bound on a terminal wealth level and then study a power-utility function with a certain lower bound on a terminal wealth level. After establishing a general statement of optimal strategies, we concentrate on one factor model of the term structure of interest rates (Brennan and Schwartz [4]) and calculate the optimal strategy.

3.1 Power Utility without Lower Bound

For the case with $\overline{Z} = 0$, the HJB equation for problem (10) is given as follows:

$$\sup_{\varphi \in \mathbb{R}} \quad J_w(w, r, t) \mu_W + J_r(w, r, t) \mu_r(r, t) + J_t(w, r, t) + \frac{1}{2} J_{ww}(w, r, t) \sigma_W^2 + J_{wr}(w, r, t) \sigma_W \sigma_r(r, t) + \frac{1}{2} J_{rr}(w, r, t) (\sigma_r(r, t))^2 = 0,$$
(11)

with the boundary condition

$$J(w, r, T) = \frac{w^{\alpha}}{\alpha}.$$
(12)

By standard homogeneity arguments, we can naturally conjecture that the function J is separable:

$$J(w,r,t) = \frac{w^{\alpha}}{\alpha} f(r,t).$$
(13)

Substituting (13) into the HJB equation (12) and taking the first order conditions, the optimal portfolio strategy φ is given by

$$\varphi = \frac{1}{1-\alpha} \frac{\mu_{\Lambda}(t,\tau) - r}{(\sigma_{\Lambda}(t,\tau))^2} + \frac{1}{1-\alpha} \frac{f_r(r,t)}{f(r,t)} \frac{\sigma_r(r,t)}{\sigma_{\Lambda}(t,\tau)}.$$
 (14)

The first term of (14) is the usual mean-variance portfolio that is held to control instantaneous risk-return combination of investor's portfolio. The second term of (14) is the hedging portfolio that is held to hedge against unfavorable movement of the state variable r. Since the mean variance portfolio does not depend on investor's terminal horizon, it is sometimes called as myopic term. The hedging portfolio depends on investor's terminal horizon through f and f_r , whose values are not trivially obtained.

In order to study function f in detail, we apply the martingale approach. Problem (10) is transformed into

$$\sup_{Z \in \mathcal{F}_T} E\left[u_T(Z)\right]$$

s.t. $E\left[\pi_T Z\right] \leqslant w_0.$ (15)

The Lagrangian \mathcal{L} for this problem is defined by

$$\mathcal{L}(Z,\lambda) = \sup_{Z,\lambda} E\left[\frac{Z^{\alpha}}{\alpha}\right] - \lambda \left(E[\pi_T Z] - w_0\right), \tag{16}$$

with a scalar Lagrange multiplier $\lambda > 0$. The complementary slackness condition is given by $E[\pi_T Z] = w_0$. The first-order conditions for optimality are, state-by-state,

$$(Z^*)^{\alpha-1} - \lambda \pi_T = 0.$$

Then we have

$$Z^* = (\lambda \pi_T)^{\frac{1}{\alpha - 1}}$$
 and $w_0 = E[\pi_T Z^*] = E[\pi_T (\lambda \pi_T)^{\frac{1}{\alpha - 1}}].$

Solving this equation with respect to λ , we have

$$\lambda^* = w_0^{\alpha - 1} E[\pi_T^{\frac{\alpha}{\alpha - 1}}]^{1 - \alpha}.$$

It then follows that

$$Z^* = w_0 \frac{\pi_T^{\frac{1}{\alpha-1}}}{E[\pi_T^{\frac{\alpha}{\alpha-1}}]}.$$
 (17)

Substituting (17) into (15), the value $V(w_0, r_0)$ in (10) is given by

$$V(w_0, r_0) = E[u(Z^*)] = \frac{w_0^{\alpha}}{\alpha} E[\pi_T^{\frac{\alpha}{\alpha-1}}]^{1-\alpha}.$$
 (18)

We can see from (13) that

$$f(r_0, 0) = E[\pi_T \frac{\alpha}{\alpha - 1}]^{1 - \alpha},$$
(19)

which is estimated by evaluating $E[\pi_T \frac{\alpha}{\alpha-1}]^{1-\alpha}$ using Monte Carlo simulation.

In order to evaluate optimal portfolio strategy (14), it is necessary to estimate f_r . We can estimate f for many values of r and approximate f_r by taking differences between two grid points. However, to evaluate function ffor many points is time consuming. Furthermore it may not be easy to find a good grid size that approximates f_r with sufficient accuracy. We can resolve these difficulties by using the stochastic flow technique. The value of f and f_r are estimated at the same time.

Let r^x be a process that satisfies

$$dr_t^x = \mu_r(r_t^x, t) dt + \sigma_r(r_t^x, t) dB(t)$$
(20)

with $r_0^x = x$. That is, r^x is a process that starts at x and satisfies (2). Define $Y_t = \partial r_t^x / \partial x$. Then Y satisfies

$$dY_t = \frac{\partial \mu_r(r_t^x, t)}{\partial r_t^x} Y_t dt + \frac{\partial \sigma_r(r_t^x, t)}{\partial r_t^x} Y_t dB_t, \qquad Y_0 = 1.$$
(21)

It follows from (5) that

$$\frac{\partial \pi_T}{\partial x} = \pi_T \left(-\int_0^T Y_t \, dt - \int_0^T \frac{\partial \theta(r_t^x, t)}{\partial r_t^x} Y_t \, dB_t - \int_0^T \theta(r_t^x, t) \frac{\partial \theta(r_t^x, t)}{\partial r_t^x} Y_t \, dt \right). \tag{22}$$

Then we have

$$f_x(x,0) = (1-\alpha)E\left[\pi_T^{\frac{\alpha}{\alpha-1}}\right]^{-\alpha}E\left[\frac{\alpha}{\alpha-1}\pi_T^{\frac{1}{\alpha-1}}\frac{\partial\pi_T}{\partial x}\right]$$

and

$$\frac{f_x(x,0)}{f(x,0)} = (1-\alpha)E\left[\pi_T^{\frac{\alpha}{\alpha-1}}\right]^{-1}E\left[\frac{\alpha}{\alpha-1}\pi_T^{\frac{1}{\alpha-1}}\frac{\partial\pi_T}{\partial x}\right].$$

Using equation (1), the optimal portfolio strategy is given as follows.

Proposition 3.1. The optimal strategy for problem (10) with $\overline{Z} = 0$ is given by

$$\varphi_{0}^{*} = \frac{1}{1-\alpha} \frac{\Lambda_{0,\tau}}{\frac{\partial \Lambda_{0,\tau}}{\partial x}} \left[\frac{\theta_{0}}{\sigma_{r}(r_{0},0)} + (1-\alpha) \frac{E\left[\frac{\alpha}{\alpha-1}\pi_{T}^{\frac{1}{\alpha-1}}\frac{\partial\pi_{T}}{\partial x}\right]}{E\left[\pi_{T}^{\frac{\alpha}{\alpha-1}}\right]} \right]$$

$$= \frac{1}{\frac{\partial}{\partial x}\log\Lambda_{0,\tau}} \frac{\theta_{0}}{(1-\alpha)\sigma_{r}(r_{0},0)} + \frac{\frac{\partial}{\partial x}\log(E[\pi_{T}^{\frac{\alpha}{\alpha-1}}]^{1-\alpha})}{(1-\alpha)\frac{\partial}{\partial x}\log\Lambda_{0,\tau}}$$

$$\equiv MVP + HP_{1},$$
(23)

where $\partial \pi_T / \partial x$ is given by the system (20), (21), and (22).

Proof. This is a special case of Proposition 3.3, whose proof is in Appendix.

As we can see from (14), MVP is large if expected excess return divided by the volatility, that is, unit risk premium is large. We can see from (23) that unit risk premium of the bond is large if market price of risk is large, the volatility σ_r of r is small, and $\log \Lambda_{0,\tau}$ is not so sensitive with respect to the current short rate level.

The numerator of HP₁ is the derivative of $\log f(r_0, 0)$ with respect to initial short rate r_0 , where $f(r_0, 0)$ is the multiplier of the value function in (19). Both sign and size of HP₁ depends on the sensitivity of f with respect to the short rate. The denominator is the derivative of the rate of return of bond with respect to r_0 . Thus HP₁ is large when the relative change in multiplier f is large or the relative change in bond price $\Lambda_{0,\tau}$ is small.

For numerical examples, we specify a model of term-structure. We use Brennan-Schwartz model (Brennan-Schwartz [4]) in the following numerical examples. Drift μ_r and volatility σ_r of the short rate process is assumed to satisfy

$$\mu_r(r,t) = a_1 + a_2 r \quad \text{and} \quad \sigma_r(r,t) = br, \tag{24}$$

where a_1, a_2 , and b are constant. It is also assumed that r satisfies

$$dr(t) = (\hat{a}_1 + \hat{a}_2 r_t)dt + \hat{b} r_t d\hat{B}_t$$

under the equivalent probability measure Q, where \hat{a}_1 , \hat{a}_2 , and \hat{b} are constant. Since \hat{B} satisfies (3), the following equations are satisfied:

$$\hat{a}_1 = a_1,$$

 $\hat{a}_2 + \hat{b} heta_t = a_2,$
 $\hat{b} = b.$

Thus the market price of risk is constant in our example:

$$\theta_t = \frac{a_2 - \hat{a}_2}{b} \equiv \theta. \tag{25}$$

Proposition 3.2. In the case of Brennan-Schwartz model, the optimal strategy is given by

$$\begin{split} \varphi_0^* &= \frac{1}{1-\alpha} \frac{\Lambda_{0,\tau}}{\frac{\partial \Lambda_{0,\tau}}{\partial x}} \left[\theta \frac{1}{b x} + (1-\alpha) \frac{E\left[\frac{\alpha}{\alpha-1} \pi_T^{\frac{1}{\alpha-1}} \frac{\partial \pi_T}{\partial x}\right]}{E\left[\pi_T^{\frac{\alpha}{\alpha-1}}\right]} \right] \\ &= \frac{\Lambda_{0,\tau}}{\frac{\partial \Lambda_{0,\tau}}{\partial x}} \frac{\theta}{(1-\alpha) b x} + \frac{\Lambda_{0,\tau}}{\frac{\partial \Lambda_{0,\tau}}{\partial x}} \frac{E\left[\frac{\alpha}{\alpha-1} \pi_T^{\frac{1}{\alpha-1}} \frac{\partial \pi_T}{\partial x}\right]}{E\left[\pi_T^{\frac{\alpha}{\alpha-1}}\right]} \\ &\equiv MVP + HP_1, \end{split}$$

where the market price of risk θ is constant. The system of $\partial \pi_T / \partial x$ is given by

$$\begin{cases} d\pi_t = -\pi_t (r_t^x dt + \theta dB_t), \\ \pi_0 = 1. \end{cases}$$
(26)

$$\begin{cases} dr_t^x &= (a_1 + a_2 r_t^x) dt + b r_t^x dB_t, \\ r_0^x &= x. \end{cases}$$
(27)

$$\begin{cases} dY_t = a_2 Y_t dt + b Y_t dB_t, \\ Y_0 = 1. \end{cases}$$
(28)

$$\frac{\partial \pi_T}{\partial x} = \pi_T \left(-\int_0^T Y_t dt \right). \tag{29}$$

Proof. This is a special case of Proposition 3.3, whose proof is in Appendix.

3.2 Power Utility with Lower Bound

For the case with $\overline{Z} > 0$, we assume that $E[\pi_T \overline{Z}] \leq w_0$. In other words, we suppose that \overline{Z} is attainable by the current wealth level. Otherwise there is no solution to investor's problem. For the case with $\overline{Z} > 0$, the value function is not separable with respect to w and r. The optimal portfolio strategy is more complicated than the case with $\overline{Z} = 0$.

Proposition 3.3. The optimal portfolio strategy φ_0^* at time 0 of problem (10) is given by

$$\varphi_0^* = \frac{1}{1 - \alpha} \frac{\Lambda_{0,\tau}}{\frac{\partial \Lambda_{0,\tau}}{\partial x}} \left[(1 - \chi) \theta_0 \frac{1}{\sigma_r} + (1 - \alpha) \left((1 - \chi) \frac{E\left[\frac{\alpha}{\alpha - 1} \pi_T^{\frac{1}{\alpha - 1}} \frac{\partial \pi_T}{\partial x}\right]}{E\left[\pi_T^{\frac{\alpha}{\alpha - 1}}\right]} + \chi \frac{E\left[\frac{\partial \pi_T}{\partial x}\right]}{E[\pi_T]} \right) \right],$$
(30)

where χ is defined by

$$\chi = \frac{E[\bar{Z}\pi_T]}{w_0}.$$

 $\partial \pi_T / \partial x$ is given by the system (20), (21), and (22).

Proof. See Appendix.

In the case of Brennan-Schwartz term structure model, as in proposition 3.2, we get the following result.

Proposition 3.4. In the case of Brennan-Schwartz model, the optimal portfolio strategy of problem (10) is given as follows:

$$\varphi_{0}^{*} = \frac{1}{1-\alpha} \frac{\Lambda_{0,\tau}}{\frac{\partial\Lambda_{0,\tau}}{\partial x}} \left[(1-\chi)\theta \frac{1}{br_{t}} + (1-\alpha) \left((1-\chi) \frac{E\left[\frac{\alpha}{\alpha-1}\pi_{T}\frac{1}{\alpha-1}\frac{\partial\pi_{T}}{\partial x}\right]}{E\left[\pi_{T}^{\frac{\alpha}{\alpha-1}}\right]} + \chi \frac{E\left[\frac{\partial\pi_{T}}{\partial x}\right]}{E[\pi_{T}]} \right) \right]$$

$$= (1-\chi) \frac{\Lambda_{0,\tau}}{\frac{\partial\Lambda_{0,\tau}}{\partial x}} \frac{\theta}{(1-\alpha)br_{t}} + (1-\chi) \frac{\Lambda_{0,\tau}}{\frac{\partial\Lambda_{0,\tau}}{\partial x}} \frac{E\left[\frac{\alpha}{\alpha-1}\pi_{T}\frac{1}{\alpha-1}\frac{\partial\pi_{T}}{\partial x}\right]}{E\left[\pi_{T}^{\frac{\alpha}{\alpha-1}}\right]} + \chi \frac{\Lambda_{0,\tau}}{\frac{\partial\Lambda_{0,\tau}}{\partial x}} \frac{E\left[\frac{\partial\pi_{T}}{\partial x}\right]}{E[\pi_{T}]}$$

$$\equiv (1-\chi)MVP + (1-\chi)HP_{1} + \chi HP_{2}.$$
(31)

The market price of risk $\theta(t)$ takes constant value

$$\theta = \frac{a_2 - \hat{a}_2}{b}$$

The system of $\partial \pi_T / \partial x$ is given by (26), (27), (28), and (29).

When there is a lower bound on the terminal wealth level, the mean variance portfolio is given by $(1 - \chi)$ times the mean variance portfolio in (23) for the case $\overline{Z} = 0$. If initial wealth level w_0 is equal to the present value of the minimum wealth level \overline{Z} and $\chi = 1$, then the mean variance portfolio is zero. On the other hand, when $\chi < 1$, investors have sufficient money to invest in the risky asset so that they can obtain higher expected utility.

The hedging portfolio consists of two parts:

$$(1-\chi)\mathrm{HP}_{1} = (1-\chi)\frac{\frac{\partial}{\partial x}\log(E[\pi_{T}\frac{\alpha}{\alpha-1}]^{1-\alpha})}{(1-\alpha)\frac{\partial}{\partial x}\log\Lambda_{0,\tau}}$$

and

$$\chi \text{HP}_2 = \chi \frac{\frac{\partial}{\partial x} \log E[\pi_T]}{\frac{\partial}{\partial x} \log \Lambda_{0,\tau}}.$$

The first component of the hedging portfolio is also given by $(1 - \chi)$ times the hedging portfolio (23) for the case $\overline{Z} = 0$. If χ is less than one but is close to one, the first component of hedging portfolio is small.

The numerator of HP₂ is the relative change in the present value of Z as r_0 changes. The denominator is the relative change in bond price $\Lambda_{0,\tau}$ as r_0 changes. Investors hold long-term bond more than $w_0\chi$, that is, more than the present value of \overline{Z} , if the present value of \overline{Z} is more sensitive to r_0 than to $\Lambda_{0,\tau}$. Thus HP₂ is hedging demand against changes in the present value of \overline{Z} . The hedging demand is large if the price of available bond is less sensitive to the current short rate level.

 HP_2 is easily interpreted if we consider the bond whose maturity is same as investor's time horizon, that is, $\tau = T$. It follows from $\Lambda_{0,T} = E[\pi_T]$ that $\text{HP}_2 = \chi$. Therefore HP_2 suggests investors to buy the bond with maturity T so that the bond delivers \overline{Z} at time T. Then the optimal portfolio is given by a combination of the bond that delivers \overline{Z} at T and the optimal portfolio of the investor whose utility function has no lower bound:

$$\varphi_0^* = (1 - \chi) \left(\text{MVP} + \text{HP}_1 \right) + \chi.$$

The weight between two parts depends on the ratio χ of the present value of \overline{Z} and the current wealth level w_0 . If $E[\pi_T \overline{Z}]$ is close to w_0 and the minimum wealth level \overline{Z} is not easily attained, then investors spend almost all of their wealth to buy the bond that delivers \overline{Z} . On the other hand, if $E[\pi_T \overline{Z}]$ is sufficiently smaller than w_0 , then investors should invest almost all of their wealth to the mean variance portfolio and HP₁.

4 Term Structure Model and Numerical Analysis

In our numerical examples, we compute the optimal portfolio strategy using Monte Carlo simulation. Our main purpose is to study properties of the optimal portfolio for reasonable parameter combinations. It is easier to elaborate simulation if we choose a simple term structure model. On the other hand, a simple model is not necessarily good from empirical point of view. Recent zero-interest-rate-policy by Bank of Japan also makes it difficult to choose a model.

For example, given recent zero-interest-rate-policy by Bank of Japan, Vasicek [27] model may not be a good choice since simulated sample paths of nominal interest rates are likely to take negative values under reasonably estimated parameter values. The term structure model by Cox, Ingersoll, and Ross (CIR) [11] is less complicated but allows time-varying market price of risk. However, to generate random sample paths of CIR model is difficult when the short rate is close to zero. Because of the square root coefficient in the volatility of CIR model, a randomly generated sample path often takes negative value even if we take each time step small. Furthermore, the existence of the stochastic flow of the short rate is not guaranteed by standard sufficient conditions if there are square root terms in coefficients of the short rate process. (See, for example, Protter [25].) Since Brennan-Schwartz model is not only simple but also is not suffered by these problems, we use it in the following numerical example.

Unfortunately, Brennan-Schwartz model is not so successful from empirical point of view. For example, Chan, Karolyi, Longstaff, and Sanders [17] conclude that Dothan [13] and CIR variable-rate securities model (CIR VR) [10] perform much better than other commonly used models including Brennan-Schwartz [4]. Our results are in fact consistent to their conclusion, and estimated parameters are not so stable. However, we use Brennan-Schwartz model, assigning higher priority to its advantages in numerical simulation.

4.1 The Econometric Approach

Following Chan, Karolyi, Longstaff, and Sanders [17], the parameters of the continuous-time model are estimated using a discrete-time econometric specification

$$r_{t+\Delta} - r_t = (a_1 + a_2 r_t) \Delta t + \epsilon_{t+1} \sqrt{\Delta t}.$$
(32)

In order to use the Maximum Likelihood (ML) method, it is assumed that

$$\epsilon_{t+1} = br_t Z_t, \quad Z_t \sim N(0, 1). \tag{33}$$

Let $\{Y_i\}$ be the sample data. Let p be the parameter vector with elements a_1, a_2 , and b. The maximum likelihood estimator \hat{p} is given as a solution to the following problem:

$$\hat{p} = \arg\max_{\theta} \frac{1}{T} \sum_{i=1}^{T} \left[-\log((bY_{i-1})^2) - \left(\frac{(Y_i - Y_{i-1})\frac{1}{\sqrt{\Delta t}} - (a_1 + a_2Y_{i-1})\Delta t}{bY_{i-1}}\right)^2 \right]$$

4.2 The Data

Japanese uncollateralized average overnight call rate data for our study were obtained from Datastream. We study daily data that cover the period from 10th April 1989 to 30th March 2001, providing 3,125 observations in total. All interest rates are annualized. Figure 1 presents the time series data of the call rates. It is obvious that there are some structural breaks during the sample period. Thus ten sample sets are created, which start at the same date and end at different date.

Table 1 shows means, standard deviations, and first six autocorrelations of the overnight call rate and daily changes for each sample set. The autocorrelations of daily changes are generally small and are not consistently positive or negative, which suggest that call rates are stationary.

4.3 Empirical Results of Parameter Estimation

Table 2 reports sample estimates and t-statistics for our ten sample sets. Parameters k, r^* , and σ are defined as follows:

$$dr_t = (a_1 + a_2 r_t)dt + br_t dB_t$$
$$= k(r^* - r_t)dt + \sigma r_t dB_t.$$

In other words,

$$k = -a_2, \quad r^* = -\frac{a_1}{a_2}, \text{ and } \sigma = b.$$

The hypothesis that a parameter is zero is rejected at the 95% confidence level if an absolute value of *t*-statistic is greater than 1.96. At the 90% confidence level, the hypothesis is rejected if an absolute value of *t*-statistics is greater than 1.645. In all case, hypotheses are rejected at the 95% confidence level.

However it is worth noting that parameter k is negative for sample sets 3 and 4. Because k is a mean-reverting speed of the short rate process, negative k suggests that the short rate is not mean-reverting. For sample sets 8, 9, and 10, mean-reverting speed parameters k are much larger than other cases. For these sample cases, the call rates are above 4% at the starting date, but r^* are less than 1%. Changes of monetary policy by Bank of Japan during the period may be the reason why estimated parameters take these values.³ Unfortunately, estimated parameters are not so stable in our sample cases, and in the empirical analysis of the optimal portfolio strategy we should pay careful attention to parameter selection.

³To investigate whether these are structural breaks, we could introduce dummy variables in our model. See Chan, Karolyi, Longstaff, and Sanders [17].

4.4 Calibration

Before we compute the optimal portfolio strategy given estimated parameter, we need to calibrate the model and determine the market price of risk θ in (25). In the following procedure, we find a value of θ so that the yield spread is 1%.

- The initial value r_0 of the short rate in our simulation is $r_0 = r^*$.
- The maturity τ of the discount bond is $\tau = 10$.
- Given a constant \hat{a}_2 , we perform Monte-Carlo integration to calculate the bond price maturing at τ with the following discretized model.

$$\theta = \frac{a_2 - \hat{a}_2}{b}.$$

$$\begin{cases} \Delta \pi_t = -\pi_t (r_t \Delta t + \theta \sqrt{\Delta t} \epsilon_t), \\ \pi_{t+\Delta t} = \pi_t + \Delta \pi_t, \\ \pi_0 = 1. \end{cases}$$

$$\begin{cases} \Delta r_t = (a_1 + a_2 r_t) \Delta t + b r_t \sqrt{\Delta t} \epsilon_t, \\ r_{t+\Delta t} = r_t + \Delta r_t, \\ r_0 = \text{given.} \end{cases}$$

$$\Lambda_{0,\tau} = \frac{\sum_{n=1}^N \pi_\tau^{(n)}}{N},$$

where *n* indicates the index of the sample path and *N* is the number of simulation. We also Calculate the yield of bond y_{τ} by $y_{\tau} = -\log \Lambda_{0,\tau}/\tau$.

• We compute $\Lambda_{0,\tau}$ and y_{τ} repeatedly, and find value \hat{a}_2 so that $y_{\tau} \approx r_0 + 1\%$.

Table 3 reports bond prices and yields for sample set 1,2, and 5 - 10. We omitted the case 3 and 4, because the short rate process is not mean-reverting in those cases.

5 Empirical properties of the optimal portfolio

In this section, we compute the optimal portfolio strategy using parameters that are estimated in the previous section. We concentrate on studying the properties of the optimal portfolio strategy, in particular of the hedging demand. Investors maximize expected utility that is obtained from their terminal wealth. We further assume that the risk aversion coefficient is $\alpha = -5.0$ and the investment horizon is T = 1.

Since our main purpose is to study properties of optimal portfolio strategy, we choose Brennan-Schwartz model so that computational cost is smaller. Unfortunately, as we have seen in Table 2, estimated parameters for Brennan-Schwartz model are not very reasonable for some cases. In order to find a parameter combination that is reasonable from both empirical and theoretical point of view, we compute the optimal portfolio for each case. Table 4 reports the optimal strategy, MVP, and HP_1 . For sample sets from 5 to 10, MVP is so large that the optimal portfolio is highly leveraged. Although we do not discuss here whether estimated parameter combinations for these cases are reasonable or not from the view point of econometrics, such a large size of MVP suggests that Brennan-Schwartz model is not very appropriate for those sample periods. In order to get a reasonable portfolio for these sample periods, we may need to use other term structure models that allow some structural breaks during sample periods. Sizes of optimal portfolio for sample set 1 and 2 are relatively reasonable, and we use sample set 2 for the following analysis.

We first consider power-utility function with no lower bound on the terminal wealth level. We find that the optimal portfolio strategy with no lower bound on the terminal wealth is highly leveraged and is not realistic. In particular, the mean variance portfolio is very large for many parameter combinations reflecting that long-term bond is too attractive from myopic point of view. When there is a lower bound on the terminal wealth level, the size of optimal portfolio is more realistic, because the lower bound restricts the size of the mean variance portfolio.

5.1 Power Utility with no Lower Bound

The optimal portfolio strategy as well as bond prices for various parameter combinations are calculated. Table 5 reports the optimal portfolio of the investor with $\alpha = -5.0$ and T = 1 for various values of the current short rate r_0 . Since the higher discount rate implies the lower bond price, the bond price $\Lambda_{0,\tau}$ is low for the higher short rate r_0 and the first derivative $\partial \Lambda_{0,\tau}/\partial r_0$ is negative. We can see that, when the short rate r_0 is large, $|\partial \Lambda_{0,\tau}/\partial r_0|$ is small and the bond price is less sensitive to the short rate.

It is interesting to see that

$$\frac{\Lambda_{0,\tau}}{\frac{\partial\Lambda_{0,\tau}}{\partial r_0}}$$

is not very sensitive to r_0 in this example. Then it follows from (1) and (24) that $\sigma_{\Lambda}(0,\tau)$ depends almost linearly on r. The volatility of bond price is high when the short rate is high. The mean variance portfolio is given by

$$MVP = \frac{\Lambda_{0,\tau}}{\frac{\partial \Lambda_{0,\tau}}{\partial r_0}} \frac{\theta}{(1-\alpha)b r_0},$$
(34)

and θ is constant by our assumption. We can thus conclude that the meanvariance portfolio is small when the short rate is high. In other words, the bond is less attractive when the short rate is high, because the volatility of bond price is also high.

The value function $f(r_0, 0)$ in the equation

$$J(w,r,t) = \frac{w^{\alpha}}{\alpha}f(r,t)$$

is decreasing with respect to r_0 , which implies that the remaining utility level is low when the short rate is high. It is also interesting to see that f_r/f are negative and almost constant for our example. The hedging portfolio is given by

$$\mathrm{HP}_{1} = \frac{1}{1-\alpha} \frac{\Lambda_{0,\tau}}{\frac{\partial\Lambda_{0,\tau}}{\partial r_{0}}} \frac{f_{r}(r_{0},0)}{f(r_{0},0)} = \frac{\Lambda_{0,\tau}}{\frac{\partial\Lambda_{0,\tau}}{\partial r_{0}}} \frac{E\left[\frac{\alpha}{\alpha-1}\pi_{T}\frac{1}{\alpha-1}\frac{\partial\pi_{T}}{\partial r_{0}}\right]}{E\left[\pi_{T}\frac{\alpha}{\alpha-1}\right]}$$

Since $\Lambda_{0,\tau}/\frac{\partial \Lambda_{0,\tau}}{\partial r_0}$ is negative and is not sensitive to r_0 , the hedging portfolio is positive and is not very sensitive to the current short rate level.

Table 6 shows the optimal portfolio for various risk aversion coefficients. As the investor becomes more risk averse and $|\alpha|$ is larger, the mean variance portfolio is smaller. In particular, the investor with $\alpha = -1000$ who may accept little instantaneous risk has small mean-variance portfolio compared to other cases. It is interesting that the hedging portfolio has a certain size even for such a risk averse investor with $\alpha = -1000$. The term f_r/f is not dominated by the term $1/(1-\alpha)$. It is also interesting to see that the hedging portfolio is positive for $\alpha < 0$ and negative for $\alpha > 0$. As is well known, logutility investors are 'myopic' in the sense that they ignore shifts in the state variable and care only about the instantaneous risk structure. The log-utility case is knife-edge case in the sense that more-risk-averse-investor holds the bond but less-risk-averse-investor sells short the bond.

Table 7 reports how investor's time horizon affects the optimal portfolio. We compute the optimal portfolio for investors with T = 1, 2, 3, 4, 5, and 9. The mean variance portfolio is same for all cases since it is a myopic term. The hedging portfolio is increasing with respect to T. We can conclude that the hedging portfolio is more important if the investor's time horizon is longer.

Table 8 reports the sensitivity of the optimal portfolio to parameter b of the short rate process, which determines a volatility of the short rate. Since a volatility $\sigma_{\Lambda}(0,\tau) = b \frac{\partial}{\partial r_0} \Lambda_{0,\tau}$ of the bond price is negative, the short rate r and the bond price Λ is negatively correlated if we simply ignore the drift terms of both processes. Our results suggest that, when a volatility b of the short rate is large, the absolute value of $\sigma_{\Lambda}(<0)$ is large. The larger volatility of the bond price process then implies the smaller size of the mean variance portfolio, since instantaneous risk is large. On the other hand, the hedging portfolio is not so sensitive to the volatility b of short rate. This is another evidence that the hedging portfolio is not held because of the instantaneous risk structure.

Table 9 also reports the sensitivity of the optimal portfolio to parameter a_2 , that is, mean-reverting speed of the short rate r. Although $\Lambda_{0,\tau}/\frac{\partial \Lambda_{0,\tau}}{\partial r_0}$ is not so sensitive to a_2 in this example, it follows from (34) and (25) that the mean variance portfolio is sensitive to the market price of risk θ and thus a_2 . Thus the mean variance portfolio is sensitive to parameter a_2 . This may be one of the reason why the mean variance portfolio is so large for the parameter sets from 5 to 10 in Table 2. On the other hand, the hedging portfolio is not so sensitive to a_2 , which implies that the hedging portfolio is not very sensitive to a shift in the instantaneous risk structure.

5.2 Power Utility with Lower Bound

Table 10-14 report the optimal portfolio for the case similar to Table 5-9 but $\bar{Z}/w_0 = 0.9$, where \bar{Z} is a minimum lower bound on the terminal wealth.

Table 10 reports the optimal strategy of the investor with $\alpha = -5.0$ and T = 1 for various short rates r_0 . As we can expect from (31), the mean variance portfolio is smaller because of the lower bound \overline{Z} . The size of the hedging portfolio is similar to the case without the lower bound, which suggests that χHP_2 is about the same size that the HP_1 is scaled down by $(1 - \chi)$. Similar to the previous subsection, the higher discount rate implies the smaller mean-variance portfolio. The hedging portfolio is increasing with respect to the short rates r_0 , but it is less sensitive to r_0 than the mean-variance portfolio.

Table 11 shows the optimal portfolio for various α . As in the previous subsection, the mean variance portfolio is a decreasing function of α , and the hedging portfolio is an increasing function of α in our parameter set. The size of the mean variance portfolio is much smaller because of the lower bound \overline{Z} . The size of the hedging portfolio also depends on \overline{Z} . The hedging portfolio with $\alpha = -0.5$ and $\overline{Z} = 0$ is smaller than those with $\alpha = -5$ and -10. On the other hand, the size of the hedging portfolio with $\alpha = -0.5$ and $\overline{Z} = 0.9w_0$ is similar to those with $\alpha = -5$ and -10. This is not surprising because HP₂ hedges against the risk of falling short of \overline{Z} and HP₂ may not be so sensitive to the risk aversion as HP₁.

Table 12 shows the effect of the time horizon. We compute the optimal portfolio for T = 1, 2, 3, 4, 5, and 9. With the lower bound \overline{Z} , the mean variance portfolio is not constant with respect to the terminal horizon T. Since χ is defined by

$$\chi = \frac{E\left[\bar{Z}\pi_T\right]}{w_0}$$

and is a function of the investment horizon T, the mean variance portfolio

$$\varphi_{\rm mvp} = (1-\chi) \frac{\mu_{\Lambda} - r}{(1-\alpha)\sigma_{\Lambda}^2}$$

is also a function of the time horizon T. In this sense, the mean-variance portfolios is not myopic if there is a lower bound on the terminal wealth. When T is large, the present value of \overline{Z} is small. Thus restrictions on the optimal portfolio by the lower bound \overline{Z} may not be so strong when T is large. This would be the reason why both the mean-variance portfolio and the hedging portfolio are increasing with respect to T in our example. The lower bound \overline{Z} , however, still has a strong impact on the optimal portfolio even for the case with T = 9.

Table 13 reports the sensitivity of the optimal strategy to b. As in the case without lower bound, the mean variance portfolio is sensitive to b and is decreasing with respect to b. On the other hand, the hedging portfolio is not so sensitive to b. Again, this may be an evidence that hedging portfolio is not held because of the instantaneous risk structure.

Table 14 shows the sensitivity of the optimal portfolios with respect to the parameter a_2 , the mean-reverting speed of the short rate process. The mean variance portfolio is sensitive to the parameter a_2 . Both the mean variance portfolio and the hedging portfolio are increasing functions with respect to a_2 , but the hedging portfolio is not so sensitive to a_2 .

6 Conclusion

We study the optimal portfolio strategy for a zero-coupon bond and a riskless asset, using Japanese interest rate data. A simple one-factor model of the term structure of interest rates, that is, Brennan and Schwartz model is used for our numerical example. For numerical examples, investors are assumed to have power-utility function that is defined on the wealth at a finite timehorizon. The optimal portfolio is computed using the technique of stochastic flows and Monte Carlo simulation. As shown in the previous literature, the hedging portfolio is not negligible, and the optimal portfolio depends on investor's terminal horizon. Mean variance portfolio is very sensitive to parameter values. The hedging portfolio is not so sensitive to model's parameter values, but depends on investor's attitude towards risk.

We also study the case where investors have minimum bound on their wealth at terminal horizon. When there is a zero-coupon bond maturing at investor's terminal horizon, the investor first holds the bond so that minimum wealth bound is guaranteed. This intuitive motivation to hold zero-coupon bond comes from the hedging portfolio, which also shows the importance of dynamic optimal portfolio. Then the investor invests remaining money in the portfolio that is for the case with no lower bound. The optimal portfolio is less sensitive to parameters and is more realistic.



This figure plots Japanese overnight call rate(uncollateralized, average) from 1989/4/10 to 2001/3/30, obtained from Datastream. Daily interest rates are expressed in annualized form. There are some structural breaks during the period.

Figure 1: The Japan call rate (uncollateral, overnight, middle)

No.	End date	Variables	Ν	Mean	Std. Dev.	$ ho_1$	$ ho_2$	$ ho_3$	$ ho_4$	$ ho_5$	$ ho_6$
1	03/31/1992	r_t	777	0.067836	0.011163	0.989	0.979	0.973	0.966	0.960	0.955
		$r_{t+1} - r_t$	776	$1.45\mathrm{E} ext{-}05$	0.001420	-0.188	-0.179	0.034	-0.047	-0.104	0.166
2	03/31/1993	r_t	1038	0.061044	0.015410	0.993	0.989	0.985	0.981	0.978	0.975
		$r_{t+1} - r_t$	1037	-1.54E-05	0.001380	-0.255	-0.121	0.050	-0.042	-0.104	0.135
3	03/31/1994	r_t	1299	0.054283	0.019369	0.997	0.995	0.993	0.991	0.989	0.988
		$r_{t+1} - r_t$	1298	0.0000	0.001440	-0.209	-0.087	0.040	-0.032	-0.076	0.098
4	03/31/1995	r_t	1560	0.048855	0.021429	0.997	0.996	0.995	0.994	0.993	0.992
		$r_{t+1} - r_t$	1559	-1.76E-05	0.001445	-0.331	-0.082	0.027	-0.024	-0.062	0.078
5	03/29/1996	r_t	1820	0.042979	0.024557	0.998	0.997	0.996	0.995	0.994	0.993
		$r_{t+1} - r_t$	1819	-2.20 E-05	0.001350	-0.324	-0.082	0.027	-0.026	-0.060	0.077
6	03/31/1997	r_t	2081	0.038188	0.026221	0.998	0.998	0.997	0.996	0.996	0.995
		$r_{t+1} - r_t$	2080	-1.82E-05	0.001266	-0.323	-0.082	0.026	-0.027	-0.059	0.076
7	03/31/1998	r_t	2342	0.034455	0.026872	0.999	0.998	0.998	0.997	0.997	0.996
		$r_{t+1} - r_t$	2341	-1.62E-05	0.001201	-0.321	-0.083	0.024	-0.027	-0.057	0.074
8	03/31/1999	r_t	2603	0.031301	0.027186	0.999	0.998	0.998	0.997	0.997	0.996
		$r_{t+1} - r_t$	2602	-1.71E-05	0.001146	-0.320	-0.083	0.022	-0.027	-0.056	0.074
9	03/31/2000	r_t	2865	0.028465	0.027413	0.999	0.998	0.998	0.998	0.997	0.997
		$r_{t+1} - r_t$	2864	-1.56 E-05	0.001095	-0.320	-0.082	0.022	-0.027	-0.056	0.074
10	03/30/2001	\overline{r}_t	3125	0.026223	0.027284	0.999	0.999	0.998	0.998	0.998	0.997
		$r_{t+1} - r_t$	3124	$-1.40 ext{E} - 05$	0.001049	-0.320	-0.082	0.022	-0.027	-0.056	0.073

Each sample set starts at the same date, 4/10/1989, and ends at different date. Means, standard deviations, and first six autocorrelations are presented for r_t and $r_{t+1} - r_t$, which are sampled at daily basis.

Table 1: Summary Statistics for Japan call rate (daily data)

No.	a_1	a_2	b	k	r^*	σ
1	0.222674	-3.23722	0.381611	3.237215	0.068786	0.381611
	(5.643344)	(-4.75456)	(254.1506)			
2	0.031086	-0.57307	0.41342	0.573068	0.054245	0.41342
	(1.516653)	(-1.3405)	(348.9367)			
3	-0.03475	0.54069	0.102757	-0.54069	0.064277	0.102757
	(-44.3525)	(29.19314)	(14084.08)			
4	-0.0175	0.156607	0.103061	-0.15661	0.111719	0.103061
	(-35.1566)	(11.03123)	(25655.19)			
5	0.003292	-0.55671	0.105539	0.556709	0.005913	0.105539
	(65.20101)	(-109.065)	(47704.36)			
6	0.006399	-0.76108	0.106818	0.761084	0.008408	0.106818
	(180.3298)	(-153.773)	(47278.52)			
7	0.006483	-0.97090	0.117463	0.970897	0.006677	0.117463
	(250.5568)	(-188.544)	(46979.07)			
8	0.108104	-10.9635	0.106685	10.96348	0.009860	0.106685
	(79937.28)	(-7273.83)	(92021.86)			
9	0.101600	-13.0078	0.100213	13.00781	0.007811	0.100213
	(564872.6)	(-16722.4)	(6448521.0)			
10	0.10192	-14.9529	0.100238	14.95294	$0.00\overline{6816}$	0.100238
	(660425.3)	(-21392.6)	(6879570.0)			

These are estimated parameters using ML. The model (Brennan-Schwartz) is as follows:

$$dr_t = (a_1 + a_2 r_t)dt + br_t dB_t$$

= $k(r^* - r_t)dt + \sigma r_t dB_t.$

The values in parentheses are t-statistics, testing the hypothesis that the parameter is equal to 0.

Table 2: Estimates for the Japanese call rates

sample set	$r_0 = r^*$	\hat{a}_2	a_1	a_2	b_2	$\Lambda_{ au}$	$y_{ au}$
1	6.88~%	-2.9870	0.2227	-3.2372	0.3816	0.4546	7.88~%
2	5.42~%	-0.4420	0.0311	-0.5731	0.4134	0.5264	6.42~%
5	0.59~%	-0.4787	0.0033	-0.5567	0.1055	0.8530	1.59~%
6	0.84~%	-0.6822	0.0064	-0.7611	0.1068	0.8318	1.84~%
7	0.67~%	-0.8838	0.0065	-0.9709	0.1175	0.8465	1.67~%
8	0.99~%	-10.8836	0.1081	-10.8635	0.1067	0.8199	1.99~%
9	0.78~%	-12.9327	0.1016	-13.0078	0.1002	0.8366	1.78~%
10	0.68~%	-14.8778	0.1019	-14.9529	0.1002	0.8450	1.68~%

In the calibration process, we find an \hat{a}_2 so that $r_0 + 1\% = y_{\tau}$ with $\tau = 10$. Since the short rate is not mean-reverting, we ommit case 3 and 4.

Table 3: Calibration of the Brennan-Schwartz model for each sample set

sample set	r_0	φ	$arphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)
1	6.88~%	13.1730	12.3929	0.7800	-0.1155
2	5.42~%	1.4216	1.1083	0.3133	-0.1440
5	0.59~%	96.8309	96.5089	0.3220	-0.1528
6	0.84~%	94.6782	94.2649	0.4133	-0.1509
7	0.67~%	140.3714	139.8829	0.4885	-0.1520
8	0.99~%	1290.1660	1289.3345	0.8316	-0.1490
9	0.78~%	2062.0472	2061.2154	0.8318	-0.1505
10	0.68~%	2717.7672	2716.9353	0.8319	-0.1512

Optimal strategies are calculated for each case in table 3. Investor's terminal horizon is T = 1 and risk aversion coefficient $\alpha = -5$. There is no lower bound condition on the terminal wealth and $\overline{Z} = 0$. Maturity of the zero-coupon bond is $\tau = 10$. We take 500 periods per unit of time. The number of simulation is 10,000.

Table 4: Optimal strategies for sample sets

r_0	φ	$\varphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)	$\Lambda_{0, au}$	$\partial \Lambda_{0, au}/\partial r_0$
5.25%	1.4574	1.1443	0.3131	-0.1450	0.5284	-1.1241
5.50%	1.4068	1.0934	0.3134	-0.1435	0.5256	-1.1170
5.75%	1.3606	1.0469	0.3137	-0.1421	0.5228	-1.1099
6.00%	1.3183	1.0043	0.3140	-0.1407	0.5200	-1.1029
6.25%	1.2794	0.9652	0.3143	-0.1393	0.5173	-1.0960
6.50%	1.2435	0.9290	0.3146	-0.1379	0.5145	-1.0891
6.75%	1.2103	0.8955	0.3148	-0.1365	0.5118	-1.0823

This table presents the optimal portfolio for investors with the expected utility function $E[W_T^{\alpha}/\alpha]$, where $\alpha = -5.0$ and T = 1. $\hat{a}_2 = -0.4420$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14). φ_{mvp} is mean-variance portfolio and φ_{hp} is hedging portfolio:

$$\varphi_{\rm mvp} = \frac{1}{1-\alpha} \frac{\mu_{\Lambda}(0,\tau) - r}{(\sigma_{\Lambda}(0,\tau))^2}, \qquad \varphi_{\rm hp} = \frac{1}{1-\alpha} \frac{f_r(r,0)}{f(r,0)} \frac{\sigma_r(r,0)}{\sigma_{\Lambda}(0,\tau)}.$$

The optimal portfolio φ is given as the sum of φ_{mvp} and φ_{hp} .

Table 5: Optimal portfolio for various r_0

α	φ	$arphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)	$\Lambda_{0, au}$	$\partial \Lambda_{0, au}/\partial r_0$
0.90	64.3387	66.4979	-2.1592	1.7657	0.5264	-1.1191
0.0000001	6.6498	6.6498	0.0000	1000000.1049	0.5264	-1.1191
-0.50	4.4550	4.4332	0.1218	-1.9127	0.5264	-1.1191
-5.00	1.4216	1.1083	0.3133	-0.1440	0.5264	-1.1191
-10.00	0.9478	0.6045	0.3433	-0.0537	0.5264	-1.1191
-1000.00	0.3858	0.0066	0.3792	-0.0000	0.5264	-1.1191

This table presents the optimal portfolio for investors with the expected utility function $E[W_T^{\alpha}/\alpha]$, where T = 1. $\hat{a}_2 = -0.4420$, and $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14). φ_{mvp} is mean-variance portfolio and φ_{hp} is hedging portfolio:

$$\varphi_{\rm mvp} = \frac{1}{1-\alpha} \frac{\mu_{\Lambda}(0,\tau) - r}{(\sigma_{\Lambda}(0,\tau))^2}, \qquad \varphi_{\rm hp} = \frac{1}{1-\alpha} \frac{f_r(r,0)}{f(r,0)} \frac{\sigma_r(r,0)}{\sigma_{\Lambda}(0,\tau)}.$$

The optimal portfolio φ is given as the sum of $\varphi_{\rm mvp}$ and $\varphi_{\rm hp}$.

Table 6: Optimal portfolio for various α

Т	φ	$arphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)	$\Lambda_{0, au}$	$\partial \Lambda_{0, au}/\partial r_0$
1	1.4216	1.1083	0.3133	-0.1440	0.5264	-1.1191
2	1.6158	1.1083	0.5075	-0.1026	0.5264	-1.1191
3	1.7352	1.1083	0.6269	-0.0721	0.5264	-1.1191
4	1.8098	1.1083	0.7015	-0.0520	0.5264	-1.1191
5	1.8537	1.1083	0.7454	-0.0368	0.5264	-1.1191
9	1.9131	1.1083	0.8048	-0.0086	0.5264	-1.1191

This table presents the optimal portfolio for investors with the expected utility function $E[W_T^{\alpha}/\alpha]$, where $\alpha = -5.0$. $\hat{a}_2 = -0.4420$, and $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14). φ_{mvp} is mean-variance portfolio and φ_{hp} is hedging portfolio:

$$\varphi_{\rm mvp} = \frac{1}{1-\alpha} \frac{\mu_{\Lambda}(0,\tau) - r}{(\sigma_{\Lambda}(0,\tau))^2}, \qquad \varphi_{\rm hp} = \frac{1}{1-\alpha} \frac{f_r(r,0)}{f(r,0)} \frac{\sigma_r(r,0)}{\sigma_{\Lambda}(0,\tau)}.$$

The optimal portfolio φ is given as the sum of $\varphi_{\rm mvp}$ and $\varphi_{\rm hp}$.

Table 7: Optimal portfolio for various T

b	φ	$arphi_{ m mvp}$	φ_{hp}	V(w,r)	$\Lambda_{0, au}$	$\partial \Lambda_{0, au}/\partial r_0$
0.30	2.3695	2.0623	0.3072	-0.1383	0.5210	-1.1304
0.35	1.8357	1.5263	0.3094	-0.1415	0.5240	-1.1287
0.40	1.4926	1.1803	0.3124	-0.1435	0.5260	-1.1216
0.45	1.2602	0.9441	0.3161	-0.1450	0.5276	-1.1115

This table presents the optimal portfolio for investors with the expected utility function $E[W_T^{\alpha}/\alpha]$, where $\alpha = -5.0$ and T = 1. $\hat{a}_2 = -0.4420$, $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14). $\varphi_{\rm mvp}$ is mean-variance portfolio and $\varphi_{\rm hp}$ is hedging portfolio:

$$\varphi_{\rm mvp} = \frac{1}{1-\alpha} \frac{\mu_{\Lambda}(0,\tau) - r}{(\sigma_{\Lambda}(0,\tau))^2}, \qquad \varphi_{\rm hp} = \frac{1}{1-\alpha} \frac{f_r(r,0)}{f(r,0)} \frac{\sigma_r(r,0)}{\sigma_{\Lambda}(0,\tau)}.$$

The optimal portfolio φ is given as the sum of $\varphi_{\rm mvp}$ and $\varphi_{\rm hp}$.

Table 8: Optimal portfolio for various b

a_2	φ	$\varphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)	$\Lambda_{0, au}$	$\partial \Lambda_{0, au}/\partial r_0$
-0.5500	1.2256	0.9121	0.3135	-0.1460	0.5257	-1.1189
-0.5600	1.3104	0.9970	0.3134	-0.1451	0.5261	-1.1192
-0.5700	1.3954	1.0821	0.3133	-0.1442	0.5264	-1.1192
-0.5800	1.4808	1.1675	0.3133	-0.1433	0.5265	-1.1188
-0.5900	1.5665	1.2532	0.3133	-0.1423	0.5266	-1.1179

This table presents the optimal portfolio for investors with the expected utility function $E[W_T^{\alpha}/\alpha]$, where $\alpha = -5.0$ and T = 1. $\hat{a}_2 = -0.4420$, and $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14). φ_{mvp} is mean-variance portfolio and φ_{hp} is hedging portfolio:

$$\varphi_{\rm mvp} = \frac{1}{1-\alpha} \frac{\mu_{\Lambda}(0,\tau) - r}{(\sigma_{\Lambda}(0,\tau))^2}, \qquad \varphi_{\rm hp} = \frac{1}{1-\alpha} \frac{f_r(r,0)}{f(r,0)} \frac{\sigma_r(r,0)}{\sigma_{\Lambda}(0,\tau)}.$$

The optimal portfolio φ is given as the sum of φ_{mvp} and φ_{hp} .

Table 9: Optimal portfolio for various a_2

r_0	φ	$arphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)	$\Lambda_{0, au}$	$\partial \Lambda_{0, au}/\partial r_0$
5.25~%	0.5405	0.1710	0.3694	-0.0217	0.5284	-1.1241
5.50~%	0.5350	0.1653	0.3696	-0.0217	0.5256	-1.1170
5.75~%	0.5299	0.1601	0.3699	-0.0217	0.5228	-1.1099
6.00~%	0.5254	0.1553	0.3701	-0.0218	0.5200	-1.1029
6.25~%	0.5212	0.1509	0.3703	-0.0218	0.5173	-1.0960
6.50~%	0.5173	0.1468	0.3705	-0.0218	0.5145	-1.0891
6.75~%	0.5138	0.1430	0.3708	-0.0218	0.5118	-1.0823

This table presents the optimal portfolio for investors with the expected utility function $E[(Z - \overline{Z})^{\alpha}/\alpha]$, where $\alpha = -5.0$, T = 1, and $\overline{Z}/w_0 = 0.9$. $\hat{a}_2 = -0.4420$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14).

 φ_{mvp} is mean-variance portfolio and φ_{hp} is hedging portfolio:

$$\varphi_{\rm mvp} = (1-\chi) \frac{\mu_{\Lambda} - r}{(1-\alpha)\sigma_{\Lambda}^2}, \qquad \varphi_{\rm hp} = \chi \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\right]} + (1-\chi) \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\alpha}{\alpha-1}\pi_T\frac{1}{\alpha-1}\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\frac{\alpha}{\alpha-1}\right]}.$$

The optimal portfolio φ is given as the sum of $\varphi_{\rm mvp}$ and $\varphi_{\rm hp}$.

Table 10: Optimal portfolio for various r_0 with the lower bound \overline{Z}

α	φ	$arphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)	$\Lambda_{0, au}$	$\partial \Lambda_{0, au}/\partial r_0$
0.90	10.0166	10.0195	-0.0030	0.2660	0.5264	-1.1191
0.0000001	1.3243	1.0020	0.3224	1506744.7685	0.5264	-1.1191
-0.50	1.0087	0.6680	0.3407	-0.2882	0.5264	-1.1191
-5.00	0.5366	0.1670	0.3696	-0.0217	0.5264	-1.1191
-10.00	0.4652	0.0911	0.3741	-0.0081	0.5264	-1.1191
-1000.00	0.3805	0.0010	0.3795	-0.0000	0.5264	-1.1191

This table presents the optimal portfolio for investors with the expected utility function $E[(Z - \overline{Z})^{\alpha}/\alpha]$, where T = 1 and $\overline{Z}/w_0 = 0.9$. $\hat{a}_2 = -0.4420$, and $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14).

 $\varphi_{\rm mvp}$ is mean-variance portfolio and $\varphi_{\rm hp}$ is hedging portfolio:

$$\varphi_{\rm mvp} = (1-\chi) \frac{\mu_{\Lambda} - r}{(1-\alpha)\sigma_{\Lambda}^2}, \qquad \varphi_{\rm hp} = \chi \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\right]} + (1-\chi) \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\alpha}{\alpha-1}\pi_T\frac{1}{\alpha-1}\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\frac{\alpha}{\alpha-1}\right]}.$$

The optimal portfolio φ is given as the sum of φ_{mvp} and φ_{hp} .

Table 11: Optimal portfolio for various α with the lower bound \overline{Z}

T	φ	$arphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)
1	0.5366	0.1670	0.3696	-0.0217
2	0.8191	0.2222	0.5969	-0.0206
3	1.0108	0.2772	0.7336	-0.0180
4	1.1415	0.3246	0.8169	-0.0152
5	1.2342	0.3725	0.8617	-0.0124
9	1.4490	0.5459	0.9031	-0.0042

This table presents the optimal portfolio for investors with the expected utility function $E[(Z - \overline{Z})^{\alpha}/\alpha]$, where $\alpha = -5.0$ and $\overline{Z}/w_0 = 0.9$. $\hat{a}_2 = -0.4420$, and $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14).

 $\varphi_{\rm mvp}$ is mean-variance portfolio and $\varphi_{\rm hp}$ is hedging portfolio:

$$\varphi_{\rm mvp} = (1-\chi) \frac{\mu_{\Lambda} - r}{(1-\alpha)\sigma_{\Lambda}^2}, \qquad \varphi_{\rm hp} = \chi \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\right]} + (1-\chi) \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\alpha}{\alpha-1}\pi_T\frac{1}{\alpha-1}\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\frac{\alpha}{\alpha-1}\right]}$$

The optimal portfolio φ is given as the sum of φ_{mvp} and φ_{hp} . χ is the function of the investment horizon, T:

$$\chi = \frac{E[\bar{Z}\pi_T]}{w_0}.$$

Therefore, $\varphi_{\rm mvp}$ is not constant with respect to T.

Table 12: Optimal portfolio for various T with the lower bound \overline{Z}

b	φ	$\varphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)
0.3000	0.6740	0.3115	0.3625	-0.0209
0.3500	0.5952	0.2303	0.3650	-0.0213
0.4000	0.5464	0.1779	0.3685	-0.0216
0.4500	0.5150	0.1422	0.3728	-0.0218

This table presents the optimal portfolio for investors with the expected utility function $E[(Z-\overline{Z})^{\alpha}/\alpha]$, where $\alpha = -5.0$, T = 1, and $\overline{Z}/w_0 = 0.9$. $\hat{a}_2 = -0.4420$, and $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14).

 φ_{mvp} is mean-variance portfolio and φ_{hp} is hedging portfolio:

$$\varphi_{\rm mvp} = (1-\chi) \frac{\mu_{\Lambda} - r}{(1-\alpha)\sigma_{\Lambda}^2}, \qquad \varphi_{\rm hp} = \chi \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\right]} + (1-\chi) \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\alpha}{\alpha-1}\pi_T\frac{1}{\alpha-1}\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\frac{\alpha}{\alpha-1}\right]}.$$

The optimal portfolio φ is given as the sum of $\varphi_{\rm mvp}$ and $\varphi_{\rm hp}$.

Table 13: Optimal portfolio for various b with the lower bound \overline{Z}

a_2	φ	$arphi_{ m mvp}$	$arphi_{ m hp}$	V(w,r)
-0.55	0.5066	0.1373	0.3692	-0.0220
-0.56	0.5195	0.1501	0.3694	-0.0219
-0.57	0.5325	0.1630	0.3695	-0.0217
-0.58	0.5457	0.1760	0.3697	-0.0216
-0.59	0.5590	0.1890	0.3700	-0.0215

This table presents the optimal portfolio for investors with the expected utility function $E[(Z - \overline{Z})^{\alpha}/\alpha]$, where $\alpha = -5.0$ and $\overline{Z}/w_0 = 0.9$. $\hat{a}_2 = -0.4420$, and $r_0 = 5.42\%$. The number of sample path for Monte Carlo simulation is 10,000, and time length of each step is $\Delta t = 1/500$. The maturity of zero-coupon bond is $\tau = 10$. The optimal portfolio φ is given by (14).

 $\varphi_{\rm mvp}$ is mean-variance portfolio and $\varphi_{\rm hp}$ is hedging portfolio:

$$\varphi_{\rm mvp} = (1-\chi) \frac{\mu_{\Lambda} - r}{(1-\alpha)\sigma_{\Lambda}^2}, \qquad \varphi_{\rm hp} = \chi \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\right]} + (1-\chi) \frac{\Lambda_{0,\tau}}{\frac{\partial}{\partial x}\Lambda_{0,\tau}} \frac{E\left[\frac{\alpha}{\alpha-1}\pi_T\frac{1}{\alpha-1}\frac{\partial\pi_T}{\partial x}\right]}{E\left[\pi_T\frac{\alpha}{\alpha-1}\right]}.$$

The optimal portfolio φ is given as the sum of $\varphi_{\rm mvp}$ and $\varphi_{\rm hp}$.

Table 14: Optimal portfolio for various a_2 with the lower bound \overline{Z}

A Proof of Proposition 3.3

Proof. The Hamilton-Jacobi-Bellman (HJB) equation for the problem (10) is given by

$$\sup_{\varphi \in \mathbb{R}} \quad J_w(w, r, t)(w\varphi(\mu_{\Lambda}(t) - r_t) + r_t w) + J_r(w, r, t)\mu_r(r, t) + J_t(w, r, t) + \frac{1}{2}J_{ww}(w, r, t)w^2\sigma_{\Lambda}(t)^2\varphi^2 + J_{wr}(w, r, t)w\varphi\sigma_{\Lambda}(t)\sigma_r(r, t) + \frac{1}{2}J_{rr}(w, r, t)(\sigma_r(r, t))^2 = 0,$$
(35)

with the boundary condition

$$J(w, r, T) = u_T(w) = \frac{(w - \overline{Z})^{\alpha}}{\alpha}.$$
(36)

We assume that there exists a solution $J \in C^{2,2,1}(\mathbb{R}_+ \times \mathbb{R}_+ \times [0,T])$ to HJB equation and that J satisfies technical conditions so that

$$V(w,r) = J(w,r,0)$$

It follows from the first order condition for the optimality that the optimal strategy φ^* satisfies

$$\varphi_t^* = -\frac{1}{J_{ww}(w,r,t)w\sigma_{\Lambda}(t)^2} \left(J_w(w,r,t)(\mu_{\Lambda}(t)-r_t) + J_{wr}(w,r,t)\sigma_{\Lambda}(t)\sigma_r(r,t)\right)$$
(37)

At time 0, the solution is given by

$$\varphi_0^* = -\frac{1}{J_{ww}(w,r,0)w\sigma_\Lambda(0)^2} \left(J_w(w,r,0)(\mu_\Lambda(0) - r_0) + J_{wr}(w,r,0)\sigma_\Lambda(0)\sigma_r(r,0) \right)$$
(38)

We can use the martingale approach. By standard arguments, we can rewrite (10) into

$$\sup_{Z \in \mathcal{F}_T} E\left[\frac{(Z - \bar{Z})^{\alpha}}{\alpha}\right]$$
s.t. $E\left[\pi_T Z\right] \leqslant w_0.$
(39)

By the Saddle Point Theorem, the control φ^* solves (10) if and only if there is a scalar Lagrange multiplier $\lambda > 0$ such that φ^* solves the unconstrained problem

$$\sup_{Z\in\mathcal{F}_T,\lambda}\mathcal{L}(Z,\lambda),$$

where

$$\mathcal{L}(Z,\lambda) = E\left[\frac{(Z-\bar{Z})^{\alpha}}{\alpha} - \lambda\left(E[\pi_T Z] - w_0\right)\right].$$

The first order conditions for optimality are, state-by-state,

$$(Z^* - \bar{Z})^{\alpha - 1} = \lambda^* \pi_T.$$

Then we have

$$Z^* = \bar{Z} + (\lambda^* \pi_T)^{\frac{1}{\alpha - 1}}.$$

By the complementary slackness condition,

$$w_0 = E[\pi_T Z^*].$$

Therefore we get

$$(\lambda^*)^{\frac{1}{\alpha-1}} = \frac{w_0 - ZE[\pi_T]}{E[\pi_T^{\frac{\alpha}{\alpha-1}}]}.$$

Then the value V(w,r) of a state (w,r) is given by

$$V(w,r) = J(w,r,0) = \frac{1}{\alpha} \left(E\left[\pi_T^{\frac{\alpha}{\alpha-1}}\right] \right)^{1-\alpha} (w - \bar{Z}E[\pi_T])^{\alpha}.$$

In order to compute the optimal portfolio, we need to estimate V_w, V_{ww} , and V_{wr} . We consider r^x , the short rate process starting at x. Then the first and second derivatives of V are given by:

$$V_{w}(w,x) = \left(E\left[\pi_{T}^{\frac{\alpha}{\alpha-1}}\right]\right)^{1-\alpha} \left(w - \bar{Z}E\left[\pi_{T}\right]\right)^{\alpha-1},$$

$$V_{ww}(w,x) = (\alpha-1) \left(E\left[\pi_{T}^{\frac{\alpha}{\alpha-1}}\right]\right)^{1-\alpha} \left(w - \bar{Z}E\left[\pi_{T}\right]\right)^{\alpha-2},$$

$$V_{wr}(w,x) = (1-\alpha) \left(E\left[\pi_{T}^{\frac{\alpha}{\alpha-1}}\right]\right)^{-\alpha} E\left[\frac{\alpha}{\alpha-1}\pi_{T}^{\frac{1}{\alpha-1}}\frac{\partial\pi_{T}}{\partial x}\right] \left(w - \bar{Z}E\left[\pi_{T}\right]\right)^{\alpha-1}$$

$$+ (1-\alpha) \left(E\left[\pi_{T}^{\frac{\alpha}{\alpha-1}}\right]\right)^{1-\alpha} \left(w - \bar{Z}E\left[\pi_{T}\right]\right)^{\alpha-2} \left(\bar{Z}E\left[\frac{\partial\pi_{T}}{\partial x}\right]\right).$$

In the last equation, we used Fubini's Theorem and stochastic flow technique. $\partial \pi_T / \partial x$ is given by the following system.

$$\begin{cases} dr_t^x &= \mu_r(r_t^x, t)dt + \sigma_r(r_t^x, t)dB_t, \\ r_0^x &= x, \end{cases}$$
$$\begin{cases} d\pi_t &= -\pi_t(r_t dt + \theta_t dB_t), \\ \pi_0 &= 1. \end{cases}$$

$$\begin{cases} dY_t &= \frac{\partial \mu_r(r_t^x, t)}{\partial r_t^x} Y_t dt + \frac{\partial \sigma_r(r_t^x, t)}{\partial r_t^x} Y_t dB_t, \\ Y_0 &= 1, \end{cases}$$
$$\frac{\partial \pi_T}{\partial x} = \pi_T \left(-\int_0^T Y_t dt - \int_0^T \frac{\partial \theta(r_t^x, t)}{\partial r_t^x} Y_t dB_t - \int_0^T \theta(r_t^x, t) \frac{\partial \theta(r_t^x, t)}{\partial r_t^x} Y_t dt \right).$$
We define χ by
$$\chi = \frac{E\left[\pi_T \bar{Z}\right]}{w_0},$$

that is, χ is a ratio of the present value of \overline{Z} to the current wealth level w_0 . Then the optimal portfolio at time 0 is given by

$$\varphi_0^* = \frac{1}{1-\alpha} \frac{\Lambda_{0,\tau}}{\frac{\partial \Lambda_{0,\tau}}{\partial x}} \left[(1-\chi)\theta_0 \frac{1}{\sigma_r} + (1-\alpha) \left((1-\chi) \frac{E\left[\frac{\alpha}{\alpha-1} \pi_T^{\frac{1}{\alpha-1}} \frac{\partial \pi_T}{\partial x}\right]}{E\left[\pi_T^{\frac{\alpha}{\alpha-1}}\right]} + \chi \frac{E\left[\frac{\partial \pi_T}{\partial x}\right]}{E[\pi_T]} \right) \right]$$

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