BONNET SURFACES WITH NON-FLAT NORMAL BUNDLE IN THE HYPERBOLIC FOUR-SPACE

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Abstract

We study surfaces in the hyperbolic four-space admitting isometric deformations preserving the length of the mean curvature vector and especially focus on the case that surfaces are non-minimal and have non-flat normal bundle.

1. Introduction

There has been a long history of study of surfaces in 3-dimensional space forms admitting isometric deformations preserving the mean curvature (see [1, 5, 6, 7] and references therein), which can be traced back to the following result due to Bonnet [2].

Proposition 1.1. If a surface in a 3-dimensional space form has constant mean curvature and is not totally umbilic, then it admits isometric deformations preserving the mean curvature.

In the previous paper [4] the author studied surfaces in 4-dimensional space forms admitting isometric deformations preserving the length of

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the mean curvature vector, called Bonnet surfaces after Bonnet's work, and obtained a generalization of Chen-Yau’s reduction theorem for surfaces with parallel mean curvature vector [3].

In this paper we study Bonnet surfaces in the hyperbolic 4-space and obtain an example of non-minimal surfaces with non-flat normal bundle.

2. Preliminaries

We denote the hyperbolic 4-space of curvature $c < 0$ by $H^4(c)$, which is described as follows:

$$H^4(c) = \{ x \in \mathbb{R}^5 \mid \langle x, x \rangle = \frac{1}{c} \},$$

where $\langle , \rangle$ is the Lorentzian inner product on $\mathbb{R}^5$ with signature $(1, 4)$.

We assume that surfaces are sufficiently smooth. Any surface in $H^4(c)$ is given by a conformal immersion $F$ from a Riemann surface $M$ to $H^4(c)$. Using a local holomorphic coordinate $z$, we write the induced metric on $M$ as $e^{\omega} dzd\bar{z}$.

Let $N_1$ and $N_2$ be orthogonal unit normals to $F$. Then the Gauss-Weingarten equations are

$$\begin{align*}
F_{zz} &= \omega_2 F_x + Q_1 N_1 + Q_2 N_2, \\
F_{\bar{z}z} &= -\frac{1}{2} ce^{\omega} F + \frac{1}{2} H_1 e^{\omega} N_1 + \frac{1}{2} H_2 e^{\omega} N_2, \\
(N_1)_z &= -H_1 F_z - 2Q_1 e^{-\omega} F_{\bar{z}} + AN_2, \\
(N_2)_z &= -H_2 F_z - 2Q_2 e^{-\omega} F_{\bar{z}} - AN_1,
\end{align*}$$

where

$$\langle F_{zz}, N_i \rangle = Q_i, \quad \langle F_{\bar{z}z}, N_i \rangle = \frac{1}{2} H_i e^{\omega} \quad (i = 1, 2), \quad \langle (N_1)_z, N_2 \rangle = A.$$  

The quartic differential $(Q_1^2 + Q_2^2)dz^4$ is independent of the choice of $z$, $N_1$ and $N_2$ as well as the function $H_1^2 + H_2^2$, which is the length of the
mean curvature vector. The compatibility conditions for (2.1) give the Gauss-Codazzi-Ricci equations:

\[
\begin{align*}
\omega_z + \frac{1}{2} (H_1^2 + H_2^2 + c)e^{\omega} - 2(|Q_1|^2 + |Q_2|^2)e^{-\omega} &= 0, \\
(Q_1)_z &= \frac{1}{2} (H_1)_z e^{\omega} + \overline{\alpha} Q_2 - \frac{1}{2} \alpha H_2 e^{\omega}, \\
(Q_2)_z &= \frac{1}{2} (H_2)_z e^{\omega} - \overline{\alpha} Q_1 + \frac{1}{2} \alpha H_1 e^{\omega}, \\
A_z - \overline{A}_z &= 2(Q_1 \overline{Q}_2 - Q_2 \overline{Q}_1)e^{-\omega},
\end{align*}
\]

(2.2)

which show that minimal surfaces or surfaces with parallel mean curvature vector are Bonnet surfaces (cf. [4]). Note that the normal bundle of minimal surfaces is non-flat in general. On the other hand, non-minimal surfaces with parallel mean curvature vector have flat normal bundle and are contained in some totally geodesic or umbilic 3-dimensional space form as surfaces with constant mean curvature, which is known as Chen-Yau’s reduction theorem [3]. We remark that the same equations as (2.1) and (2.2) hold for surfaces in the simply connected, complete, 4-dimensional space form of curvature $c \geq 0$.

3. Bonnet Surfaces with Non-flat Normal Bundle

We consider a surface $F : M \to H^4(c)$ such that $Q_2 = \alpha Q_1$ for some $\alpha \in C$. Then the second and third equations of (2.2) give a linear equation for $\alpha H_1 - H_2$ if and only if $\alpha = \pm \sqrt{-1}$:

\[
(aH_1 - H_2)_z = -\alpha \Lambda (\alpha H_1 - H_2).
\]

(3.1)

In the following we put $\alpha = \pm \sqrt{-1}$. Exchanging the orthogonal unit normals, if necessary, we may assume that $\alpha = \sqrt{-1}$. Let $B : M \to C$ be a function such that $A = (\log B)_z$. Then (3.1) can be solved explicitly:

\[
\sqrt{-1}H_1 - H_2 = \overline{f} B^{-\sqrt{-1}},
\]

(3.2)

where $f : M \to C$ is a holomorphic function. We consider the case that the mean curvature vector never vanishes. Then changing $B$, if necessary, we may assume that $f = 1$. Since $H_1$ and $H_2$ are real-valued, (3.2) is
equivalent to

\[ H_1 = -\frac{\sqrt{-1}}{2} (B^{-\sqrt{-1}} - \overline{B}^{\sqrt{-1}}), \quad H_2 = -\frac{1}{2} (B^{-\sqrt{-1}} + \overline{B}^{\sqrt{-1}}). \]

From the second equation of (2.2), we have the linear equation for \( Q_1 \):

\[ (Q_1)_\tau = \sqrt{-1} \frac{\overline{B}_\tau}{B} Q_1 + \frac{1}{4} \overline{B}^{\sqrt{-1}} \left( \log \frac{B}{\overline{B}} \right)_z e^{\omega}. \]  

(3.3)

Solutions of (3.3) are given by \( Q_1 = P B^{\sqrt{-1}} \), where \( P : M \to \mathbb{C} \) is a function such that

\[ P_\tau = \frac{1}{4} \left( \log \frac{B}{\overline{B}} \right)_z e^{\omega}. \]

Let \( r \) and \( \theta \) be a positive or real valued functions on \( M \) respectively such that

\[ B = re^{\sqrt{-1} \theta}, \quad |B^{-\sqrt{-1}}| = e^{\theta}. \]

Then a direct computation leads to the following:

**Proposition 3.1.** The equations (2.2) are equivalent to

\[
\begin{aligned}
\omega_\pi + \frac{1}{2} (e^{2\theta} + c)e^{\omega} - 4|P|^2 e^{2\theta - \omega} &= 0, \\
P_\pi &= \frac{\sqrt{-1}}{2} 0_e^{\omega}, \\
0_\pi &= -2|P|^2 e^{2\theta - \omega}.
\end{aligned}
\]  

(3.4)

**Remark 3.2.** Note that the normal bundle of \( F \) is flat if and only if \( \theta_\pi = 0 \). In this case it is easy to see that \( F \) is a totally umbilic surface in some totally geodesic or umbilic 3-dimensional space form.

Now we assume that \( F \) is a Bonnet surface without umbilic points. Then \( H_1 \) and \( H_2 \) are deformed under the deformations as

\[ H_1 \to H_1 \cos \lambda - H_2 \sin \lambda, \quad H_2 \to H_1 \sin \lambda + H_2 \cos \lambda, \]

where \( \lambda \) is a deformation parameter. However transforming \( N_1 \) and \( N_2 \)
as
\[ N_1 \rightarrow N_1 \cos \lambda + N_2 \sin \lambda, \quad N_2 \rightarrow -N_1 \sin \lambda + N_2 \cos \lambda, \]
we may assume that \( H_1 \) and \( H_2 \) are invariants under the deformations. In the following we consider the case that \( A \) is invariant under the deformations and \( F \) is simple, i.e., the deformations are given by the transformations:
\[ Q_1 \rightarrow \mu Q_1, \quad Q_2 \rightarrow \mu Q_2 \] (3.5)
for some function \( \mu : M \rightarrow \mathbb{C} \) with \( |\mu| = 1 \) (cf. [4]). Then from the second equation of (3.4), we have
\[ |(\mu - 1)P|_{\bar{z}} = 0. \]
Since \( F \) contains no umbilic points, we have \( Q_1 \neq 0 \) and hence \( P \neq 0 \). In particular from the third equation of (3.4), the normal bundle of \( F \) is non-flat. Then we have the following analog of the result due to Graustein [7] for Bonnet surfaces in 3-dimensional space forms.

**Proposition 3.3.** Changing the holomorphic coordinate, if necessary, we may assume that
\[ P = \frac{\sqrt{-1}}{g + \bar{g}} \] (3.6)
for some holomorphic function \( g : M \rightarrow \mathbb{C} \).

From the second equation of (3.4) and (3.6), we have
\[ \frac{\bar{g}z}{(g + \bar{g})^2} = \frac{1}{2} \theta e^{\alpha}. \] (3.7)
Hence if we assume \( g_z \neq 0 \) and put
\[ w = \int \frac{dz}{g_z}, \quad s = w + \bar{w}, \quad t = w - \bar{w}, \] (3.8)
it is easy to verify that \( \theta \) is a function of \( s \) only.

**Theorem 3.4.** Let \( r \) be a positive valued function on \( M \) and \( g \) be a holomorphic function on \( M \) such that \( g_z \neq 0 \) and \( g + \bar{g} \neq 0 \). Then a
surface $F : M \to H^4(c)$ given by

$$e^{\varphi} = -\frac{2^2|g_z|^2}{(g + g)^2\theta_s}, \quad H_1 = -e^{\varphi} \sin r, \quad H_2 = -e^{\varphi} \cos r,$$

$$Q_1 = \frac{\sqrt{-1}e^{\varphi + \sqrt{-1}\log r}}{g + \bar{g}}, \quad Q_2 = \sqrt{-1}Q_1, \quad A = \frac{r_z}{r} + \sqrt{-1}\frac{\theta_s}{\bar{g}_z}$$

is a Bonnet surface with non-flat normal bundle, where $s$ is a real parameter given by (3.8)

$$\theta = -\frac{1}{2} \log \left( \beta e^{-cs} - \frac{1}{c} \right), \quad \beta > 0. \quad (3.9)$$

Moreover the deformations of $F$ are given by (3.5) with

$$\mu = \frac{1 - 2\sqrt{-1}u\bar{g}}{1 + 2\sqrt{-1}ug}, \quad u \in \mathbb{R}.$$

**Proof.** We continue the above argument to solve (3.4). From (3.6), (3.7), (3.8) and the third equation of (3.4), we have

$$e^{\varphi} = -\frac{2^2|g_z|^2}{(g + g)^2\theta_s} = -\frac{2^2|g_z|^2e^{2\varphi}}{(g + g)^2\theta_s}. \quad (3.10)$$

Hence we have $\theta_s < 0$ and

$$\theta_{ss} = \theta_se^{2\varphi}. \quad (3.11)$$

From (3.6), (3.8), (3.10) and the first equation of (3.4), we have

$$2 - \frac{e^{2\varphi} + c}{\theta_s} = \frac{(g + g)^2}{|g_z|^4} \left\{ \left( \frac{\theta_{ss}}{\theta_s} \right)_s - 2\theta_se^{2\varphi} \right\}. \quad (3.12)$$

Since the right-hand side of (3.12) vanishes from (3.11), we can integrate (3.11) as

$$\theta_s = \frac{1}{2}(e^{2\varphi} + c). \quad (3.13)$$

Since $c < 0$, we can obtain solutions of (3.13) with $\theta_s < 0$ as (3.9). \qed
References


