GEOMETRY OF THE SPACE OF CLOSED CURVES IN THE COMPLEX HYPERBOLA

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Abstract. We study the geometry of the space of closed curves in the complex hyperbola from a viewpoint of Riemannian or symplectic geometry. We show that it has natural flat Kähler structure. Moreover, we define natural Hamiltonian actions on the universal cover of the space.

1. Introduction

A one-parameter family of curves in some appropriate space is called a motion of curves, which has a long history in differential geometry (cf. [2]). Under a motion of curves, we can regard geometric quantities, such as the curvature or the torsion, as varying according to the time evolution equation. For example, it is known that the curvature of curves in the Euclidean 2-space evolve according to the modified Korteweg–de Vries (mKdV) equation under special motions, and Pinkall [3] showed that the space of closed centroaffine curves in the centroaffine plane possesses a natural symplectic structure and the centroaffine curvature evolves according to the Korteweg–de Vries (KdV) equation when the flow is generated by a Hamiltonian function given by the total centroaffine curvature. In a previous paper [1], we showed that special motions of curves in the complex hyperbola are linked with the Burgers hierarchy, which can be formulated as a Hamiltonian system and also leads to the difference Burgers hierarchy via the discretization.

In this paper we study more about geometry of the space of closed curves in the complex hyperbola from a viewpoint of Riemannian or symplectic geometry. We show that it has natural flat Kähler structure. Moreover, we define natural Hamiltonian actions on the universal cover of the space.

2. A Riemannian viewpoint

The complex hyperbola *C* is the subset of the complex 2-plane \mathbb{C}^2 given by

$$C = \{(z, w) \in \mathbb{C}^2 \mid zw = 1\}.$$

In this paper curves in *C* are assumed to be sufficiently smooth and immersed. Let \mathcal{M} be the space of all closed curves in *C*, i.e. the set of all curves from S^1 to *C*, where S^1 is the unit circle. The tangent space of \mathcal{M} at $\gamma = (z, w) \in \mathcal{M}$ can be identified with the set

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 $\{\alpha \gamma' \mid \alpha : S^1 \to \mathbb{C}\}$ (see [1]). Note that \mathcal{M} has an almost complex structure defined by

$$\alpha \gamma' \mapsto \sqrt{-1} \alpha \gamma',$$

where $\alpha : S^1 \to \mathbb{C}$. Then we can define a Hermitian metric *h* on \mathcal{M} by

$$h(\alpha_1\gamma', \alpha_2\gamma') = \operatorname{Re} \int_{S^1} \alpha_1 \overline{\alpha_2} |\tau|^2 ds$$

where $\alpha_1, \alpha_2 : S^1 \to \mathbb{C}$ and $\tau = z'/z$.

Let $\gamma = \gamma(s, t)$ be a one-parameter family of a closed curve in *C*, i.e. γ is a map from the product of S^1 and an interval *I* to *C*. Then there exists a function $\alpha : S^1 \times I \to \mathbb{C}$ such that $\gamma_t = \alpha \gamma_s$ and we have

$$\gamma_{st} = \tau^2 \alpha \gamma + (\alpha_s + \sqrt{-1}\kappa\alpha)\gamma_s, \qquad (2.1)$$

$$\tau_t = \tau \alpha_s + \tau_s \alpha, \tag{2.2}$$

where $\kappa = -\sqrt{-1}\tau_s/\tau$, called the curvature of γ (see [1]). Let $\beta\gamma_s$ be a vector field along $\gamma(\cdot, t)$. From (2.1) and (2.2), we have

$$(\beta\gamma_s)_t = \tau^2 \alpha \beta \gamma + \{\beta_t + (\log \tau)_t \beta\} \gamma_s.$$
(2.3)

Taking the tangential part of (2.3), we can define a connection ∇ on \mathcal{M} by

$$\nabla_{\alpha\gamma_s}(\beta\gamma_s) = \{\beta_t + (\log\tau)_t\beta\}\gamma_s.$$

PROPOSITION 2.1. *The connection* ∇ *is the Levi-Civita connection for h.*

Proof. Let $\gamma = \gamma(s, t)$ be a one-parameter family of a closed curve in *C*. It is easy to see that ∇ is a metric connection, i.e.

$$\frac{\partial}{\partial t}h(\beta_1\gamma_s,\,\beta_2\gamma_s)=h(\nabla_{\alpha\gamma_s}(\beta_1\gamma_s),\,\beta_2\gamma_s)+h(\beta_1\gamma_s,\,\nabla_{\alpha\gamma_s}(\beta_2\gamma_s)),$$

where $\beta_1 \gamma_s$ and $\beta_2 \gamma_s$ are vector fields along $\gamma(\cdot, t)$.

Let $\gamma = \gamma(s, t_1, t_2)$ be a two-parameter family of a closed curve in *C*. Putting $\gamma_{t_1} = \alpha \gamma_s$, $\gamma_{t_2} = \beta \gamma_s$, we have

$$\nabla_{\alpha\gamma_s}(\beta\gamma_s) = \nabla_{\beta\gamma_s}(\alpha\gamma_s), \quad [\alpha\gamma_s, \beta\gamma_s] = 0,$$

which implies that ∇ is a torsion-free connection.

THEOREM 2.2. *The connection* ∇ *is flat.*

Proof. Let $\gamma = \gamma(s, t_1, t_2)$ be a two-parameter family of a closed curve in *C* and $\beta \gamma_s$ a vector field along $\gamma(\cdot, t_1, t_2)$. Putting $\gamma_{t_1} = \alpha_1 \gamma_s$, $\gamma_{t_2} = \alpha_2 \gamma_s$, we have

$$\nabla_{\alpha_{1}\gamma_{s}}\nabla_{\alpha_{2}\gamma_{s}}(\beta\gamma_{s}) = \nabla_{\alpha_{1}\gamma_{s}}[\{\beta_{t_{2}} + (\log\tau)_{t_{2}}\beta\}\gamma_{s}] \\ = \{\beta_{t_{2}t_{1}} + (\log\tau)_{t_{2}t_{1}} + (\log\tau)_{t_{2}}\beta_{t_{1}} + (\log\tau)_{t_{1}}\beta_{t_{2}} + (\log\tau)_{t_{1}}(\log\tau)_{t_{2}}\beta\}\gamma_{s}.$$

Hence, we have

$$\nabla_{\alpha_1\gamma_s}\nabla_{\alpha_2\gamma_s}(\beta\gamma_s)=\nabla_{\alpha_2\gamma_s}\nabla_{\alpha_1\gamma_s}(\beta\gamma_s),$$

which implies that ∇ is flat, i.e.

$$(\nabla_{\alpha_1\gamma_s}\nabla_{\alpha_2\gamma_s}-\nabla_{\alpha_2\gamma_s}\nabla_{\alpha_1\gamma_s}-\nabla_{[\alpha_1\gamma_s,\alpha_2\gamma_s]})(\beta\gamma_s)=0.$$

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Example 2.3. Let γ be a geodesic in \mathcal{M} , i.e. $\nabla_{\gamma_t} \gamma_t = 0$. A direct computation shows that

$$\gamma(s, t) = (e^{f(s)t + g(s)}, e^{-f(s)t - g(s)}),$$

where f(s) and g(s) are just functions of $s \in S^1$.

3. A symplectic viewpoint

We denote the fundamental two-form of h by ω , i.e.

$$\omega(\alpha_1\gamma',\alpha_2\gamma') = \operatorname{Im} \int_{S^1} \alpha_1 \overline{\alpha_2} |\tau|^2 \, ds,$$

where $\alpha_1, \alpha_2 : S^1 \to \mathbb{C}$.

PROPOSITION 3.1. The two-form ω is closed.

Proof. Let $\gamma = \gamma(s, t_1, t_2, t_3)$ be a three-parameter family of a closed curve in C. Note that

$$\tau_{t_1} = \tau(\alpha_1)_s + \sqrt{-1\kappa\tau\alpha_1}.$$

Putting $\gamma_{t_i} = \alpha_i \gamma_s$ (*i* = 1, 2, 3), we have

$$\frac{\partial}{\partial t_1}\omega(\alpha_2\gamma_s,\alpha_3\gamma_s) = \int_{S^1} \operatorname{Im}\{(\alpha_2)_{t_1}\overline{\alpha_3} + \alpha_2(\overline{\alpha_3})_{t_1} + (\alpha_1)_s\alpha_2\overline{\alpha_3} + \sqrt{-1}\kappa\alpha_1\alpha_2\overline{\alpha_3} + (\overline{\alpha_1})_s\alpha_2\overline{\alpha_3} - \sqrt{-1}\overline{\kappa\alpha_1}\alpha_2\overline{\alpha_3}\}|\tau|^2 \, ds.$$

If we put

$$\beta_{ij} = (\alpha_i)_{t_j} + \alpha_i (\alpha_j)_s$$
 (*i*, *j* = 1, 2, 3),

the integrand is equal to

$$\frac{|\tau|^2}{2\sqrt{-1}}(\beta_{21}\overline{\alpha_3} + \overline{\beta_{31}}\alpha_2 + \sqrt{-1}\kappa\alpha_1\alpha_2\overline{\alpha_3} - \sqrt{-1}\overline{\kappa\alpha_1}\alpha_2\overline{\alpha_3} - \overline{\beta_{21}}\alpha_3 - \beta_{31}\overline{\alpha_2} + \sqrt{-1}\overline{\kappa\alpha_1\alpha_2}\alpha_3 - \sqrt{-1}\kappa\alpha_1\overline{\alpha_2}\alpha_3).$$

Note that $\beta_{ij} = \beta_{ji}$ since we have

$$\gamma_{t_i t_j} = \tau^2 \alpha_i \alpha_j \gamma + \{(\alpha_i)_{t_j} + \alpha_i (\alpha_j)_s + \sqrt{-1} \kappa \alpha_i \alpha_j\} \gamma_s.$$

Hence, we have

$$\frac{\partial}{\partial t_1}\omega(\alpha_2,\alpha_3) + \frac{\partial}{\partial t_2}\omega(\alpha_3,\alpha_1) + \frac{\partial}{\partial t_3}\omega(\alpha_1,\alpha_2) = 0$$

which implies that ω is closed.

In the previous paper [1], we introduced Hamiltonian functions H_n ($n \in \mathbb{N}$) on \mathcal{M} defined by

$$H_n(\gamma) = \frac{(\sqrt{-1})^{n-1}}{2} \int_{S^1} \tau^{(n-1)} \overline{\tau} \, ds \quad (\gamma \in \mathcal{M})$$

and showed that the Hamiltonian vector field X_n for H_n with respect to the symplectic form ω is given by

$$(X_n)_{\gamma} = (-\sqrt{-1})^n \frac{D_s^n \tau}{\tau} \gamma'.$$
(3.1)

PROPOSITION 3.2. Functions $\{H_n\}_{n \in \mathbb{N}}$ are involutive.

Proof. From (3.1) we have

$$\omega(X_n, X_m) = \operatorname{Im} \int_{S^1} (-\sqrt{-1})^{n-m} (D_s^n \tau) (D_s^m \overline{\tau}) \, ds.$$

We may assume that $n \ge m$. If n - m = 2k - 2 ($k \in \mathbb{N}$), integrating by parts, we have

$$\omega(X_n, X_m) = \operatorname{Im} \int_{S^1} (D_s^{m+k-1}\tau) (D_s^{m+k-1}\overline{\tau}) \, ds$$
$$= 0.$$

If n - m = 2k - 1, we have

$$\omega(X_n, X_m) = \operatorname{Im} \int_{S^1} (-\sqrt{-1}) (D_s^{m+k-1}\tau) (D_s^{m+k}\overline{\tau}) \, ds$$

= $\frac{1}{2\sqrt{-1}} \int_{S^1} \{-\sqrt{-1} (D_s^{m+k-1}\tau) (D_s^{m+k}\overline{\tau}) - \sqrt{-1} (D_s^{m+k-1}\overline{\tau}) (D_s^{m+k}\tau)\} \, ds$
= 0.

Let \widetilde{M}_0 be the set of all closed curves in *C* with argument such that the rotation number is zero, i.e., the universal cover of the connected component of \mathcal{M} homotopic to constant maps. Then we can define the following three kinds of symplectic actions of S^1 or \mathbb{R} on \widetilde{M}_0 :

- (1) $\lambda \cdot (z, w) = (\lambda z, \lambda^{-1}w)$ $(\lambda \in S^1, (z, w) \in \widetilde{M}_0);$ (2) $\lambda \cdot (z, w) = (e^{\lambda}z, e^{-\lambda}w)$ $(\lambda \in \mathbb{R}, (z, w) \in \widetilde{M}_0);$
- (3) $\lambda \cdot (z, w) = (z^{\lambda}, w^{\lambda}) \quad (\lambda \in S^1, (z, w) \in \widetilde{M}_0).$

THEOREM 3.3. The actions (1)–(3) are Hamiltonian, whose moment maps $\mu_1 \sim \mu_3$ are given by

$$\mu_1(\gamma)(\sqrt{-1}\theta) = \theta \int_{S^1} (\operatorname{Re}\log z + C) \, ds,$$
$$\mu_2(\gamma)(\theta) = -\theta \int_{S^1} (\operatorname{Im}\log z + C) \, ds,$$
$$\mu_3(\gamma)(\sqrt{-1}\theta) = \frac{\theta}{2} \int_{S^1} (|\log z|^2 + C) \, ds,$$

respectively, where $C \in \mathbb{R}$ *.*

Proof. In the case of (1), the fundamental vector field <u>A</u> for $\sqrt{-1\theta} \in \text{Lie } S^1$ is given by

$$\underline{A}_{\gamma} = \sqrt{-1} \, \frac{\theta}{\tau} \, \gamma'.$$

Let $\gamma = \gamma(s, t)$ be a one-parameter family of a closed curve in C. Putting $\gamma_t = \alpha \gamma_s$, we have

$$\begin{split} \omega(\underline{A}_{\gamma}, \alpha \gamma') &= \operatorname{Im} \int_{S^1} \sqrt{-1} \theta \overline{\alpha \tau} \, ds \\ &= \frac{\theta}{2} \int_{S^1} \left(\frac{\overline{\alpha z_s}}{\overline{z}} + \frac{\alpha z_s}{z} \right) ds \\ &= \frac{\theta}{2} \int_{S^1} \{ (\log \overline{z})_t + (\log z)_t \} \, ds \\ &= \theta \int_{S^1} (\operatorname{Re} \log z)_t \, ds. \end{split}$$

Hence, the action is Hamiltonian and the moment map is given by μ_1 .

In the case of (2), the fundamental vector filed <u>A</u> for $\theta \in \text{Lie } \mathbb{R}$ is given by

$$\underline{A}_{\gamma} = \frac{\theta}{\tau} \gamma'.$$

A similar computation to that in the case of (1) shows that the action is Hamiltonian and the moment map is given by μ_2 .

In the case of (3), the fundamental vector filed <u>A</u> for $\sqrt{-1\theta} \in \text{Lie } S^1$ is given by

$$\underline{A}_{\gamma} = \sqrt{-1} \, \frac{\theta \, \log z}{\tau} \, \gamma'$$

We can also show that the action is Hamiltonian and the moment map is given by μ_3 .

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