REVISED PROOF OF SKOLEM'S THEOREM*

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In [N 91], the author intended to present an easy finitary proof of Skolem's Theorem but unfortunately it turned out to contain some serious errors. This corrected version is self-contained and readers' knowledge of [N 91] is not assumed. Skolem's Theorem is the following statement: In the classical predicate logic, let \( f \) be a \( k \)-ary function symbol not contained in a formula \( \forall x_1 \ldots \forall x_k \exists y A(x_1, \ldots, x_k, y) \supset B \). Then

\[
\forall x_1 \ldots \forall x_k \exists y A(x_1, \ldots, x_k, y) \supset B
\]

is valid if and only if

\[
\forall x_1 \ldots \forall x_k A(x_1, \ldots, x_k, f(x_1, \ldots, x_k)) \supset B
\]

is valid.

We use the logical symbols \( =, \land, \lor, \supset, \forall \) and \( \exists \). Different letters are used for free variables and for bound variables. Any set consisting of function symbols and predicate symbols is called a language. The language obtained by adding function symbols \( f, g, \ldots \) and predicate symbols \( P, Q, \ldots \) to any language \( \mathcal{L} \) is written \( \mathcal{L} \cup \{f, g, \ldots, P, Q, \ldots\} \). Terms and formulae are constructed according to the usual syntactic rules. Any formula of the form \( (A \supset B) \land (B \supset A) \) is abbreviated as \( A \equiv B \). For any formula \( A(a) \), the formula

\[
\exists x A(x) \land \forall x \forall y (A(x) \land A(y) \supset x = y)
\]

is abbreviated as \( \exists! x A(x) \). If two formal expressions \( A \) and \( B \) differ only in their bound variables, \( A \) and \( B \) are congruent [K 52, §33] or \( A \) is an alphabetical variant of \( B \) [T 75, §3].

A cedent is a sequence of zero or more formulae separated by commas. A sequent is an expression of the form

\[ \Gamma \rightarrow \Theta \]

where \( \Gamma \) and \( \Theta \) are any decents. Partition of cedent is defined as follows:

1. If \( \Gamma \) is the empty cedent then \([\Gamma; \Gamma]\) is the only partition of \( \Gamma \).
2. If \([\Gamma_1; \Gamma_2]\) is a partition of \( \Gamma \) then \([\Gamma_1; A; \Gamma_2]\) and \([\Gamma_1; \Gamma_2; A]\) are partitions of \( \Gamma, A \).

A partition of a sequent \( \Gamma \rightarrow \Theta \) is an ordered pair (of sequents)

\[ [\Gamma_1 \rightarrow \Theta_1; \Gamma_2 \rightarrow \Theta_2] \]

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For any formula or a cedent or a sequent $S$, let

$$\text{FV}(S) = \text{the set of free variables occurring in } S,$$

$$\text{BV}(S) = \text{the set of bound variables occurring in } S,$$

$$\text{Func}(S) = \text{the set of function symbols occurring in } S.$$

For any formula or a cedent or a sequent $S$, $\text{Pred}^+(S)$ (resp. $\text{Pred}^-(S)$) denotes the set of predicate symbols occurring positively (resp. negatively) in $S$ and $\text{Pred}(S)$ denotes the set of all predicate symbols occurring in $S$.

Let $\mathcal{L}$ be a language. A term $t$ is an $\mathcal{L}$-term if $\text{Func}(t) \subseteq \mathcal{L}$. If $\text{Func}(A) \cup \text{Pred}(A) \subseteq \mathcal{L}$ then $A$ is an $\mathcal{L}$-formula. The language $\text{Func}(A) \cup \text{Pred}(A)$ is called the language of $A$ and denoted as $\mathcal{L}(A)$. Similar notations and terminologies are used for cedents and sequents.

For any formal expression $S$ and for any (free or bound) variable $v$, $S[v := t]$ denotes the result of replacing all occurrences of variable $v$ in $S$ with $t$. When a formula is denoted as $A(a)$, the expression $A(a)[a := t]$ is abbreviated as $A(t)$.

The system $LK_e$ (LK with equality) is an extension of $LK$ obtained by adding the following schemata for initial sequents:

\[
\begin{align*}
\vdash t = t \\
\vdash s = t, A(s) \rightarrow A(t)
\end{align*}
\]

where $a$ is a free variable, $s$ and $t$ are terms and $A(a)$ is an atomic formula. This system is equivalent to $LK_e$ in [T 75, §7]. $LK_e$ is also equivalent to the system $LKG$ [N 66], which is an extension of $LK$ obtained with additional inference schemata

\[
\begin{align*}
\vdash t = t, \Gamma \rightarrow \Theta \\
\vdash \Gamma \rightarrow \Theta, A(s), A(t) \rightarrow A \\
\vdash s = t, \Gamma, \Delta \rightarrow \Theta, A
\end{align*}
\]

where $a$ is a free variable, $s$ and $t$ are terms and $A(a)$ is an atomic formula.

A derivation (or a proof figure) is defined as usual. A sequent $S$ is $LK$-provable and denoted as $\vdash S$ is there exists an $LK$-derivation $\mathcal{H}$ of a sequent $S$ and if all sequents in $\mathcal{H}$ are $\mathcal{L}$-sequents, then $S$ is $LK$-provable in $\mathcal{L}$ and denoted as $\mathcal{L}\vdash S$. Corresponding terminologies and notations are used also for $LK_e$. If the equality symbol $=$ is not contained in $\mathcal{L}$, then $LK_e$-provability in $\mathcal{L}$ is clearly equivalent to $LK$-provability in $\mathcal{L}$.

If $A(a_1, \ldots, a_n)$ is a formula and $x_1, \ldots, x_n$ are distinct bound variables not occurring in this formula, then the formal expression

$$\lambda x_1, \ldots, x_n. A(x_1, \ldots, x_n)$$

is an $n$-ary abstract [T 75, §20]. If $V$ is $\lambda x_1, \ldots, x_n. A(x_1, \ldots, x_n)$ and if $t_1, \ldots, t_n$ are terms then $V(t_1, \ldots, t_n)$ denotes the formula $A(t_1, \ldots, t_n)$. For any $n$-ary abstract $V$, define
\[ \text{Func}(V) = \text{Func}(V(a_1, \ldots, a_n)), \]
\[ \text{Pred}(V) = \text{Pred}(V(a_1, \ldots, a_n)) \]

where \( a_1, \ldots, a_n \) are free variables not occurring in \( V \). An abstract \( V \) is an \( \mathcal{L} \)-abstract if \( \text{Func}(V) \cup \text{Pred}(V) \subseteq \mathcal{L} \).

For any \( k \)-ary predicate symbol \( P \) and any \( k \)-ary abstract \( V \), the result of substituting \( V \) for \( P \) in a formula or a cedent or a sequent \( S \) is denoted as \( S[P:=V] \).

The key idea of our proof is replacing a function symbol by a predicate symbol \([K 52, S74]\). Let \( \mathcal{L} \) be a language, let \( f \) be a \( k \)-ary function symbol not contained in \( \mathcal{L} \) and let \( F \) be a \((k+1)\)-ary predicate symbol not contained in \( \mathcal{L} \). The \((f;F)\)-transformation applies to \( \mathcal{L} \cup \{f\}\)-terms and to \( \mathcal{L} \cup \{f\}\)-formulae. Any \( \mathcal{L} \cup \{f\}\)-term is transformed into a unary \( \mathcal{L} \cup \{=, F\}\)-abstract and any \( \mathcal{L} \cup \{f\}\)-formula is transformed into an \( \mathcal{L} \cup \{=, F\}\)-formula.

Definition is by the following induction.

1. \( a^* \) is \( \lambda u.(u=a) \).
2. \( f(t_1, \ldots, t_k)^* \) is
   \[ \lambda u.\exists x_1 \ldots \exists x_k(f^*(t_1)(x_1) \land \ldots \land F^*(x_1, \ldots, x_k, u)) \]
   where \( u, x_1, \ldots, x_k \in BV(t_1^*) \cup \ldots \cup BV(t_k^*) \).
3. \( g(t_1, \ldots, t_n)^* \) is
   \[ \lambda u.\exists x_1 \ldots \exists x_n(g^*(t_1)(x_1) \land \ldots \land g^*(x_1, \ldots, x_n)) \]
   where \( u, x_1, \ldots, x_n \in BV(t_1^*) \cup \ldots \cup BV(t_n^*) \).
4. \( P(t_1, \ldots, t_n)^* \) is
   \[ \exists x_1 \ldots \exists x_n(P^*(t_1)(x_1) \land \ldots \land P^*(x_1, \ldots, x_n)) \]
   where \( x_1, \ldots, x_n \in BV(t_1^*) \cup \ldots \cup BV(t_n^*) \).
5. \( (A \land B)^* \) is \( A^* \land B^* \), \( (A \lor B)^* \) is \( A^* \lor B^* \), \( (A \supset B)^* \) is \( A^* \supset B^* \) and \( (\neg A)^* \) is \( \neg A^* \).
6. \( (\forall x.A(x))^* \) is \( \forall y.A^*(y) \) and \( (\exists x.A(x))^* \) is \( \exists y.A^*(y) \) where \( y \) is any bound variable such that \( y \in BV(A(a)^*) \).

Example. In case of \( k=1 \), \((f(a)=b)^*\) is (any alphabetical variant of)
\[ \exists x_1 \exists x_2 \exists y z(z=a \land F(z, x)) \land y=b \land x=y. \]

For any \((k+1)\)-ary predicate symbol \( F \), the existence condition [Mo 82] \( \text{Ex}(F) \) is the formula
\[ \forall x_1 \ldots \forall x_k \exists y F(x_1, \ldots, x_k, y) \]
and the uniqueness condition [Mo 82] \( \text{Un}(F) \) is the formula
\[ \forall x_1 \ldots \forall x_k \forall y \forall z(F(x_1, \ldots, z_k, y) \land F(x_1, \ldots, x_k, z) \supset y=z). \]

Lemma 1. An \( \mathcal{L} \)-sequent is \( \mathcal{L} \)-provable if and only if it is \( \mathcal{L} \)-provable in \( \mathcal{L} \). An \( \mathcal{L} \)-sequent is \( \mathcal{L}K \)-provable if and only if it is \( \mathcal{L}K \)-provable in \( \mathcal{L} \cup \{=\} \).

Proof. The first part of Lemma is a direct consequence of Gentzen's cut-elimination theorem [G 35]. The latter part follows from cut-elimination theorem of \( \mathcal{L}K \) \([T 75, \S 7]\) or cut-elimination theorem of \( \mathcal{L}KG \) \([N 66]\).

Lemma 2. Let \( P \) be a \( k \)-ary predicate symbol, \( V \) be a \( k \)-ary \( \mathcal{L} \)-abstract and \( S \) be any \( \mathcal{L} \)-sequent. If \( \mathcal{L} \vdash S \) then \( \mathcal{L} \vdash S[P:=V] \). If \( \mathcal{L} \vdash \not S \) then \( \mathcal{L} \vdash \not S[P:=V] \).
Proof. Case \(LK\): By cut-elimination theorem, there exists an cut-free \(LK\)-derivation \(\mathcal{H}\) of \(S\). Applying redesignation of free variables [G 35, III 3.10], \(\mathcal{H}\) can be converted into a cut-free \(LK\)-derivation \(\mathcal{H}'\) of \(S\) such that no eigenvariable of \(\mathcal{H}'\) occurs in \(V\). Substitute \(V\) for \(P\) in every sequent of \(\mathcal{H}'\). The result of substitution is easily verified to be an \(LK\)-derivation of \(S[P := V]\). Similarly for Case \(LK_e\). □

Now let a \(k\)-ary function symbol \(f\) and a \((k + 1)\)-ary predicate symbol \(F\) be fixed. We state some Lemmas concerning the \((f; F)\)-transformation.

**Lemma 3.** For any \(\mathcal{L}\)-term \(t\),

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F) \rightarrow \exists ! xt^*(x). \] □

**Proof.** By induction on the structure of \(t\). □

**Lemma 4.** For any free variable \(a\), any \(\mathcal{L}\)-term \(t\) and any \(\mathcal{L}\)-formula \(A\),

\[ \mathcal{L} \cup \{ = \} \vdash_e t^*(a) \equiv a = t, \]

\[ \mathcal{L} \cup \{ = \} \vdash_e A^* \equiv A. \] □

**Lemma 5.** For any free variable \(a\) and any \(\mathcal{L}\)-terms \(s\) and \(t\),

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), t^*(a) \rightarrow s^*(b) \equiv s[a := t]^*(b). \] □

**Proof.** By induction on the structure of \(s\). □

**Lemma 6.** If \(= \in \mathcal{L}, F \in \mathcal{L}, a\) is a free variable, \(t\) is a \(\mathcal{L}\)-term and \(A\) is a \(\mathcal{L}\)-formula, then

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), t^*(a) \rightarrow A^* \equiv A[a := t]^*. \] □

**Proof.** By induction on the structure of \(A\). □

**Lemma 7.** If \(= \in \mathcal{L}, F \in \mathcal{L}, a\) is a free variable, \(t\) is an \(\mathcal{L}\)-term and \(A(a)\) is an \(\mathcal{L}\)-formula, then

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), \forall x A(x)^* \rightarrow A(t)^*, \]

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), A(t)^* \rightarrow \exists x A(x)^*. \] □

**Proof.** Let \(a \in \text{FV}(\forall x A(x)) \cup \text{FV}(t)\). By Lemma 6,

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), t^*(a) \rightarrow A(a)^* \equiv A(t)^*. \]

Hence

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), t^*(a), A(a)^* \rightarrow A(t)^*, \]

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), t^*(a), \forall x A(x)^* \rightarrow A(t)^*, \]

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), \exists x t^*(x), \forall x A(x)^* \rightarrow A(t)^*. \]

By Lemma 3,

\[ \mathcal{L} \cup \{ =, F \} \vdash_e \text{Ex}(F), \text{Un}(F), \forall x A(x)^* \rightarrow A(t)^*. \]
The latter half is proved similarly. □

**Lemma 8.** For any \( \mathcal{L} \cup \{ f \} \)-sequent \( \Gamma \rightarrow \Theta \), if \( \mathcal{L} \cup \{ f \} \vdash \Gamma \rightarrow \Theta \) then

\[
\mathcal{L} \cup \{ \varepsilon, F \} \vdash \varepsilon \text{Ex}(F), \text{Un}(F), \Delta^* \rightarrow \Theta^*.
\]

Proof. By cut-elimination theorem, there exists a cut-free \( \mathcal{L} \cup \{ f \} \)-LK-derivation \( \mathcal{H} \) of \( \Gamma \rightarrow \Theta \). It suffices to prove by induction that

\[
\mathcal{L} \cup \{ \varepsilon, F \} \vdash \varepsilon \text{Ex}(F), \text{Un}(F), \phi^* \rightarrow \Psi^*
\]

for any sequent \( \phi \rightarrow \Psi \) occurring in \( \mathcal{H} \). The statement is evident for initial sequents. We divide cases according to the inferences whose lower sequent is \( \phi \rightarrow \Psi \).

Case \((\rightarrow \exists)\). Let the inference be

\[
\frac{\Delta \rightarrow \Lambda, \, A(t)}{\Delta \rightarrow \Lambda, \, \exists x A(x)}
\]

Since \( \mathcal{L} \cup \{ f \} \vdash \Delta \rightarrow \Lambda, A(t) \),

\[
\mathcal{L} \cup \{ \varepsilon, F \} \vdash \varepsilon \text{Ex}(F), \text{Un}(F), \Delta^* \rightarrow \Lambda^*, A(t)^*
\]

by the inductive hypothesis. By Lemma 7,

\[
\mathcal{L} \cup \{ \varepsilon, F \} \vdash \varepsilon \text{Ex}(F), \text{Un}(F), A(t)^* \rightarrow \exists x A(x)^*,
\]

hence

\[
\mathcal{L} \cup \{ \varepsilon, F \} \vdash \varepsilon \text{Ex}(F), \text{Un}(F), \Delta^* \rightarrow \Lambda^*, \exists x A(x)^*.
\]

Case \((\forall \rightarrow)\). Similarly by Lemma 7.

All the other cases are straightforward. □

**Theorem 9** (Lyndon). For any partition \( [\Gamma_1 \rightarrow \Theta_1; \Gamma_2 \rightarrow \Theta_2] \) of an LK-provable sequent \( \Gamma \rightarrow \Theta \), if

\[
\text{Pred}^-(\Gamma_1 \rightarrow \Theta_1) \cap \text{Pred}^+(\Gamma_2 \rightarrow \Theta_2) \models \phi
\]

or

\[
\text{Pred}^+(\Gamma_1 \rightarrow \Theta_1) \cap \text{Pred}^-(\Gamma_2 \rightarrow \Theta_2) \models \phi
\]

then there exists a formula \( C \) satisfying the following properties:

1. \( \vdash \Gamma_1 \rightarrow \Theta_1, \ C \) and \( \vdash \neg C, \Gamma_2 \rightarrow \Theta_2 \).
2. \( \text{FV}(C) \subseteq \text{FV}(\Gamma_1 \rightarrow \Theta_1) \cap \text{FV}(\Gamma_2 \rightarrow \Theta_2) \).
3. \( \text{Pred}^+(C) \subseteq \text{Pred}^-(\Gamma_1 \rightarrow \Theta_1) \cap \text{Pred}^+(\Gamma_2 \rightarrow \Theta_2) \).
4. \( \text{Pred}^-(C) \subseteq \text{Pred}^+(\Gamma_1 \rightarrow \Theta_1) \cap \text{Pred}^-(\Gamma_2 \rightarrow \Theta_2) \). □

Any formula \( C \) satisfying (1)–(4) is called a Lyndon interpolant of the partition \( [\Gamma_1 \rightarrow \Theta_1; \Gamma_2 \rightarrow \Theta_2] \).

Lyndon’s proof is not finitary but a finitary proof can be carried out with Maehara’s method [Ma 73, §8.3], [T 75, §6].
Theorem 10. If a sequent \( S \) contains no equality symbol and if \( \vdash \varepsilon S \) then \( \vdash S \). \( \square \)

Proof. This is an easy application of cut-elimination theorem of \( LK \) with equality [Ma 73, §6.6], [T 75, §7], [N 66]. \( \square \)

Theorem 11 (Skolem). Let \( f \) be a \( k \)-ary function symbol not occurring in

\[
\forall x_1 \ldots \forall x_k \exists y A(x_1, \ldots, x_k, y), A, A.
\]

If

\[
\vdash \forall x_1 \ldots \forall x_k A(x_1, \ldots, x_k, f(x_1, \ldots, x_k)), A \rightarrow A
\]

then

\[
\vdash \forall x_1 \ldots \forall x_k \exists y A(x_1, \ldots, x_k, y), A \rightarrow A.
\] \( \square \)

Proof. For the sake of simplicity in notation, we assume \( k = 1 \). Let \( \mathcal{L} = \mathcal{L}(\forall x \exists y A(x, y), A, A) \) and let \( F \) be a 2-ary predicate symbol not contained in \( \mathcal{L} \). Assume

\[
\vdash \forall x A(x, f(x)), A \rightarrow A.
\]

Whence follows

\[
\mathcal{L} \cup \{ f \} \vdash \forall x A(x, f(x)), A \rightarrow A \quad (1)
\]

by Lemma 1. By Lemma 8,

\[
\mathcal{L} \cup \{ =, F \} \vdash \varepsilon \exists x(F, \exists x A(x, f(x))\}, A^* \rightarrow A^*. \quad (2)
\]

Because \( f(a)^*(b) \) is \( \exists x(x = a \land F(x, b)) \),

\[
\{ =, F \} \vdash \varepsilon f(a)^*(b) \equiv F(a, b).
\] \( (3) \)

\[
\mathcal{L} \cup \{ = \} \vdash \varepsilon \exists x A(x, f(x)) \equiv A(a, b) \quad (4)
\]

and

\[
\mathcal{L} \cup \{ =, F \} \vdash \varepsilon \exists x(F, Un(F), f(a)^*(b) \rightarrow A(a, b)^* \equiv A(a, f(a)) \quad (5)
\]

follows immediately from Lemmas 4 and 6 respectively. From (3), (4) and (5) we obtain successively

\[
\mathcal{L} \cup \{ =, F \} \vdash \varepsilon \exists x(F, Un(F), F(a, b) \rightarrow A(a, f(a))^* \equiv A(a, b),
\]

\[
\mathcal{L} \cup \{ =, F \} \vdash \varepsilon \exists x(F, Un(F), F(a, b) \rightarrow A(a, f(a))^* \equiv F(a, b) \land A(a, b),
\]

\[
\mathcal{L} \cup \{ =, F \} \vdash \varepsilon \exists x(F, Un(F), F(a, b) \rightarrow A(a, f(a))^* \equiv y(F(a, y) \land A(a, y)),
\]

\[
\mathcal{L} \cup \{ =, F \} \vdash \varepsilon \exists x(F, Un(F), Ex(F) \rightarrow A(a, f(a))^* \equiv y(F(a, y) \land A(a, y)),
\]

\[
\mathcal{L} \cup \{ =, F \} \vdash \varepsilon \exists x(F, Un(F) \rightarrow \forall x A(x, f(x)) \equiv \forall x \exists y(F(x, y) \land A(x, y)). \quad (6)
\]

By Lemma 4,

\[
\mathcal{L} \cup \{ = \} \vdash B^* \equiv B
\]
for any member \( B \) of \( \Delta, \Lambda \). Therefore

\[ \mathcal{L} \cup \{ =, F \} \vdash \text{Ex}(F), \text{Un}(F), \forall xA(x, f(x))^*, \Delta \rightarrow \Lambda \]  

(7)

follows from (2). From (6) and (7) follows

\[ \mathcal{L} \cup \{ =, F \} \vdash \forall x\exists y(F(x, y) \wedge A(x, y)), \text{Ex}(F), \text{Un}(F), \Delta \rightarrow \Lambda. \]  

(8)

Consider the partition

\[ \text{Un}(F) \quad \vdash \forall x\exists y(F(x, y) \wedge A(x, y)), \text{Ex}(F), \Delta \rightarrow \Lambda \]

of this sequent. Then

\[ \text{Pred}^{-}(\text{Un}(F) \rightarrow \{ = \}) = \{ = \}, \quad \text{Pred}^{+}(\text{Un}(F) \rightarrow \{ = \}) = \{ F \} \]

and

\[ F \notin \text{Pred}^{+}(\forall x\exists y(F(x, y) \wedge A(x, y)), \text{Ex}(F), \Delta \rightarrow \Lambda), \]

Let \( C \) be a Lyndon interpolant of this partition. Then \( C \) satisfies \( \text{Pred}(C) \subset \{ = \}, \)

\[ \mathcal{L} \cup \{ =, F \} \vdash \text{Un}(F) \rightarrow C \]  

(9)

and

\[ \mathcal{L} \cup \{ =, F \} \vdash \forall x\exists y(F(x, y) \wedge A(x, y)), \text{Ex}(F), \Delta \rightarrow \Lambda. \]  

(10)

Substitute \( \lambda uv.(u = v) \) for \( F \) in (9) and apply Lemma 2. Then

\[ \mathcal{L} \cup \{ = \} \vdash \forall x\forall y\forall z(x = y \wedge x = z \supset y = z) \rightarrow C, \]

hence

\[ \mathcal{L} \cup \{ = \} \vdash C. \]  

(11)

From (10), (11) and

\[ \mathcal{L} \cup \{ F \} \vdash \forall x\exists y(F(x, y) \wedge A(x, y)) \rightarrow \text{Ex}(F), \]

it follows

\[ \mathcal{L} \cup \{ =, F \} \vdash \forall x\exists y(F(x, y) \wedge A(x, y)), \Delta \rightarrow \Lambda. \]

By substitution of \( \lambda uv. A(u, v) \) for \( F \), we obtain

\[ \mathcal{L} \cup \{ = \} \vdash \forall x\exists y(A(x, y) \wedge A(x, y)), \Delta \rightarrow \Lambda. \]

Hence

\[ \mathcal{L} \cup \{ = \} \vdash \forall x\exists y A(x, y), \Delta \rightarrow \Lambda. \]  

(12)

By Lemma 10, we conclude

\[ \mathcal{L} \vdash \forall x\exists y A(x, y), \Delta \rightarrow \Lambda. \]

Remark. Another proof is sketched in [Mo 82], which can be stated as follows. From (8) we have
\[ \mathcal{L} \cup \{=, F\} \vdash \forall \forall \forall (F(x,y) \supset A(x,y)), \text{Ex}(F), \text{Un}(F), \mathcal{A} \rightarrow \Lambda. \] (13)

Since \( F \not\in \text{Pred}^{-}(\forall \forall \forall (F(x,y) \supset A(x,y)), \mathcal{A} \rightarrow \Lambda) \),
\[ \mathcal{L} \cap \{=, F\} \vdash \forall \forall \forall (F(x,y) \supset A(x,y)), \text{Ex}(F), \mathcal{A} \rightarrow \Lambda \] (14)

by [Mo 82, Theorem 1]. We obtain (12) by substituting \( \lambda uv. A(u,v) \) for \( F \).

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References


