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AN INTERACTION-BASED FOUNDATION OF AGGREGATE INVESTMENT FLUCTUATIONS

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This study demonstrates that the interactions of firm-level indivisible investments give rise to aggregate fluctuations without aggregate exogenous shocks. When investments are indivisible, aggregate capital is determined by the number of firms that invest. I develop a method to derive the closed-form distribution of the number of investing firms when each firm's initial capital level varies stochastically. This method shows that idiosyncratic shocks may lead to non-vanishing aggregate fluctuations when the number of firms tends to infinity. I incorporate this mechanism in a dynamic general equilibrium model with indivisible investment and predetermined goods prices. The model features no aggregate exogenous shocks, and the fluctuation is driven by idiosyncratic productivity shocks. Numerical simulations show that the model generates aggregate fluctuations comparable to the business cycles in magnitude and correlation structure under standard calibration.
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1. INTRODUCTION

This study offers a novel mechanism by which idiosyncratic micro-level shocks affect aggregate outcomes. The possibility that idiosyncratic shocks might contribute to aggregate fluctuations has traditionally been discounted in macroeconomic research because such shocks are expected to cancel each other out when the number of agents is large. However, recent literature has identified examples where idiosyncratic shocks influence aggregate fluctuations. For example, Gabaix (2011) demonstrated that the actions of individual agents influence aggregate outcomes when the agents are asymmetric and their size distribution has a fat tail. Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) demonstrated a similar effect when the influence vector of an agent's action on other agents' actions is characterized by a fat-tailed distribution.

In this study, I demonstrate how firm-level productivity shocks affect aggregate investments. The mechanism I identify allows for the aggregate effects even when agents are symmetric. I consider the situation where firms' investments are indivisible and strategically complementary. In this situation, the actions of

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a few firms can trigger discrete responses of a stochastic fraction of other firms. This study shows that the aggregate fluctuations may persist even when the number of agents tends to infinity under a particular degree of complementarity. Under this condition, which may be called a "criticality condition," one agent's action induces a similar action by another agent on average. This condition was highlighted by Jovanovic (1987) as allowing idiosyncratic shocks to generate aggregate risks.

This study builds on that by Nirei (2006), who showed that the aggregate size of the discrete responses follows a power-law distribution with exponential truncation. The truncation point was determined by the degree of strategic complementarity. It was shown that as the number of agents tends to infinity, the variance of aggregate outcomes converges to 0 more slowly in the case where actions are discrete than in an economy with continuous actions. The present study extends these results by characterizing the aggregate fluctuations under a critical level of complementarity. The critical level of complementarity results in a power-law distribution without exponential truncation and thus serves as a new source of aggregate fluctuations.

I consider monopolistic firms competing by producing differentiated intermediate goods. This economy features aggregate demand externality as in Blanchard and Kiyotaki (1987), where an increase in aggregate demand proportionally shifts the demand schedule for each good. Given a technology with constant returns to scale, the aggregate capital level is indeterminate in the production sector if firm-level capital is continuously adjusted. By incorporating indivisible investments, I obtain two advantages that do not arise in the case of continuous investments. First, the equilibrium aggregate capital level is locally unique. Second, the distribution of aggregate investment fluctuations is analytically derived. Multiple equilibria may exist under the complementarity. However, I obtain the aggregate fluctuations by selecting the least volatile equilibria and not through the use of extreme equilibria.

Three results arise from this paper. First, I develop an equilibrium model of investments with exogenous factor prices and derive an asymptotic distribution function of aggregate capital fluctuation when the number of firms tends to infinity. As in Nirei (2006), the distribution has a heavier tail than the normal distribution. Second, I show that under the critical level of complementarity and with particular equilibrium selections, the variance of aggregate fluctuations does not vanish at the infinite limit of the number of firms or vanishes much more slowly than the central limit theorem predicts. I obtain the latter claim when I select the least volatile equilibrium. I obtain the former claim when I select the equilibrium that is least volatile in the same direction as the sum of the idiosyncratic shocks. Third, I develop a dynamic general equilibrium model where the critical level of complementarity arises even with endogenously determined factor prices. The first and second results are shown with exogenous factor prices, whereas the third result is shown without this assumption. Moreover, I quantitatively demonstrate that the dynamic general equilibrium model with indivisible capital can generate aggregate fluctuations comparable to business cycles in magnitude and correlation structure, if I additionally assume a predetermined price-setting behavior. Given predetermined goods prices, the investment fluctuations can propagate to consumption and output fluctuations.

Scholars working on interaction-based models have tackled the question of how to analyze aggregate fluctuations that arise from discrete, or more generally, non-linear, actions at the micro level. These models have suggested the possibility of endogenous fluctuations arising from the non-linearity of micro-level actions (e.g., Durlauf (1993); Glaeser, Sacerdote, and Scheinkman (1996); Brock and Hommes (1997); Brock and Durlauf (2001)). In macroeconomics, the so-called (S,s) literature concentrates on the case where pricing or investment incurs fixed costs and thus exhibits non-linearity at the micro level. Typically, an aggregate (S,s) model features a continuum of firms as in Thomas (2002). This modeling choice precludes the possibility that interactions of "granular" firms give rise to aggregate fluctuations—a feature of interaction-based models. While I draw on the (S,s) literature in some respects, the aggregate fluctuation results presented in this paper are obtained using a model with a large but finite number of firms. The intuition of the results is analogous to that of interaction-based models.

This study contributes to the ongoing debate on the origins of business cycle fluctuations in three ways. First, I provide a microfoundation for the investment-specific technology shocks that influence business cycles in dynamic general equilibrium models empirically demonstrated by researchers including Fisher (2006) and Justiniano, Primiceri, and Tambalotti (2010). Second, this paper shares its motivation to explain aggregate fluctuations in the absence of aggregate shocks with the literature on sunspot equilibria (Galí (1994); Wang and Wen (2008)). However, this study differs from the sunspot literature in that the agents' expectation system is dynamically determinate in this study. Unlike sunspot models, the equilibrium outcome is locally unique because of the discreteness of micro-level decisions. Third, this study extends the literature that emphasizes the role of fat-tailed distributions that allow idiosyncratic shocks to induce aggregate fluctuations (Gabaix (2011); Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012)). The fluctuation mechanism in this paper is most closely related to self-organized criticality models (Bak, Chen, Scheinkman, and Woodford (1993)). In those models, an individual action causes an "avalanche" of other actions, and the size of the avalanche follows a fat-tailed distribution. While these models feature locally interacting firms, this study is concerned with firms that interact globally (i.e., with all other firms) in goods markets in dynamic general equilibrium.

The rest of this paper is organized as follows. Section 2 uses an equilibrium model of investments with exogenous factor prices and analytically characterizes the aggregate fluctuations that arise from threshold behaviors and a critical level of complementarity without aggregate shocks. Section 3 presents a dynamic general equilibrium model with indivisible capital, a technology with constant returns to scale, and predetermined goods price-setting. Under this specification, the dynamic general equilibrium model generates the criticality condition and aggregate fluctuations even with endogenously determined factor prices. Moreover, numerical simulations of the model with a finite number of firms show that equilibrium paths mimic the business cycles in the magnitude of standard deviations and correlations. Section 4 presents my conclusions. All proofs are in the Appendices.

2. ANALYTICAL RESULTS

In this section, I present my main theoretical results showing that aggregate fluctuations can occur without aggregate exogenous shocks when firms' investments follow a threshold rule. I assume that the real wage and interest rate are exogenously given. This is a simplifying assumption adopted in this section to develop the theory of aggregate fluctuations. This assumption is relaxed in Section 3, which shows that the same aggregate fluctuations can occur in a dynamic general equilibrium model where the real wage and interest rate are determined endogenously.

2.1. *Firm's investment decision*

I consider an economy with N distinct intermediate goods, each produced by a monopolist. These goods can be produced using capital and labor. Specifically, if the producer of good i uses k_i units of capital and l_i units of labor in period t, it will produce $y_{i,t} = a_{i,t} k_{i,t}^{\alpha} l_{i,t}^{\gamma}$ units of intermediate good i, where $\alpha + \gamma \leq 1$. The productivity $a_{i,t}$ is stochastic and i.i.d. across i and t with a bounded support. These intermediate goods can be combined to produce a final good by a competitive goods producer, where $y_{i,t}$ units of each respective intermediate good will yield $Y_t = \left(\sum_{i=1}^{N} y_{i,t}^{(\eta-1)/\eta}/N\right)^{\eta/(\eta-1)}$ units of the final good. This final good can be converted one-for-one into capital that can be used by intermediate goods producers. Given this structure, I can express the production Y_t in terms of aggregate capital across the N intermediate goods producers. To see this, note that demand for each intermediate good by the final goods producer will be given by $y_{i,t} = (p_{i,t}/P_t)^{-\eta} Y_t$, where $P_t \equiv \left(\sum_{i=1}^{N} p_{i,t}^{1-\eta}/N\right)^{1/(1-\eta)}$ denotes aggregate price and is normalized to 1. Given this demand for its input, each intermediate goods producer will set labor demand optimally as $l_{i,t} = (c_L/w_t) p_{i,t} y_{i,t}$, where $c_L \equiv (1 - 1/\eta)\gamma$. Substituting these into the respective production functions shows that final goods output can be expressed as $Y_t = (c_L/w_t)^{\gamma/(1-\gamma)} K_t^{\alpha/(1-\gamma)}$, where $K_t \equiv \left(\sum_{i=1}^{N} a_{i,t}^{\rho/\alpha} k_{i,t}^{\rho}/N\right)^{1/\rho}$ is a productivity-weighted aggregate[1] of the capital of all intermediate goods producers and $\rho \equiv (1 - 1/\eta)\alpha/(1 - c_L)$.

[1] K_t corresponds to aggregate capital when aggregate productivity is properly defined and normalized to 1. The analysis in this paper does not depend on the level of aggregate productivity.

Firm i owns physical capital $k_{i,t}$, which accumulates as $k_{i,t+1} = (1-\delta)k_{i,t}+x_{i,t}$. I consider the case where the firm's investment decision is restricted to a discrete set: $k_{i,t+1} \in \{\lambda^{\kappa}(1-\delta)k_{i,t}\}_{\kappa=0,\pm1}$, where $\lambda(1-\delta) > 1$. Capital $k_{i,t+1}$ is chosen to be either the depreciated level $(1-\delta)k_{i,t}$, the depreciated level multiplied by indivisibility parameter λ, or the depreciated level multiplied by λ^{-1}. This discrete constraint is equivalent to assuming that the firm can only choose a gross investment rate $x_{i,t}/k_{i,t}$ of 0, $(\lambda-1)(1-\delta)$, or $(\lambda^{-1}-1)(1-\delta)$—that is, inaction, lumpy investment, or lumpy divestment, respectively. This constraint reflects the firm's capital choice in the short term, where investments in physical assets such as equipment and structure are indivisible. Firms lack incentives to make substantial capital adjustments in sufficiently short time horizons when both productivity shocks and depreciation are relatively small and the environment is stationary.[2]

In this section, I assume that factor prices are exogenously given. This assumption is relaxed in Section 3 where factor prices are determined endogenously. I further assume that all firms know the productivity profile $(a_{i,t+1})_{i=1}^{N}$ in period t. This assumption simplifies the analysis of a firm's investment decision.

Firm i chooses capital $k_{i,t+1}$ in period t in order to maximize the expected discounted sum of the dividend stream, $\mathrm{E}_t\left[\sum_{\tau=t}^{\infty}\left(\prod_{s=t+1}^{\tau} R_s^{-1}\right) d_{i,\tau}\right]$, where $d_{i,t} = p_{i,t}y_{i,t} - w_t l_{i,t} - x_{i,t}$ denotes the dividend and R_t denotes the inverse of the discount factor given to the firm. Using the firm's labor demand schedule, the objective function is written as an expected discounted sum of $\pi(k_{i,t+1})$, where

(1)

$$\pi(k_{i,t+1}) = (1-c_L)(c_L/w_{t+1})^{\frac{\gamma}{1-\gamma}} K_{t+1}^{\frac{\alpha/\eta}{(1-\gamma)(1-c_L)}} a_{i,t+1}^{\rho/\alpha} k_{i,t+1}^{\rho} - (R_{t+1}-1+\delta)k_{i,t+1}.$$

$\pi(\cdot)$ is strictly concave because $\rho < 1$, which holds true in that $\alpha+\gamma \leq 1$ and $\eta > 1$. Thus, there exists a unique $k_{i,t+1}^{*}$ that satisfies $\pi(k_{i,t+1}^{*}) = \pi(\lambda k_{i,t+1}^{*})$.

Because π is concave, there exists a threshold $\underline{k}_{i,t+1}$ below which it is optimal for firm i to invest, as well as another threshold $\bar{k}_{i,t+1}$ above which it is optimal to disinvest. I assume that the support of $a_{i,t}$ and the shifts in factor prices are sufficiently small so that $|\log k_{i,t+2}^{*} - \log(k_{i,t+1}^{*}(1-\delta))|$ is no greater than $\log\lambda$. This boundedness condition is satisfied in the environment assumed in Section 2.4. Under this condition, firm i's problem reduces to a maximization of π with respect to $k_{i,t+1}$ that is chosen from the discrete set.[3] Firm i is indifferent

[2] It is possible to extend the choice set to $\{\lambda^{\pm\kappa}(1-\delta)k_{i,t}\}$, $\kappa = 0, 1, \ldots, \bar{\kappa}$, for a finite $\bar{\kappa}$. For some initial capital profile, the equilibrium is a corner solution where capital takes the boundary value of the choice set. Thus, if $\bar{\kappa}$ is taken to infinity, an equilibrium may not exist with exogenous factor prices and finite N.

[3] It must be noted that the choice set for $k_{i,t+2}$ depends on $k_{i,t+1}$. However, this dependence does not affect the optimal threshold for $k_{i,t+1}$ under the boundedness condition, because the new option for $k_{i,t+2}$ that is gained by not following the optimal threshold rule is dominated by the choices available when the optimal threshold is followed.

between investment and inaction at an optimal threshold, and hence $\pi(\underline{k}_{i,t+1}) = \pi(\lambda\underline{k}_{i,t+1})$. Similarly, i is indifferent between divestment and inaction at $\bar{k}_{i,t+1}$, and hence $\pi(\bar{k}_{i,t+1}) = \pi(\lambda^{-1}\bar{k}_{i,t+1})$. Therefore, I obtain the optimal thresholds as $\underline{k}_{i,t+1} = k^*_{i,t+1}$ and $\bar{k}_{i,t+1} = \lambda k^*_{i,t+1}$. By solving the optimal condition $\pi(k^*_{i,t+1}) = \pi(\lambda k^*_{i,t+1})$, the lower threshold $k^*_{i,t+1}$ is obtained as follows:

$$
\begin{aligned}
(2) \qquad & k^*_{i,t+1} = b_{i,t+1} K^{\phi}_{t+1}, \\
& b_{i,t+1} \equiv \left(\frac{\lambda^\rho - 1}{\lambda - 1}\right)^{\frac{1}{1-\rho}} \left((1-c_L)\left(\frac{c_L}{w_{t+1}}\right)^{\frac{\gamma}{1-\gamma}} \frac{a^{\rho/\alpha}_{i,t+1}}{R_{t+1} - 1 + \delta}\right)^{\frac{1}{1-\rho}},
\end{aligned}
$$

where $\phi \equiv \alpha/[(1-\gamma)\{\eta-(\eta-1)(\alpha+\gamma)\}]$. The parameter $\phi \in (0,1]$ determines the strength of the positive feedback from aggregate capital to individual investment decisions, and thus represents the degree of strategic complementarity between investments. In particular, $\phi = 1$ holds when $\alpha + \gamma = 1$. Note that $k^*_{i,t+1}$ is decreasing in λ.

At the heart of the aggregate fluctuations arising from idiosyncratic shocks in this model lie the complementarity and non-linearity of firm-level investment decisions. The firms' investment choices exhibit complementarity with each other because of aggregate demand externality. The capital decision $k_{i,t+1}$ is non-linear because of indivisibility and the threshold policy. The average capital level K_{t+1} affects threshold $k^*_{i,t+1}$ continuously, but it may or may not induce an adjustment in capital $k_{i,t+1}$. Individual capital is insensitive to a small perturbation in average capital, whereas an average response amounts to the size of the perturbation multiplied by ϕ.

2.2. *Random gap distribution*

The gap between a firm's capital and the threshold, normalized by indivisibility, is denoted by $s_{i,t} = (\log k_{i,t} - \log k^*_{i,t})/\log\lambda$. In this section, I derive the closed-form distribution of the fluctuations of aggregate capital K_{t+1} when the initial capital profile $(k_{i,t})_{i=1}^N$ varies stochastically. Specifically, I assume that $s_{i,t}$ is a uniform random variable with support $[0,1)$. In Section 2.6, I show that $s_{i,t}$ converges to the uniform distribution as $t \to \infty$, independent across i, when λ and δ are heterogeneous across i. This implies that the probability of drawing a particular profile $(s_{i,t})_{i=1}^N$ from an N-dimensional jointly uniform distribution corresponds to the likelihood of the profile of the gap between a firm's capital and the threshold being realized over the long run.

2.3. *Equilibrium selection*

For each realization of the gap and productivity profiles $(s_{i,t}, a_{i,t+1})_{i=1}^N$, and given aggregate capital K_{t+1}, the capital profile in the next period $(k_{i,t+1})_{i=1}^N$ is

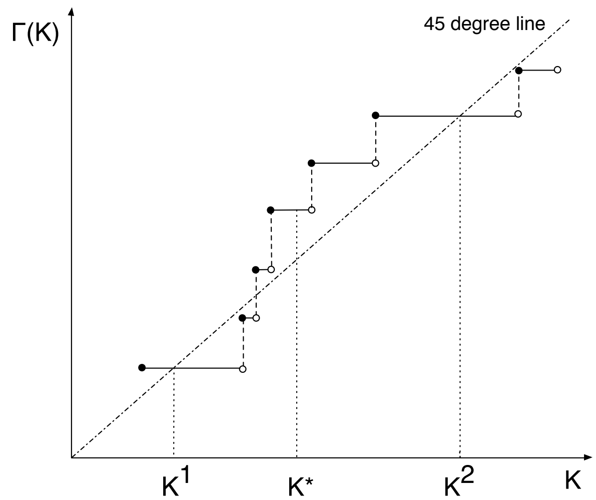

FIGURE 1.— Aggregate reaction function Γ. K^1 is selected using Equilibrium Selection 1 (ES1) because $|\log K^1 - \log K^*| < |\log K^2 - \log K^*|$. K^2 is selected using Equilibrium Selection 2 (ES2) because $\text{sign}(\log K^2 - \log K^*) = \text{sign}(\log \Gamma(K^*) - \log K^*)$.

determined using the threshold rule (2). An aggregate reaction function is then defined by aggregating the firms' capital decision, given K, as

$$\Gamma\left(K; (k_{i,t}, a_{i,t+1})_{i=1}^N\right) = \left(\sum_{i=1}^N \left(a_{i,t+1}^{1/\alpha}(1-\delta)k_{i,t}\lambda^{\kappa_{i,t+1}}\right)^\rho /N\right)^{1/\rho}, \tag{3}$$

where

$$\kappa_{i,t+1} = \begin{cases} 1 & \text{if } (1-\delta)k_{i,t} < b_{i,t+1}K^\phi \\ 0 & \text{if } b_{i,t+1}K^\phi \le (1-\delta)k_{i,t} < \lambda b_{i,t+1}K^\phi \\ -1 & \text{if } \lambda b_{i,t+1}K^\phi \le (1-\delta)k_{i,t} \end{cases}. \tag{4}$$

Note that K enters Γ via the threshold rule (4). As depicted in Figure 1, Γ is a step function and non-decreasing in K.

The equilibrium aggregate capital is a fixed point of Γ. If the mapping from a profile $(s_{i,t}, a_{i,t+1})_{i=1}^N$ to the fixed point were one-to-one, the distribution function of aggregate growth $\log K_{t+1} - \log K_t$ would be determined using the joint distribution function of $(s_{i,t}, a_{i,t+1})_{i=1}^N$. However, as depicted in Figure 1, multiple fixed points may exist because of the indivisibility of capital. Thus, to obtain the distribution for fluctuations of $\log K_{t+1} - \log K_t$, an equilibrium selection mechanism is required.

I determine the equilibrium by selecting the fixed point of Γ that is closest to some benchmark level of aggregate capital denoted by K^*_{t+1}. For $\phi < 1$, I define K^*_{t+1} as the fixed point of the aggregate reaction function when a continuum of firms exists:

$$K^*_{t+1} = \left(\int_0^1 \left(a_{i,t+1}^{1/\alpha} (1-\delta) k_{i,t} \lambda^{\kappa_{i,t+1}} \right)^{\rho} di \right)^{1/\rho}, \tag{5}$$

where $k_{i,t} = \lambda^{\tilde{s}_{i,t}} b_{i,t} K_t^{\phi}$ and $\tilde{s}_{i,t}$ is a uniform random variable. Here, $\log \Gamma$ plotted against $\log K$ converges to a line with slope ϕ as $N \to \infty$ because each step size of $\log \Gamma$ shrinks as $1/N$. Thus, K^*_{t+1} uniquely exists for $\phi < 1$. For $\phi = 1$, $\log \Gamma$ plotted against $\log K$ coincides with the 45-degree line as $N \to \infty$, and thus, K^*_{t+1} becomes indeterminate in (5). This reflects the fact that, if a continuum of firms exists, the aggregate capital level is indeterminate in the production sector under $\phi = 1$ and given factor prices. Thus, when $\phi = 1$, I set K^*_{t+1} exogenously. The analysis in this section holds for any choice of K^*_{t+1} when $\phi = 1$. Note that R_{t+1}, R_t, w_{t+1}, w_t, and K_t are exogenously given. For $\phi = 1$, R_{t+1} and w_{t+1} are restricted by the condition $\int_0^1 \lambda^{\rho \tilde{s}_{i,t+1}} b_{i,t+1}^{\rho} di = 1$, where $\tilde{s}_{i,t+1}$ is a uniform random variable. In Section 3, I develop a dynamic general equilibrium model in which R_{t+1} and w_{t+1} are endogenously determined, and K^*_{t+1} is uniquely determined as $K^*_{t+1} = \mathrm{E}(K_{t+1} \mid K_t)$ under rational expectations of factor prices for the case $\phi = 1$.

With this K^*_{t+1}, I define an equilibrium selection as follows.

Definition 1 *Equilibrium Selection 1 (ES1) selects the equilibrium aggregate capital* K_{t+1} *for each realization of* $(k_{i,t}, a_{i,t+1})_{i=1}^N$ *that attains the minimum of* $|\log K_{t+1} - \log K^*_{t+1}|$ *among all* K_{t+1} *solving* $K_{t+1} = \Gamma(K_{t+1}; (k_{i,t}, a_{i,t+1})_{i=1}^N)$.

Using this equilibrium selection, I construct the least-volatile fluctuations of aggregate capital in equilibrium that are possible. In other words, fluctuations due to multiple equilibria are excluded from the selected equilibrium. This is a strategic assumption made in this paper in order to demonstrate that idiosyncratic shocks with non-linear behaviors alone can generate non-vanishing aggregate fluctuations even when I exclude the possibility of a large shift in aggregate capital that arises from purely informational coordination among firms.

To facilitate the analysis of this equilibrium, I define another selection mechanism as an auxiliary.

Definition 2 *Equilibrium Selection 2 (ES2) selects the equilibrium aggregate capital* K_{t+1} *for each realization of* $(k_{i,t}, a_{i,t+1})_{i=1}^N$ *that attains the minimum of* $|\log K_{t+1} - \log K^*_{t+1}|$ *among all* K_{t+1} *solving* $K_{t+1} = \Gamma(K_{t+1}; (k_{i,t}, a_{i,t+1})_{i=1}^N)$ *and satisfying* $\mathrm{sign}(\log K_{t+1} - \log K^*_{t+1}) = \mathrm{sign}(\log \Gamma(K^*_{t+1}) - \log K^*_{t+1})$.

ES2 adds a condition to ES1. Namely, ES2 selects the equilibrium aggregate capital that is closest to K^*_{t+1} in the direction the firms are induced by idiosyncratic shocks to adjust to under K^*_{t+1}. In Figure 1, this mechanism selects K^2. Some properties are known about this mechanism. Vives (1990) showed that the equilibrium selected using this mechanism is the convergent point of the best-response dynamics $K_{u+1} = \Gamma(K_u)$ starting at K^*. Cooper (1994) supported the use of this selection mechanism in macroeconomics on the grounds that the best-response dynamics are a realistic process in a situation where many agents interact with each other and make decisions only with aggregate-level information.

In the following, I characterize the fluctuations of K_{t+1} provided that the investment follows the threshold rule (2), the gap $s_{i,t}$ follows a uniform distribution, and K^*_{t+1} is given. These three premises are established in a dynamic general equilibrium model in Section 3.

2.4. *Results*

The equilibrium aggregate capital growth rate, $\log K_{t+1} - \log K_t$, consists of an anticipated part, $\log K^*_{t+1} - \log K_t$, and an unanticipated part, $\log K_{t+1} - \log K^*_{t+1}$. The anticipated part is exogenously given. Thus, I focus on the distribution of the unanticipated growth. I concentrate on a homogeneous setup in which indivisibility and productivity are common across firms: $\lambda_i = \lambda$ and $a_{i,t} = a_{i,t+1} = 1$. In this homogeneous setup, the only source of deviation from the expected aggregate capital is the gap $s_{i,t}$. The variation of $s_{i,t}$ can be regarded as the results of past realizations of productivity shocks up to period $t-1$. A generalization to the case of heterogeneous indivisibility and productivity is discussed in Section 2.6. I use the following notation:

$$q_t \equiv \frac{\phi \log \frac{K^*_{t+1}}{K_t} - \frac{1}{1-\rho}\left(\frac{\gamma}{1-\gamma}\log\frac{w_{t+1}}{w_t} + \log\frac{R_{t+1}-1+\delta}{R_t-1+\delta}\right) - \log(1-\delta)}{\log\lambda}. \tag{6}$$

This denotes the anticipated fraction of firms that invest because of an exogenous shift in the aggregate environment. I assume that the exogenous shifts of w_t and R_t are bounded so that $q_t < 0.5$ holds for any t. In the remainder of this section, I drop the time subscript t from all variables.

Consider a sequence of economies with a number of firms $N = N_0, N_0+1, \ldots,$ for a large N_0. Let g_N denote the unanticipated growth $\log K - \log K^*$ for an economy with N firms. The main analytical result of this study is characterizing the asymptotic variance of g_N for ES1 and ES2.

To characterize the aggregate fluctuations, I use a fictitious tatonnement, which is defined by the best-response dynamics of the capital profile:

$$k_{i,1} = \begin{cases} \lambda(1-\delta)k_{i,0} & \text{if } (1-\delta)k_{i,0} < k^*_{i,0} \\ (1-\delta)k_{i,0} & \text{otherwise} \end{cases}, \tag{7}$$

$$k_{i,u+1} = \begin{cases} \lambda k_{i,u} & \text{if } k_{i,u} < k^*_{i,u} \\ k_{i,u}/\lambda & \text{if } k_{i,u} \geq \lambda k^*_{i,u} \\ k_{i,u} & \text{otherwise} \end{cases}, \tag{8}$$

where $K_u = \left(\sum_i a_{i,t+1}^{\rho/\alpha} k_{i,u}^{\rho}/N\right)^{1/\rho}$ and $k^*_{i,u} = b_{i,t+1}K_u^{\phi}$ for $u = 1, 2, \dots$ and $k^*_{i,0} = b_{i,t+1}K^{*\phi}$, respectively. Subscript u represents a step in the fictitious tatonnement. The best-response dynamics are consistent with the aggregate response function $K_{u+1} = \Gamma(K_u; (k_{i,u}, a_{i,t+1})_{i=1}^N)$.

The expected number of firms that adjust capital in the first step is Nq. Their investments may not exactly balance with aggregate capital depreciation: $\Gamma(K^*)$ may not coincide with K^*. The gap is denoted by $m_1 \equiv N(\log\Gamma(K^*) - \log K^*)/\log\lambda$. If $m_1 = 0$, K^* constitutes the equilibrium. Otherwise, the optimal threshold is updated under a new aggregate capital K_1 and the adjustments in the second step occur. This procedure is iterated until there are no more firms that newly adjust. The convergent point corresponds to the equilibrium selected by ES2 (depicted as K^2 in Figure 1).

Unanticipated growth g_N is divided into m_1 and subsequent adjustments. Subsequent adjustments after the first step are measured in the number of firms that adjust capital upward in step u, denoted by m_u for $u = 2, 3, \dots, T$. If firms adjust downward, m_u is set as negative. The series m_u is either positive or negative for all u depending on if $m_1 > 0$ or $m_1 < 0$. The total number of firms that adjust capital subsequently after the first step of tatonnement is denoted by $M \equiv \sum_{u=2}^T m_u$. Note that T is the stopping time of the tatonnement: $T \equiv \min_{u:m_u=0} u$. The equilibrium capital vector is determined by the convergent point of the dynamics, $(k_{i,T})_{i=1}^N$.

In the first step toward the characterization of g_N, I show that the capital growth due to the subsequent adjustments is asymptotically proportional to the number of firms that adjust.

Lemma 1 $N(\log K_{u+1} - \log K_u)$ *converges to* $m_{u+1}\log\lambda$ *as* $N \to \infty$ *almost surely for* $u = 1, 2, \dots, T-1$.

Lemma 1 implies that $N(\log K^2 - \log K^*)$ converges in distribution to $(m_1 + M)\log\lambda$. I then show that the number of adjusting firms in the tatonnement asymptotically follows a Poisson branching process by applying a result from my previous paper (Nirei (2006), Lemma 9).

Lemma 2 m_u *for* $u = 2, 3, \dots, T$ *asymptotically follows a branching process, in which each firm in* m_u *bears firms in step* $u+1$ *whose number follows a Poisson distribution with mean* ϕ.

A branching process is an integer stochastic process of a population in which each parent in a generation bears a random number of children in the next generation. In a Poisson branching process, the number of children born from a parent is a Poisson random variable. It is known that a branching process converges to 0 in a finite time period with probability 1 if the mean number of children born from a parent is less than or equal to 1 (Feller, 1957, p. 276). This confirms that the best-response dynamics stop in a finite time T with probability 1 when $\phi \leq 1$.

The following result is known for the cumulative population size of the Poisson branching process.

Lemma 3 *M conditional on $m_1 > 0$ follows*

$$\Pr(M = m \mid m_1) = m_1 e^{-\phi(m+m_1)} \phi^m (m + m_1)^{m-1}/m! \tag{9}$$

for $m = 0, 1, \ldots$. Conditional on $m_1 < 0$, $-M$ follows the same distribution with $|m_1|$ instead of m_1. The right tail of the distribution (9) is approximated by

$$\Pr(M = m \mid m_1) \sim (m_1 e^{(1-\phi)m_1}/\sqrt{2\pi}) e^{-(\phi-1-\log\phi)m} m^{-1.5}. \tag{10}$$

Lemma 3 is similar to my previous result (Nirei (2006), Proposition 4) except for the characteristics of m_1. Nirei (2006) was concerned with a productivity perturbation scaled as $1/N$, which resulted in a Poisson distribution of m_1. In the present model, the initial adjustments are caused by capital depreciation. Thus, the central limit theorem holds for the initial adjustment size. Defining

$$\sigma_1^2 \equiv (1 - \lambda^{-2\rho q})/(2\rho \log \lambda) - ((1 - \lambda^{-\rho q})/(\rho \log \lambda))^2, \tag{11}$$

I obtain the following result.

Lemma 4 *$m_1/\sqrt{N}$ asymptotically follows a normal distribution with mean 0 and variance σ_1^2.*

Lemmas 3 and 4 fully characterize the distribution of g_N under ES2. Using these results, I obtain the main results of this study.

Proposition 1 *Under ES1, if $\phi = 1$, $\lim_{N\to\infty} \sqrt{N}\mathrm{Var}(g_N) > 0$.*

Proposition 2 *Under ES2, if $\phi = 1$, $\mathrm{Var}(g_N)$ converges to a strictly positive constant as $N \to \infty$.*

Proposition 1 states that if $\phi = 1$, the asymptotic variance of g_N under ES1 declines no faster than $1/\sqrt{N}$. Proposition 2 states that the asymptotic variance can be non-zero under ES2.

The main idea of the proof for Proposition 2 is as follows. Lemma 4 shows that $m_1/\sqrt{N}$ asymptotically follows a normal distribution with finite variance. Hence, the mean of the absolute value $|m_1|$ scales as $\sqrt{N}$. Lemma 3 shows that $Ng_N/\log\lambda - m_1$ conditional on $m_1 = 1$ follows a power-law tailed distribution with exponent 0.5 if $\phi = 1$. Then, the variance of Ng_N conditional on $m_1 = 1$ diverges as $N^{1.5}$ because $\int^N x^2 x^{-1.5} dx \sim N^{1.5}$. Combining these two results implies that Ng_N unconditional on m_1 has variance scaling as N^2 because Ng_N can be divided into $\sqrt{N}$ sub-population sets, each of which has variance that scales as $N^{1.5}$. Hence, the variance of g_N scales as N^0 under ES2 for $\phi = 1$. Namely, g_N has a non-nil variance at the limit because the vanishing variance of initial disturbance m_1/N is multiplied by the diverging effect of subsequent propagation M that follows a power law with exponent 0.5.

Finally, Proposition 1 for ES1 is proven using the power-law distribution of M. Under ES1, g_N is determined using the minimum between $|\log K^1 - \log K^*|$ and $|\log K^2 - \log K^*|$. These two terms are independent conditional on m_1. Moreover, it can be shown that the distribution of $|\log K^2 - \log K^*|$ declines no faster than a power law with exponent 0.5 for $\phi = 1$. Combined with the power law for $\log K^1 - \log K^*$, g_N under ES1 cannot decline faster than a power law with exponent 1. This yields the desired result.

2.5. *Implications*

Proposition 1 states that the variance of g_N converges to 0 no faster than $1/\sqrt{N}$ under ES1 if $\phi = 1$. Proposition 2 shows that the variance of g_N converges to a non-zero constant under ES2 if $\phi = 1$. These results open up a theoretical possibility that indivisible investment at the micro level contributes to sizable macro-level fluctuations when the number of firms is large but finite. These results contrast with the Long-Plosser model with continuous capital adjustments, where the aggregate variance declines as fast as $1/N$. This is because idiosyncratic productivity shocks cancel each other out in aggregation, as the central limit theorem predicts (Dupor (1999)). In contrast, Proposition 1 shows that the variance of the capital growth rate decreases to 0 at a much slower rate, even if I choose the least-volatile equilibrium (ES1) when $\phi = 1$ holds.

Equation (10) shows that g_N conditional on m_1 asymptotically follows a gamma-type distribution that combines a power function $m^{-1.5}$ and an exponential function $e^{-(\phi-1-\log\phi)m}$. Here, $\phi - 1 - \log\phi > 0$ for $\phi < 1$. Because an exponential function declines faster than a power function does, the tail distribution is dominated by the exponential when $\phi < 1$. Thus, the degree of strategic complementarity ϕ determines the speed of the exponential truncation of the distribution.

Whether the tail obeys an exponential decay or a power decay has an important implication for moments of the distribution. Any k-th moment exists if the tail decays exponentially because $\int_0^\infty x^k e^{-x} dx$ is a gamma function and thus finite. In contrast, if the tail decays in power with exponent α, only moments lower than α exist because $\int^\infty x^k x^{-\alpha-1} dx$ is finite only for $k < \alpha$. When the exponent of the power law is 0.5, even the mean diverges.

The power-law tail of the propagation effect, resulting from the criticality condition $\phi = 1$, generates a macro-level fluctuation. When this condition is not met, the aggregate fluctuations eventually die down as the number of firms increases to infinity. This is because ϕ, the mean number of children per parent, determines the trend population growth in the branching process. The mean population of the n-th generation is ϕ^n given a single initial parent. The population diverges to infinity when the process is supercritical, $\phi > 1$, whereas the population decreases to 0 if the process is subcritical, $\phi < 1$. At the critical point $\phi = 1$, the population of a generation decreases to 0 with probability 1, whereas the mean cumulative population diverges to infinity.

The key environment for the power law, $\phi = 1$, can be interpreted as a critical level of complementarity of indivisible investments. By the critical level of complementarity, I mean that a proportional increase in the capital of all other firms induces the same proportional increase in the capital of a firm if the increment is much larger than the indivisibility is. Because of the indivisibility of capital, however, a shock smaller than the indivisibility does not cause a symmetric movement across firms. Thus, the firm's investment behavior at criticality can be summarized as local inertia combined with global linear complementarity.

It might appear counterintuitive that the aggregate variance does not converge to 0 when there are only idiosyncratic discrepancies in the initial capital gap. In a smoothly adjusting competitive economy, the aggregate capital level is indeterminate in the production sector if the firms' investment decisions are linearly complementary because of technology with constant returns to scale. In the present model, the equilibrium is locally unique because of the indivisibility of capital. Nonetheless, the globally indeterminate environment makes it possible for the aggregate fluctuations to reappear as the power-law distribution.

The limit of the standard deviation of g_N under ES2 in Proposition 2 is affected by $\log\lambda$. In fact, the indivisibility parameter $\log\lambda$ has an almost proportional effect on the aggregate standard deviation. This is because g_N is a product of $\log\lambda$ and M/N that has only weak dependence on λ. The almost proportional impact of $\log\lambda$ on the aggregate standard deviation implies that the indivisibility of capital provides a foundation for the sizable idiosyncratic volatility of firm-level decisions, which in turn has a one-to-one impact on aggregate volatility.

2.6. *Heterogeneous firms and uniform distribution of capital gap*

The fluctuation results can be extended to the case where the indivisibility and depreciation rates are heterogeneous across firms. Suppose that a type of firm with lumpiness λ_i and depreciation δ_i is drawn from a joint density function with finite mean and i.i.d. across i. I assume $\underline{\lambda}(1-\bar{\delta}) > 1$, where $\underline{\lambda}$ and $\bar{\delta}$ denote the lower bound of λ_i and upper bound of δ_i, respectively. Productivity $a_{i,t}$ is a random variable independent from (λ_i, δ_i) and i.i.d. across i and t. I assume that the support of $\log a_{i,t}$ is bounded as $\Pr(|\log a_{i,t+1} - \log a_{i,t}| < q\alpha(1/\rho - 1)\log\underline{\lambda}) = 1$ so that firms have no incentives to make adjustments of more than one notch. This constraint holds for sufficiently short time horizons, when both productivity change and depreciation are relatively small. I maintain that $s_{i,t}$ is a uniform random variable. I next define the following:

$$\hat{\phi} \equiv \phi \mathrm{E}\left[(\log\lambda_i)\left(\frac{\lambda_i^{\rho}-1}{\lambda_i-1}\right)^{\rho/(1-\rho)}\right]\mathrm{E}\left[\frac{1}{\log\lambda_i}\right]/\mathrm{E}\left[\left(\frac{\lambda_i^{\rho}-1}{\lambda_i-1}\right)^{\rho/(1-\rho)}\right]. \tag{12}$$

Then, I obtain the following proposition.

Proposition 3 *Under ES2 and with heterogeneous* λ_i, M *conditional on* $m_1 = 1$ *follows the same distribution as (9), where* ϕ *is replaced with* $\hat{\phi}$.

Proposition 3 shows that the power-law tail distribution with the same exponent is obtained even in the general setup where the indivisibility and depreciation rates are heterogeneous across firms. This is an important generalization for the business cycle model, as empirical studies imply large variations in the lumpiness in the investment-to-capital ratio across firms (Doms and Dunne (1998); Cooper, Haltiwanger, and Power (1999)). It is also a necessary extension for this study because the uniform distribution of $s_{i,t}$ is proven when λ_i has a non-trivial density, as shown next.

So far, I have assumed that the capital gap distribution follows a uniform distribution. In the heterogeneous setup, it can be shown that the capital gap distribution converges to the uniform distribution. This gap $s_{i,t}$ always takes a value between 0 and 1 at equilibrium under the boundedness condition of $a_{i,t+1}$ and the stationarity of w_t and R_t. The gap develops as

$$s_{i,t+1} = \left(\frac{\log(1-\delta_i) + \log k^*_{i,t} - \log k^*_{i,t+1}}{\log \lambda_i} + s_{i,t} + 1 \right) \bmod 1, \tag{13}$$

where "x mod 1" denotes the remainder after the division of x by 1. Starting from an initial state $s_{i,0}$, $s_{i,t}$ is given as the natural depreciation $t \log(1-\delta_i)$ divided by $\log \lambda_i$, plus a random variable, and taken modulo 1. This remainder converges to a uniform distribution on a unit interval (Engel, 1992, 3.1.1).

Proposition 4 *If* $(\log(1-\delta_i))/\log \lambda_i$ *has a non-degenerate density,* $s_{i,t}$ *converges in distribution to a uniform random variable in* $[0, 1)$ *as* $t \to \infty$*.*

Proposition 4 is proven similarly as in Caballero and Engel (1991). As in Caplin and Spulber (1987), the cross-section distribution of $s_{i,t}$ stays at the uniform distribution even if aggregate variables fluctuate because a shift in K_t merely rotates the distribution of $s_{i,t}$ on a circle of unit circumference.[4]

2.7. *Relation to previous research*

The power-law tail with exponent 0.5 characterizes the aggregate fluctuations even in a heterogeneous extension of the model (Section 2.6). The robustness of the exponent 0.5 results from the fact that *any* branching process with a martingale property (i.e., $\phi = 1$) brings out the power-law tail with exponent 0.5 for the cumulative population size (Harris, 1989, p. 32).[5] The robustness reflects the fact that in various models of connected non-linear dynamics, the critical level of complementarity $\phi = 1$ appears as a condition for idiosyncratic shocks to have aggregate consequences through power-law distributions. For example, in a celebrated theorem by Erdős and Rényi, the condition $\phi = 1$ corresponds to the critical point for the emergence of a "giant cluster" in a random graph (Bollobás, 1998, p. 240).

In the literature on economic fluctuations, Jovanovic (1987) demonstrated in several simple models that aggregate fluctuations could be generated by interactions of idiosyncratic shocks. Notably, he pointed out that a key condition for the aggregate risks to emerge from interacting idiosyncratic shocks is that "the effect that a unit increase in the average decision of others has on [an individual decision]" is 1.[6] This corresponds to the criticality condition $\phi = 1$.

[4]When the support of the distribution of $a_{i,t}$ is broader than what is assumed, Proposition 4 still holds if the firm's capital choice set is broadened as $\{\lambda^{\pm\kappa}(1-\delta)k_{i,t}\}$, $\kappa = 0, 1, \ldots, \bar{\kappa}$ for properly set $\bar{\kappa}$.

[5]The distribution of population size in the branching processes is closely related to the distribution of the first return time of a random walk, which has the same power-law exponent 0.5.

[6]Jovanovic (1987, p. 403)

He shows some examples in which a "multiplier" effect of an individual's action has an $\sqrt{N}$ order of magnitude. Combined with the multiplier effect, idiosyncratic shocks, which shrink in aggregation as $1/\sqrt{N}$, can generate non-vanishing aggregate fluctuations. In the case of adjustments on the extensive margin as featured in this model, the propagation effect (i.e., how many firms are affected) becomes stochastic rather than a constant multiplier. This study develops Jovanovic's insight and fully characterizes the fluctuations on the extensive margin. The analysis shows that the *variation* of the propagation effect, rather than the mean, has an order of magnitude $\sqrt{N}$.[7]

In a general model of industries with binary technological choice and complementarities, Durlauf (1993) showed that the degree of complementarities determines whether an economy has a unique equilibrium or multiple equilibria. The present model is narrower than his in that the firm's behavioral rule is parametrically specified and in that the firms interact only through aggregate capital. The analysis here, however, differs in its aim. By specifying an equilibrium selection mechanism, this model excludes the fluctuations from multiple equilibria and concentrates on the least-volatile ones. While Durlauf's paper explains long-run phenomena such as industrialization, this study is concerned with short-run fluctuations such as business cycles and derives the distribution of aggregate fluctuations.

The possibility of a power-law distribution of sectoral propagation was first pointed out by Bak, Chen, Scheinkman, and Woodford (1993). In a model of a simple supply chain with a lattice network, they obtained a power-law distribution of aggregate fluctuations. Nirei (2006) implemented their fluctuation mechanism in an equilibrium model of a globally connected network where an agent's action affects all other agents.[8] A fluctuation distribution similar to (9) was obtained in that paper. This current study extends the previous one by proving the non-zero asymptotic variance of the aggregate growth rate for the case of $\phi = 1$. The mechanism for the break from the law of large numbers is analogous to Jovanovic's: the variation of the extensive margin (the number of firms affected) implied by the distribution (9) is scaled as $\sqrt{N}$, which cancels out with the shrinking magnitude $1/\sqrt{N}$ (implied by the law of large numbers)

[7] Gabaix (2011) points out that Jovanovic's multiplier is too large, especially when the multiplier is applicable to aggregate shocks. Whether the same criticism applies to the present model depends on how the aggregate shock is modeled. For example, in the general equilibrium model in Section 3, any common shock in productivity is mitigated by shifts in factor prices and expected aggregate capital; furthermore, such a shock does not alter the fluctuation of unanticipated capital growth.

[8] A corollary difference between Bak, Chen, Scheinkman, and Woodford (1993) and Nirei (2006) occurs in the exponent of the power-law distribution, which arises from the varied network topology. Bak et al. feature a two-dimensional lattice network in which two avalanches starting from neighboring sites can overlap. This leads to a longer chain reaction and a lower power-law exponent (1/3). In contrast, market equilibrium models as in Nirei (2006) and this paper feature an essentially dimensionless network of firms. Thus, the market model corresponds to an infinite-dimension case of lattice models, which yields the cluster-volume exponent 0.5 at criticality (Grimmett, 1999, p. 256).

of the initial disturbances caused by capital depreciation. In addition, this study is placed within a standard real business cycle framework (Section 3), which underpins three key assumptions in this section: the criticality $\phi = 1$, the uniform distribution of the gap $s_{i,t}$, and the well-defined expectation formation K^*_{t+1}.

This model may be viewed as a self-organized criticality model, as advocated by Bak, Tang, and Wiesenfeld (1987). In that interpretation, the criticality in this model is the uniform distribution of $s_{i,t}$. When the density of $s_{i,t}$ at the threshold is greater than 1, a large propagation of investments ensues. When the density at the threshold is smaller than 1, little propagation occurs. In either case, diffusion effects caused by productivity $a_{i,t}$ and heterogeneous indivisibility λ_i bring the density at the threshold to 1, where the size of the propagation follows a power-law distribution. However, the key condition $\phi = 1$ is set exogenously. This study claims that the aggregate fluctuations arise from idiosyncratic shocks in an environment where individual investment thresholds are linearly dependent on aggregate capital, that is, when $\phi = 1$ is realized. While this is a restrictive condition, there are several important examples that satisfy this condition in an economy, as shown by Jovanovic (1987). In this section, I showed that the firms' investment decisions exhibit a critical level of complementarity under constant returns to scale in an equilibrium given factor prices. In the next section, I present a general equilibrium example in which the critical level of complementarity of investments continues to hold when the factor prices are determined endogenously.

3. A BUSINESS CYCLE MODEL

In this section, I construct a dynamic general equilibrium model with indivisible investments, predetermined prices of goods, and constant returns to scale. The model is shown to satisfy the premises in the previous section: (1) K^*_{t+1} is rationally determined, (2) $s_{i,t+1}$ follows a uniform distribution, and (3) $\phi = 1$. Then, the equilibrium paths are numerically simulated.

3.1. *Households*

There is a representative household that maximizes utility $\mathrm{E}_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} U(C_\tau, L_\tau)\right]$ by choosing consumption C_τ and labor supply L_τ subject to $C_\tau = w_\tau L_\tau + D_\tau$, $\forall\tau$. Here D_t denotes aggregate dividends that the household receives from firms. Each firm i owns capital and delivers dividend $d_{i,t}$. Households as shareholders instruct each firm to maximize its expected discounted sum of dividends stream $\mathrm{E}_t \left[\sum_{\tau=t}^{\infty} \Delta_{t,\tau} d_{i,\tau}\right]$. The discount factor is $\Delta_{t,\tau} \equiv \Pi_{s=t+1}^{\tau} R_s^{-1}$, where $\Delta_{t,t} = 1$ by convention and R_t is the inverse of a stochastic discount factor

$$R_t^{-1} \equiv \beta U_{C_t} / U_{C_{t-1}}. \tag{14}$$

The real wage is equal to the marginal rate of substitution between L_t and C_t by the maximization conditions:

$$w_t = -U_{L_t}/U_{C_t}. \tag{15}$$

3.2. *Firms*

The production of firms is specified as in Section 2.1. In this section, I assume constant returns to scale ($\gamma = 1-\alpha$), and I allow the indivisibility parameter λ_i to be heterogeneous across i. At the initial period, λ_i is drawn from a continuous density function over the support $(1/(1-\delta), \infty)$. I assume that the mean of $\lambda_i - 1$ is smaller than the output-to-capital ratio so that the resource constraint is always satisfied. Once λ_i is drawn, it does not change over periods. Productivity $a_{i,t}$ is an i.i.d. random variable that satisfies the boundedness condition in Section 2.6.

I also assume that firm i commits to the price $p_{i,t+1}$ of its product one period ahead. Namely, firm i decides $p_{i,t+1}$ in period t. This assumption of predetermined goods prices is necessary for the investment fluctuations to have propagation effects on other macroeconomic variables. The aggregate price is still normalized to 1. This normalization is innocuous even with the predetermined prices because all the firms decide the goods prices simultaneously. The real wage w_t is flexible. The time protocol is set as follows. At the beginning of period t, productivities $a_{i,t+1}$ are revealed to all firms. Next, firm i decides its price $p_{i,t+1}$ and capital $k_{i,t+1}$ for the next period, while next-period aggregate capital K_{t+1} and contemporaneous investment X_t are determined simultaneously. Finally, contemporaneous $y_{i,t}$, $l_{i,t}$, $d_{i,t}$, and C_t are determined given X_t.

Firm i's problem in period t is to maximize $\mathrm{E}_t\left[\sum_{\tau=t}^{\infty} \Delta_{t,\tau} d_{i,\tau}\right]$ by choosing $p_{i,t+1}$ and $k_{i,t+1}$ subject to the demand function, the production function, and the discrete constraint for capital. The optimal price is solved by maximizing the dividend in period $t+1$ as $p_{i,t+1} = (a_{i,t+1}^{1/\alpha} k_{i,t+1}/K_{t+1})^{-\alpha/(\eta(1-c_L))}$. Substituting $p_{i,t+1}$ in the demand function and aggregating across i, I obtain[9]

$$K_{t+1} = \left(\mathrm{E}_t[(w_{t+1}/c_L) Y_{t+1}^{1/(1-\alpha)}/R_{t+1}]/\mathrm{E}_t[Y_{t+1}/R_{t+1}]\right)^{(1-\alpha)/\alpha}. \tag{16}$$

Using the optimal price, firm i's problem in period t is choosing $k_{i,t+1}$ from a discrete set $\{\lambda_i^{\kappa}(1-\delta)k_{i,t}\}_{\kappa=0,\pm1}$, as in the previous section, to maximize $\pi(k_{i,t+1}) = (1-c_L)\mathrm{E}_t[Y_{t+1}/R_{t+1}] a_{i,t+1}^{\rho/\alpha}(k_{i,t+1}/K_{t+1})^{\rho} - (1-(1-\delta)\mathrm{E}_t[R_{t+1}^{-1}])k_{i,t+1}$. The optimal strategy for firm i is to invest in period t when $(1-\delta)k_{i,t}$ is below a lower threshold $k_{i,t+1}^*$, to divest when $(1-\delta)k_{i,t}$ is above an upper threshold $\lambda_i k_{i,t+1}^*$, and to not adjust otherwise. Then, proceeding as in Section 2.1, I

[9] See Appendix B for details of the derivation.

determine $k^*_{i,t+1}$ as follows:

$$k^*_{i,t+1} = b_{i,t+1} K_{t+1}, \tag{17}$$

$$b_{i,t+1} \equiv A_t \left(a_{i,t+1}^{\rho/\alpha} (\lambda_i^\rho - 1)/(\lambda_i - 1) \right)^{1/(1-\rho)}, \tag{18}$$

$$A_t \equiv \left(\frac{(1-c_L)\mathrm{E}_t[Y_{t+1}/R_{t+1}]^{\frac{1}{\alpha}}}{(1-(1-\delta)\mathrm{E}_t[R_{t+1}^{-1}])\mathrm{E}_t[(w_{t+1}/c_L) Y_{t+1}^{\frac{1}{1-\alpha}}/R_{t+1}]^{\frac{1-\alpha}{\alpha}}} \right)^{\frac{1}{1-\rho}}. \tag{19}$$

Note that A_t summarizes the environment of aggregate demand and factor prices in period $t+1$, expected conditionally on the information available to firms in period t.

3.3. *General Equilibrium*

The labor-market-clearing condition is $L_t = \sum_{i=1}^N l_{i,t}/N$. By substituting the price-setting rule in labor demand $l_{i,t} = (p_{i,t}^{-\eta} Y_t/(a_{i,t} k_{i,t}^\alpha))^{1/(1-\alpha)}$ and aggregating, I obtain the following aggregate production function:

$$Y_t = K_t^\alpha L_t^{1-\alpha}. \tag{20}$$

The final goods market clears as

$$Y_t = C_t + X_t, \tag{21}$$

where $X_t \equiv \sum_{i=1}^N x_{i,t}/N$ is aggregate investment. Because there are only a finite number N of firms, the economy experiences some fluctuations due to finite idiosyncratic shocks. I show that the fluctuation of aggregate investment X_t remains non-trivial even when N is large. When X_t differs from the expected level because of finite shocks, firms adjust their labor demand and the labor market clears by adjusting the wage. Thus, under predetermined prices, the investment fluctuation causes quantity adjustments in the hours worked, production, and consumption.

Equilibrium conditions are derived as (14, 15, 16, 17, 20, 21) and

$$K_{t+1} = \left(\sum_{i=1}^N \frac{(\lambda_i^{\kappa_{i,t+1}} (1-\delta) a_{i,t}^{1/\alpha} k_{i,t})^\rho}{N} \right)^{1/\rho}, \tag{22}$$

$$X_t = \sum_{i=1}^N (\lambda_i^{\kappa_{i,t+1}} - 1)(1-\delta) k_{i,t}/N, \tag{23}$$

$$1 = \left(\sum_{i=1}^N a_{i,t+1}^{\rho/(\alpha(1-\rho))} \left(\frac{\lambda_i^\rho - 1}{\lambda_i - 1} \right)^{\rho/(1-\rho)} \frac{\lambda_i^{\rho s_{i,t+1}}}{N} \right)^{1/\rho} A_t, \tag{24}$$

where the last condition is derived from $K_{t+1} = (\sum_{i=1}^{N} a_{i,t+1}^{\rho/\alpha}(\lambda_i^{s_{i,t+1}} k_{i,t+1}^*)^\rho/N)^{1/\rho}$.

The state space involves the distribution of the gap $s_{i,t}$, which is included in the information set for the conditional expectation in period t and affects the summations in (22–24). Because the gap profile has large dimensions N, it is difficult to solve the equilibrium exactly. Thus, I approximate the equilibrium system using the stationary distributions of $s_{i,t}$ and $a_{i,t}$ with a continuum of firms. Using this approximation, I replace the summations across i in (22–24) with integrals over the uniform distribution of $s_{i,t}$. Note that Proposition 4 applies to this model, warranting the convergence of $s_{i,t}$ to the uniform distribution.

I assume that agents use the approximated equilibrium system to form expectations of future variables, whereas the exact realizations of K_{t+1} and X_t are determined by (22, 23) while keeping the summations. Then, the threshold becomes a function of only idiosyncratic productivity and aggregate capital:

$$k_{i,t+1}^*/K_{t+1} = b_{i,t+1} = A\left(a_{i,t+1}^{\rho/\alpha}(\lambda_i^\rho - 1)/(\lambda_i - 1)\right)^{1/(1-\rho)}, \tag{25}$$

$$A \equiv \left(\int \left(\frac{\lambda_i^\rho - 1}{(\lambda_i - 1)^\rho}\right)^{1/(1-\rho)} \frac{a_{i,t+1}^{\rho/(\alpha(1-\rho))}}{\rho \log \lambda_i} di\right)^{-1/\rho}. \tag{26}$$

Note that (25) satisfies the condition $\phi = 1$ in Section 2.1. The effect of factor prices on the threshold tends to the constant, A. This constant effect of factor prices along with constant returns to scale technology results in the critical level of complementarity of the investment decision. The system of equations for the agents' expectation becomes (14, 15, 16, 20, 21) and

$$A^{1-\rho} = \frac{(1-c_L)\mathrm{E}_t[Y_{t+1}/R_{t+1}]^{1/\alpha}}{(1-(1-\delta)\mathrm{E}_t[R_{t+1}^{-1}])\mathrm{E}_t[(w_{t+1}/c_L)Y_{t+1}^{1/(1-\alpha)}/R_{t+1}]^{(1-\alpha)/\alpha}}, \tag{27}$$

$$K_{t+1} = (1-\delta)K_t + (A^{\rho-1}/\rho)X_t \text{ (which is derived in Appendix B)}, \tag{28}$$

$$X_t = \mathrm{E}_{t-1}[X_t]e^{\epsilon_t}. \tag{29}$$

Here, X_t is multiplied by $A^{\rho-1}/\rho$ in the capital accumulation because K_t is the average of $k_{i,t}$ weighted by productivity. Entering (29) is the aggregate investment shock ϵ_t, defined as the log-difference between realized and expected investments. Note that ϵ_t signifies the shock on firms' demand for investment goods. Finally, I specify the utility function using the King-Plosser-Rebelo preference

$$U(C_t, L_t) = C_t^{1-\sigma}(1-\psi L_t^\zeta)^{1-\sigma}/(1-\sigma). \tag{30}$$

The expectation system can be approximated in the first order as shown in Appendix A. Using a standard procedure, I establish the following proposition, where a bar over a variable denotes a steady-state value.

Proposition 5 *There exists a unique saddle-point path for the expectation system if $\bar{X}/\bar{Y} \leq \alpha$ holds.*

Using this proposition, the expectation system has a determinate solution. Combined with ϵ_t, the equilibrium path fluctuates around the determinate saddle-point path.

3.4. *Aggregate investment shock*

In a finite economy, the aggregate investment shock ϵ_t is defined as the log-difference between realized aggregate investment X_t and expected aggregate investment $\mathrm{E}_{t-1}[X_t]$. I determine X_t along with K_{t+1} and $k^*_{i,t+1}$, using (22, 23, 25) given exact capital $k_{i,t}$ and realized productivity $a_{i,t+1}$. I determine $\mathrm{E}_{t-1}[X_t]$, using the expectation system (14, 15, 16, 20, 21, 27, 28, 29) given K_t. The deviation of actual aggregate investment from the expected value is caused by idiosyncratic productivity shocks $a_{i,t+1}$ for a finite number of firms and the deviation of the gap distribution from the uniform distribution.

Because of the non-linear decision of $k_{i,t+1}$ with strategic complementarity across i, there can be multiple solutions for (22, 25) for each state $(k_{i,t}, a_{i,t+1})_{i=1}^N$. For those cases, I use Equilibrium Selection 1 that picks the solution that minimizes $|\epsilon_t|$ among all solutions. This selection rule picks the equilibrium capital path that minimizes the deviation from the expected one. The benchmark level of capital K^*_{t+1} that I used to select equilibria in Section 2 now corresponds to the rationally expected level of capital $\mathrm{E}_{t-1}[K_{t+1}] = (1-\delta)K_t + (A^{\rho-1}/\rho)\mathrm{E}_{t-1}[X_t]$.

3.5. *Calibration and numerical simulations*

For a benchmark calibration, I set the unit of time to quarters. The parameters for production technology and households' preferences are set as in Table I. Details on the calibration are in Appendix C. Table II reports the standard deviations and comovement structure of simulated output, consumption, investment, hours worked, and capital. As can be seen, the model is able to generate aggregate investment fluctuations to a magnitude comparable to business cycles. The fluctuations in aggregate variables are driven mostly by investment shocks ϵ_t, while movements in capital play a very small quantitative role. The standard deviation of $\log K$ is less than 0.3% in the table.[10]

TABLE I

Benchmark calibration and endogenous steady-state values

α	δ	σ	β	η	ψ	ζ	$\mathrm{E}(\lambda_i)$	$\mathrm{Std}(\log a_{i,t})$	N	$\bar{w}L/Y$	C/Y
0.26	0.02	1.5	0.99	11	1	2	1.028	0.05%	110000	0.67	0.84

The standard deviation of ϵ_t, which determines the standard deviation of $\tilde{X}$, is almost proportionally related to the size of indivisibility $\log \lambda$. This can be

[10] For the sake of comparison, the standard deviation of $\log K$ for ES2 is calculated using the analytical distribution of M. It is calculated as 1.9% under the calibrated parameter values while ignoring the dispersion of λ_i and setting $q = -\log(1-\delta)/\log \lambda$.

TABLE II

STANDARD DEVIATIONS AND CORRELATIONS OF BUSINESS CYCLE VARIABLES

	Standard deviation (%)					Correlation with $\tilde{Y}$			
	$\tilde{Y}$	$\tilde{C}$	$\tilde{X}$	$\tilde{L}$	$\tilde{K}$	$\tilde{C}$	$\tilde{X}$	$\tilde{L}$	$\tilde{K}$
Benchmark	2.28	0.86	6.60	3.06	0.26	0.822	0.979	0.795	-0.018
(std errors)	(0.09)	(0.03)	(0.28)	(0.11)	(0.01)	(0.007)	(0.001)	(0.006)	(0.009)
$\mathrm{E}[\lambda_i] = 1.056$	4.67	1.69	13.52	5.99	0.55	0.840	0.980	0.813	-0.004
$N = 350000$	2.23	0.85	6.44	3.02	0.26	0.820	0.979	0.793	-0.020
$\sigma = 3$	3.49	2.80	6.45	5.21	0.26	0.930	0.905	0.925	-0.155

seen in Table II for the case of $\mathrm{E}[\lambda_i] = 1.056$, for which the indivisibility is twice the benchmark case. This result agrees with the analytical result, suggesting that the asymptotic aggregate standard deviation of capital growth decreases almost proportionally when $\log \lambda$ is lowered. The aggregate fluctuation is subdued because capital can closely keep track of idiosyncratic productivity with little indivisibility. The size of indivisibility, rather than the size of productivity shocks, determines the magnitude of idiosyncratic volatility in this model.

Simulations with an increased N in Table II exhibit little reduction in the magnitude of fluctuations compared with the benchmark. That is, the diversification effect of large N is weak for ES1 in the calibrated range of parameter values. This implies that even though Proposition 1 does not establish a non-vanishing variance of g_N, the convergence of g_N is sufficiently slow that fluctuations for ES1 potentially match with business cycle fluctuations in terms of their magnitude.

3.6. *Discussion*

The investment shock ϵ_t propagates to other variables in two paths: K_{t+1} and Y_t. On one hand, an investment shock generates an exogenous increase in future capital K_{t+1}. This raises future labor productivity and the real wage. The prospect of increased labor productivity induces households to consume more in both the current and following periods. This effect can be seen in the saddle-point path, where the marginal utility of consumption is negatively related to capital. On the other hand, an increase in investment raises aggregate demand for contemporaneous goods, if consumption demand is unaffected. Firms respond to the increased demand by increasing labor demand, which raises the real wage. Households respond to the higher real wage by raising hours worked, which in turn raises the marginal utility of consumption when $\sigma > 1$. Thus, in order to keep the marginal utility lower so that it is on the saddle-point path, consumption demand must increase. Hence, the investment shock raises consumption and thus output. In Table II, I observe that the standard deviation of consumption relative to investment is larger when σ is greater. This result is consistent with the propagation mechanism previously described because the hours–consumption complementarity becomes larger when $\sigma - 1$ is greater, given a fixed marginal utility of consumption.

The predetermined price provides a key environment for the investment–consumption comovement, as previously described. With predetermined prices, firms are committed to accommodating demand shocks using only output. Thus, an increase in aggregate investment causes firms to hire more, which raises contemporaneous consumption. In contrast, under flexible prices, firms are able to increase their prices and suppress output when a demand shock occurs. Thus, as in Thomas (2002), an increase in investment raises factor prices and dampens production. The key difference from the flexible prices model is that the efficient hiring condition (16) holds only in terms of expectations in the predetermined prices model.

The dynamic general equilibrium model in this section underpins the assumptions made in the previous section where factor prices are exogenously given. The critical level of complementarity of investments (a key condition for non-vanishing aggregate fluctuations) is shown under the assumption of constant returns to scale, where the effect of factor prices on the investment rule becomes constant. The dynamic general equilibrium model also provides a well-defined expected aggregate capital K^*_{t+1}, which is determined by the unique saddle-point path of the expectation system. Moreover, it generates the self-organization of the gap distribution toward the uniform distribution at which the power-law propagation effects emerge. Simulated gap distributions show little deviation from the uniform distribution. Simulations using the exact gap distribution instead of the uniform distribution did not significantly improve firms' prediction power over future factor prices.

The model presented here can be a departure point for various extensions. The model can be extended by incorporating firms that adjust capital smoothly. Because the capital choice of the smoothly adjusting firm is proportional to aggregate capital given factor prices, the results of the model are not affected when such firms enter symmetrically as those with an indivisibility constraint. It is also possible to incorporate some firms that flexibly adjust goods prices. The flexible pricing does not alter the functional form of the optimal threshold, but it changes the contemporaneous response of factor prices when an aggregate investment shock occurs. Another extension is to introduce a fixed adjustment cost that endogenizes the indivisibility of investments as studied in (S,s) models. In this study, aggregate fluctuations stemming from interactions occur when the density of firms at the threshold of $s_{i,t}$ is 1. This level of density holds under the uniform distribution, which is generated by the one-sided (S,s) rule.[11] Given that capital is constantly depreciated, the one-sided rule holds for investment with a fixed adjustment cost at least in the short term, where productivity shocks are small. However, the density condition may well not hold in a general (S,s) economy. One limitation of the model presented here as a business cycle model is that it does not generate quantitatively large autocorrelation. I leave

[11] See Nirei (2006) for the distribution of aggregate capital when the threshold density is not equal to 1.

the incorporation of a mechanism that generates persistence to future research.

4. CONCLUSION

This paper characterizes the aggregate fluctuations arising from the complementarity of indivisible investments at the firm level. Analytically, I propose a method to evaluate the fluctuation of aggregate investment along the evolution of heterogeneous capital as if it were a stochastic fluctuation whose randomness arises from the stochastic configuration of relative capital levels. For each configuration, the equilibrium aggregate investment is determined as a convergent point of a fictitious best-response dynamics of firms' investment decisions. The best-response dynamics can be embedded in a branching process with a probability measure of the stochastic configuration of relative capital. This enables us to derive the distribution function of the aggregate fluctuation in a closed form.

The fluctuation in the number of investing firms is shown to follow a power-law distribution with an exponential truncation at the tail. The truncation speed is determined using the degree of strategic complementarity among firms. Under the constant returns to scale assumption, the distribution becomes a pure power law, and the standard deviation of the growth rate of aggregate capital is shown to be strictly positive even when there are an infinite number of firms. The limit of the standard deviation is shown to be directly affected by the indivisibility of firm-level investments.

I incorporate this fluctuation mechanism in a dynamic general equilibrium model, and I numerically compute equilibrium paths without making the randomness assumption of the capital configuration. Under plausible parameter values, the equilibrium path is shown to exhibit aggregate fluctuations comparable in magnitude and cross-correlation structure to business cycles. The dynamic general equilibrium model presented here does not provide a full account of business cycles because it lacks important dimensions, such as autocorrelations. Nonetheless, the model highlights the possibility that interactions of idiosyncratic shocks may cause aggregate fluctuations in a realistically calibrated environment.

APPENDIX A: PROOFS

Proof of Lemma 1

Let H_u, $u = 2, 3, \dots, T$ denote the set of firms that adjust capital in step u. Assume that the size of H_u is finite with probability 1 when $N \to \infty$, which I verify later. I consider the case $m_1 > 0$ without loss of generality. Thus, $\log k_{i,u} = \log k_{i,u-1} + \log \lambda$ for $i \in H_u$.

The Taylor series expansion of $N(\log K_{u+1} - \log K_u)$ around $(\log k_u)_{i \in H_{u+1}}$ is calculated as follows. The first derivative is $\partial N \log K_u / \partial \log k_{i,u} = (k_{i,u}/K_u)^\rho$. Thus, $\partial K_u / \partial k_{i,u}$ is of order $1/N$.[12] The second and higher derivatives with respect to $\log k_{i,u}$ are $\partial^n (k_{i,u}/K_u)^\rho / \partial \log k_{i,u}^n = \rho^n (k_{i,u}/K_u)^\rho + O(\partial K_u / \partial k_{i,u})$ for $n = 1, 2, \dots$. The second cross-derivatives, $\partial^2 \log K_u / (\partial \log k_{i,u} \partial \log k_{j,u})$,

[12] By taking y_N of order x_N, or interchangeably, $y_N = O(x_N)$, I mean that y_N / x_N converges to a finite number as $N \to \infty$.

are of order $\partial K_u/\partial k_{j,u}$, and thus, $O(1/N)$. Similarly, the higher-order cross-derivative terms with respect to the capital of h distinct firms in H_{u+1} are of order $1/N^{h-1}$. Because H_{u+1} is finite, the n-th derivative of $N \log K_u$ has a finite number of cross derivative terms for any finite n. Hence, a Taylor series expansion of $N(\log K_{u+1} - \log K_u)$ yields

$$\sum_{n=1}^{\infty} \sum_{i \in H_{u+1}} \left(\frac{k_{i,u}}{K_u}\right)^{\rho} \frac{\rho^{n-1}(\log \lambda)^n}{n!} + O(1/N) = \frac{\lambda^{\rho} - 1}{\rho} \sum_{i \in H_{u+1}} \left(\frac{k_{i,u}}{K_u}\right)^{\rho} + O(1/N),$$

where I used $\lambda^{\rho} = \lambda^0 + \sum_{n=1}^{\infty}(d^n \lambda^{\rho}/d\rho^n)|_{\rho=0}(\rho^n/n!)$. Using $k_{i,u} = k_u^* \lambda^{s_{i,u}}$, I obtain that $\sum_{i \in H_{u+1}}(k_{i,u}/K_u)^{\rho} = (\sum_{i \in H_{u+1}} \lambda^{s_{i,u}\rho})/(\sum_{i=1}^{N} \lambda^{s_{i,u}\rho}/N)$. The denominator converges to $\mathrm{E}[\lambda^{s_{i,u}\rho}]$ as $N \to \infty$ almost surely by the law of large numbers, and I have $\mathrm{E}[\lambda^{s_{i,u}\rho}] = \int_0^1 \lambda^{s_{i,u}\rho} ds_{i,u} = (\lambda^{\rho} - 1)/(\rho \log \lambda)$. The numerator, $\sum_{i \in H_{u+1}} \lambda^{s_{i,u}\rho}$, converges to m_{u+1} for every event when H_{u+1} is finite because $s_{i,u}$ is smaller than $\phi(\log K_u - \log K_{u-1})/\log \lambda$ for any $i \in H_{u+1}$. Thus, $\lambda^{s_{i,u}}$ converges to 1 as $N \to \infty$. This completes the proof.

Proof of Lemma 2

The conditional probability for firm i to invest in $u = 2, 3, \ldots, T$ is

$$\Pr(i \in H_u \mid i \notin \cup_{v=2,3,\ldots,u-1} H_v) = \frac{\phi(\log K_u - \log K_{u-1})/\log \lambda}{1 - \phi(\log K_{u-1} - \log K_0)/\log \lambda}. \tag{31}$$

Thus, m_u follows a binomial distribution with population $N - \sum_{v=2}^{u-1} m_v$ and probability (31). The mean of m_u converges to ϕm_{u-1} as $N \to \infty$, by using Lemma 1. Then, the binomial distribution of m_u converges to a Poisson distribution with mean ϕm_{u-1} for $u = 2, 3, \ldots, T$. Since a Poisson distribution is infinitely divisible, the Poisson variable with mean ϕm_{u-1} is equivalent to a m_{u-1}-times convolution of a Poisson variable with mean ϕ. Thus, the process m_u for $u = 2, 3, \ldots, T$ is a branching process with a Poisson random variable with mean ϕ, where m_2 follows a Poisson distribution with mean ϕm_1. Note that m_1 is not included in the branching process because it is not necessarily an integer.

Proof of Lemma 3

It is known that the accumulated sum $M = \sum_{u=2}^{T} m_u$ of the Poisson branching process conditional on m_2 follows an infinitely divisible distribution called the Borel-Tanner distribution (Kingman, 1993, p. 68):

$$\Pr(M = m \mid m_2) = (m_2/m)e^{-\phi m}(\phi m)^{m - m_2}/(m - m_2)! \tag{32}$$

for $m = m_2, m_2 + 1, \ldots$. By combining (32) with m_2, which follows the Poisson distribution with mean ϕm_1, and by using the binomial theorem in the summation over m_2, I obtain (9) as follows.

$$\begin{aligned} \Pr(M = m \mid m_1) &= \sum_{m_2=1}^{m} \frac{(m_2/m)e^{-\phi m}(\phi m)^{m-m_2}}{(m-m_2)!} \frac{e^{-\phi m_1}(\phi m_1)^{m_2}}{m_2!} \\ &= \frac{m_1 e^{-\phi(m+m_1)}\phi^m}{m(m-1)!} \sum_{m_2=1}^{m} \frac{(m-1)!}{(m-m_2)!(m_2-1)!} m^{m-m_2} m_1^{m_2-1} \\ &= \frac{m_1 e^{-\phi(m+m_1)}\phi^m}{m!}(m+m_1)^{m-1} \end{aligned} \tag{33}$$

Furthermore, the approximation in (10) is obtained by applying Stirling's formula $m! \sim \sqrt{2\pi} e^{-m} m^{m+0.5}$ and the fact that $(1 + m_1/m)^{m-1} \to e^{m_1}$ as $m \to \infty$.

Proof of Lemma 4

I split $m_1/\sqrt{N}$ into three terms as $\log\Gamma(K_0)-\log(\sum_{i=1}^N((1-\delta)k_{i,0})^\rho/N)^{1/\rho}$, $\log(\sum_{i=1}^N((1-\delta)k_{i,0})^\rho/N)^{1/\rho}-\log K_{-1}$, and $\log K_{-1}-\log K_0$, all multiplied by $\sqrt{N}/\log\lambda$, where (K_{-1},K_0) corresponds to (K_t,K^*_{t+1}) in the model. The second term represents depreciation and is equal to $(\sqrt{N}/\log\lambda)\log(1-\delta)$. Thus, the sum of the second and third terms yields $-q\sqrt{N}$. The first term represents the first-step adjustments induced directly by depreciation. Define H_1 as the set of firms that adjust in the first step. Using $k_{i,0}=\lambda^{s_{i,0}}k_0^*$, I obtain $K_1=(1-\delta)k_0^*((\lambda^\rho-1)\sum_{i\in H_1}\lambda^{s_{i,0}\rho}/N+\sum_{i=1}^N\lambda^{s_{i,0}\rho}/N)^{1/\rho}$ and $(\sum_{i=1}^N((1-\delta)k_{i,0})^\rho/N)^{1/\rho}=(1-\delta)k_0^*(\sum_{i=1}^N\lambda^{s_{i,0}\rho}/N)^{1/\rho}$. Hence, the first term of $m_1/\sqrt{N}$ becomes

$$\frac{\sqrt{N}}{\rho\log\lambda}\log\left((\lambda^\rho-1)\frac{\sum_{i\in H_1}\lambda^{s_{i,0}\rho}/N}{\sum_{i=1}^N\lambda^{s_{i,0}\rho}/N}+1\right). \tag{34}$$

By assumption, $s_{i,0}$ is distributed uniformly. Thus, the denominator $\sum_{i=1}^N\lambda^{s_{i,0}\rho}/N$ in (34) converges to $\int_0^1\lambda^{s_{i,0}\rho}ds_{i,0}=(\lambda^\rho-1)/(\rho\log\lambda)$ with probability 1 by the law of large numbers. Let x denote the numerator: $x\equiv\sum_{i\in H_1}\lambda^{s_{i,0}\rho}/N$. Here, $i\in H_1$ is equivalent to $0\le s_{i,0}<q$. Then, the asymptotic mean of x is $x_0=\int_0^q\lambda^{s_{i,0}\rho}ds_{i,0}=(\lambda^{\rho q}-1)/(\rho\log\lambda)$. By the central limit theorem, $\sqrt{N}(x-x_0)$ converges in distribution to the normal distribution with mean 0 and variance

$$\int_0^q(\lambda^{s_{i,0}\rho})^2\,ds_{i,0}-\left(\frac{\lambda^{\rho q}-1}{\rho\log\lambda}\right)^2=\frac{\lambda^{2\rho q}-1}{2\rho\log\lambda}-\left(\frac{\lambda^{\rho q}-1}{\rho\log\lambda}\right)^2. \tag{35}$$

I regard (34) as a function F of x. Using the delta method, I determine that $F(x)$ asymptotically follows a normal distribution with mean $F(x_0)$ and variance $F'(x_0)^2\mathrm{Avar}(x)$. Note that $F(x_0)$ is calculated as

$$\frac{\sqrt{N}}{\rho\log\lambda}\log\left(\frac{(\lambda^\rho-1)(\lambda^{\rho q}-1)/(\rho\log\lambda)}{(\lambda^\rho-1)/(\rho\log\lambda)}+1\right)=q\sqrt{N}. \tag{36}$$

This cancels out with the second and third terms of $m_1/\sqrt{N}$. Also note that $F'(x_0)^2\mathrm{Avar}(x)$ is calculated as σ_1^2 in the proposition. Then, $m_1/\sqrt{N}$ asymptotically follows a normal distribution with mean 0 and variance σ_1^2.

Proof of Proposition 1

As shown in Figure 1, K^1 is defined as a fixed point on the other side of K^2 (selected by ES2) across K^*. The interval between K^1 and K^* is divided by an interior point (denoted by K^a), where the aggregate reaction function Γ crosses the 45-degree line vertically. Define M_a as the number of firms that adjust between $\Gamma(K^*)$ and K^a, and M_b as the number of firms that adjust between K^a and K^1. Using Lemma 1, $(N/\log\lambda)(\log K^1-\log K^*)+m_1$ asymptotes to M_a+M_b.

The function $(N/\log\lambda)\Gamma(K)$ is regarded as a realized path of a Poisson process with rate 1 when the horizontal axis is rescaled by $(N/\log\lambda)\log K$. Hence, each horizontal interval between jumps of $(N/\log\lambda)\Gamma$ follows an exponential distribution with mean 1. Note that $(N/\log\lambda)|\log K^a-\log K^*|$ is a sum of the intervals that require the Poisson jumps to achieve the level m_1+M_a. Let Z_i denote an exponential random variable with mean 1. Then, m_1+M_a is equal to the minimum integer s of a discrete-time stochastic process $m_1+\sum_{i=1}^s Z_i$ to drop below s. In other words, m_1+M_a is the first-passage time of a discrete-time martingale $m_1+\sum_{i=1}^s(Z_i-1)$ passing 0.

$(1/\sqrt{N})\sum_{i=1}^s(Z_i-1)$ asymptotically follows a normal distribution with mean 0 and variance s/N for large s. Thus, $(1/\sqrt{N})(m_1+\sum_{i=1}^s(Z_i-1))$ for $s=1,2,\ldots,N$ asymptotically follows the Wiener process W_t in the interval $t\in[0,1]$ starting from $W_0=\lim_{N\to\infty}m_1/\sqrt{N}$ as N goes to infinity. The first-passage time T of the Wiener process starting from $m_1/\sqrt{N}$ to 0 follows the Inverse Gaussian distribution, with density function $(m_1/\sqrt{2\pi N})T^{-1.5}e^{-(m_1/\sqrt{N})^2/(2T)}$ (Asmussen and Albrecher, 2010, p. 42). Thus, the probability of $M_a = TN$ asymptotically becomes proportional to $(m_1/\sqrt{2\pi})M_a^{-1.5}e^{-m_1^2/(2M_a)}$ for large M_a conditional on m_1. This implies that the inverse cumulative probability of M_a does not decline faster than $M_a^{-0.5}$.

By Lemma 3 for ES2, M_b follows a power-law tail with exponent 0.5 and an initial disturbance smaller than 1. In contrast, the initial disturbance for M_a is $|m_1|$. By Lemma 4, $m_1/\sqrt{N}$ asymptotically follows $N(0, \sigma_1^2)$. Thus, $|m_1|$ scales as $\sqrt{N}$. Hence, the asymptotic variance of $M_a + M_b$ is dominated by M_a.

By combining the tail behaviors of M_a and M_b, the inverse cumulative probability of $M_a + M_b$ does not decline faster than the power law with exponent 0.5 does. By the selection rule ES1, $|g_N| = \min\{|\log K^1 - \log K^*|, |\log K^2 - \log K^*|\}$. Because the two terms in the minimization operator are independent conditional on m_1, $\Pr(|g_N| > g \mid m_1) = \Pr(|\log K^1 - \log K^*| > g \mid m_1) \Pr(|\log K^2 - \log K^*| > g \mid m_1)$. Thus, g_N conditional on m_1 has a tail that does not decay faster than the power function with exponent $0.5 + 0.5 = 1$. At the power exponent 1, the variance of g_N conditional on m_1 decreases as $\int^N x^2 x^{-2} dx/N^2 \sim 1/N$.

The variance of M_a is linear in $|m_1|$, because M_a is the first-passage time to travel $|m_1|$. For the same reason, the variance of M for ES1 is linear in $|m_1|$. Hence, $\min\{M, M_a + M_b\}$ asymptotically also has a variance linear in $|m_1|$. Because the mean of $|m_1|$ increases as $\sqrt{N}$ and the variance of g_N conditional on $m_1 = 1$ decreases as $1/N$, the variance of g_N decreases as $1/\sqrt{N}$. Therefore, when the tail distribution of g_N conditional on m_1 does not decay faster than the power law with exponent 1, the variance of g_N does not decrease faster than $1/\sqrt{N}$.

Proof of Proposition 2

Lemma 1 implies that $(\log K^2 - \log K^*)/\log\lambda$ asymptotes to $(m_1 + M)/N$, which I focus on here. Its unconditional variance $\mathrm{Var}((m_1 + M)/N)$ is decomposed as $\mathrm{E}[\mathrm{Var}(M/N \mid m_1)] + \mathrm{Var}(m_1/N + \mathrm{E}[M/N \mid m_1])$. By Lemma 4, the variance of m_1/N converges to 0. Furthermore, $|M| \leq N(1+q)$ holds because of the discrete constraint on capital choice. Therefore, $\mathrm{Var}((m_1 + M)/N)$ is always finite. In what follows, I show that this variance has a strictly positive lower bound.

The asymptotic probability distribution function for M conditional on $|m_1|$ when $\phi = 1$ is obtained using (33) as

$$p(m) \equiv \Pr(M = m \mid |m_1|) = \frac{|m_1| e^{-|m_1| - m}}{m!}(m + |m_1|)^{m-1}. \tag{37}$$

The maximum number of firms that adjust in the tatonnement process depends on the sign of m_1. This asymmetry in the upper bound of M matters for ES2, where the aggregate fluctuation does not vanish in the limit of N. With the discrete constraint on capital choice, the maximum number of investments in the tatonnement process (m_u) is $N(1-q)$, whereas that of divestments is $N(1+q)$. Thus, the distribution of M unconditional on m_1 is symmetric around 0 up to $N(1-q)$. Hence, the maximum deviation of $\mathrm{E}[M/N||m_1|]$ from 0 is obtained by modifying the integrand M/N as $M/N = 1-q$ for the event $M > N(1-q)$ and $M/N = -(1+q)$ for $M < -N(1-q)$. The maximum deviation is then evaluated as $\mathrm{E}[\sum_{m=N(1-q)+1}^{\infty} qp(m)]$. This implies

$$\begin{aligned} \mathrm{Var}(M/N \mid |m_1|) &= \mathrm{E}[(M/N)^2 \mid |m_1|] - \mathrm{E}[M/N \mid |m_1|]^2 \\ &> \sum_{m=0}^{N(1-q)} (m/N)^2 p(m) + \sum_{m=N(1-q)+1}^{\infty} (1-q)^2 p(m) - \left[\sum_{m=N(1-q)+1}^{\infty} qp(m)\right]^2. \end{aligned} \tag{38}$$

The combination of the last two terms is non-negative for any N when $q < 0.5$. The first term is evaluated using (37) as

$$\sum_{m=0}^{N(1-q)} (m/N)^2 p(m) > \sum_{m=0}^{N(1-q)} \frac{|m_1| e^{-|m_1|-m}}{\sqrt{2\pi} m^{m+0.5} e^{-m+1/(12m)}} (m+|m_1|)^{m-1} (m/N)^2$$
$$= \sum_{m=0}^{N(1-q)} \frac{|m_1|}{\sqrt{2\pi}} (1+|m_1|/m)^{m-1} e^{-|m_1|} e^{-1/(12m)} m^{-1.5} (m/N)^2, \tag{39}$$

where the first line holds according to the inequality (Feller, 1957, p. 52)

$$m! < \sqrt{2\pi} m^{m+0.5} e^{-m+1/(12m)}. \tag{40}$$

For an arbitrarily small $\epsilon_0 > 0$, there exists a large number N_{ϵ_0} such that for all $m > N_{\epsilon_0}$, $(1+|m_1|/m)^{m-1} e^{-|m_1|} e^{-1/(12m)} > 1 - \epsilon_0$ holds. Thus,

$$\sum_{m=0}^{N(1-q)} (m/N)^2 p(m) > (1-\epsilon_0) \frac{|m_1|}{\sqrt{2\pi}} \sum_{m=N_{\epsilon_0}}^{N(1-q)} m^{-1.5} (m/N)^2$$
$$> (1-\epsilon_0) \frac{|m_1|}{\sqrt{2\pi} N^2} \int_{N_{\epsilon_0}-1}^{N(1-q)} m^{0.5} dm$$
$$= \frac{(1-\epsilon_0)((1-q)^{1.5} - ((N_{\epsilon_0}-1)/N)^{1.5})}{1.5\sqrt{2\pi}} \frac{|m_1|}{\sqrt{N}}. \tag{41}$$

Hence, $\sum_{m=0}^{N(1-q)} (m/N)^2 p(m)$ converges to a number greater than (41).

Because $m_1/\sqrt{N}$ asymptotically follows $N(0, \sigma_1^2)$ by Lemma 4, I can use the formula $\mathrm{E}[|m_1|/\sqrt{N}] \to \sigma_1 \sqrt{2/\pi}$. Applying this, I find that the asymptotic variance of M/N is bounded from below by $(1-q)^{1.5} \sigma_1 / (1.5\pi)$.

Proof of Proposition 3

When $s_{i,t}$ follows a distribution uniform over the unit interval, $s_{i,u-1}$ follows the same distribution. This is because the uniform distribution is invariant to transformation by adding idiosyncratic and common shocks and by taking a modulo 1.

Define $\log \hat{\lambda}_i \equiv (\log \lambda_i) a_{i,t+1}^{\rho/\alpha} b_{i,t+1}^{\rho} / \mathrm{E}[a_{i,t+1}^{\rho/\alpha} b_{i,t+1}^{\rho}]$. The time subscript t is dropped in the rest of the proof. The heterogeneous counterpart of Lemma 1 is as follows:

$$N(\log K_u - \log K_{u-1}) = \sum_{n=1}^{\infty} \sum_{i \in H_u} \left(\frac{a_i^{1/\alpha} k_{i,u-1}}{K_{u-1}} \right)^{\rho} \frac{\rho^{n-1} (\log \lambda_i)^n}{n!} + O(1/N)$$
$$= \frac{\sum_{i \in H_u} a_i^{\rho/\alpha} b_i^{\rho} \lambda_i^{s_{i,u-1}\rho}}{\sum_{i=1}^{N} a_i^{\rho/\alpha} b_i^{\rho} \lambda_i^{s_{i,u-1}\rho} / N} \sum_{n=1}^{\infty} \frac{\rho^{n-1} (\log \lambda_i)^n}{n!} + O(1/N)$$
$$\to \sum_{i \in H_u} \log \hat{\lambda}_i, \tag{42}$$

where the last line used $\sum_{i=1}^{N} \lambda_i^{s_{i,u-1}\rho}/N \to \int_0^1 \lambda_i^{s_{i,u-1}\rho} ds_{i,u-1} = (\lambda_i^{\rho} - 1)/(\rho \log \lambda_i)$. In addition, for $i \in H_u$, $s_{i,u-1} < \phi(\log K_{u-1} - \log K_{u-2}) \to 0$ as $N \to \infty$.

The probability for firm j to be included in H_{u+1} is

$$\Pr(j \in H_{u+1} | j \notin \cup_{v=2,3,\ldots,u} H_v) = \frac{\phi(\log K_u - \log K_{u-1}) / \log \lambda_j}{1 - \phi(\log K_{u-1} - \log K_0) / \log \lambda_j}$$
$$\to \frac{\phi \sum_{i \in H_u} \log \hat{\lambda}_i}{N \log \lambda_j - \phi \sum_{h \in \cup_{v=2,3,\ldots,u} H_v} \log \hat{\lambda}_h}. \tag{43}$$

The event $j \in H_{u+1}$ asymptotically follows a Bernoulli trial with probability (43). Unconditional on realizations of $\hat{\lambda}_i$ and λ_j, the probability is equal to $\phi m_u \mathrm{E}[\log \hat{\lambda}_i]\mathrm{E}[1/(N \log \lambda_j - \phi \sum_{h \in \cup_{v=2,3,\ldots,u} H_v} \log \hat{\lambda}_h)]$. The number of firms of $j \in \cup_{v=2,3,\ldots,u} H_v$ is $\sum_{v=2}^{u} m_v$. Hence, the number of firms m_{u+1} follows a binomial distribution with this probability and population $N - \sum_{v=2}^{u} m_v$.

Suppose that the process $\sum_{v=2}^{u} m_v$ is finite with probability 1. Then, the mean of m_{u+1} with this binomial distribution converges to $\hat{\phi} m_u$ as $N \to \infty$. For $\sum_{v=2}^{u} m_v$ to be finite, m_u must be a submartingale or martingale. Thus, $\hat{\phi} \leq 1$ must hold. Hence, for $\hat{\phi} \leq 1$, m_{u+1} asymptotically follows a Poisson distribution with mean $\hat{\phi} m_u$. Because a Poisson distribution is infinitely divisible, m_{u+1} is equivalent to a m_u-times convolution of a Poisson distribution with mean $\hat{\phi}$. The rest of the proof proceeds as that for Lemma 3.

Proof of Proposition 4

Using a heterogeneous-firm counterpart of the optimal investment threshold, the right-hand side of the gap dynamics in (13) is written as a modulo 1 of

$$\frac{\log(1-\delta_i) + \log(\hat{A}_{i,t} K_t^{\phi}) - \log(\hat{A}_{i,t+1} K_{t+1}^{\phi}) + \frac{\rho}{\alpha(1-\rho)}(\log a_{i,t} - \log a_{i,t+1})}{\log \lambda_i} + s_{i,t} + 1,$$

where $\hat{A}_{i,t} \equiv \left(w_t^{\gamma/(1-\gamma)}(R_t - 1 + \delta_i)\right)^{-1/(1-\rho)}$. Then, $s_{i,t} = (tU_i + V_{i,t} + W_{i,t} + s_{i,0} + t) \mod 1 = (tU_i + V_{i,t} + W_{i,t} + s_{i,0}) \mod 1$, where $U_i \equiv (\log(1-\delta_i))/\log \lambda_i$, $V_{i,t} \equiv (\rho/(\alpha(1-\rho)))(\log a_{i,0} - \log a_{i,t})/\log \lambda_i$, and $W_{i,t} \equiv (\log(\hat{A}_{i,0} K_0^{\phi}) - \log(\hat{A}_{i,t} K_t^{\phi}))/\log \lambda_i$. Because $a_{i,t}$ is an i.i.d. bounded random variable, $V_{i,t}$ has a well-defined density that is common for any t. $W_{i,t}$ is a stationary process. U_i has a continuous density. Hence, tU_i taken modulo 1 converges in distribution to a unit uniform random variable as $t \to \infty$. Moreover, its sum with an absolutely continuous random variable, taken modulo 1, also converges to the unit uniform distribution (Engel, 1992, pp. 28-29).

Proof of Proposition 5

The expectation system (14, 15, 16, 20, 21, 27, 28, 29) can be log-linearized as follows. Let tilde denote the log difference from the steady state. In accordance with Sims (2001), for the log-difference variables, the time subscripts indicate the period in which the variable is observable to the agents. For example, a predetermined variable K_t corresponds to $\tilde{K}_{-1}$, whereas $E_{t-1}C_t$ corresponds to $E_{-1}\tilde{C}_0$. Then, I obtain

$$\tilde{K}_0 = (1-\delta)\tilde{K}_{-1} + \delta \tilde{X}_0, \tag{44}$$

$$\tilde{Y}_0 = \alpha \tilde{K}_{-1} + (1-\alpha)\tilde{L}_0, \tag{45}$$

$$\tilde{Y}_0 = (\bar{C}/\bar{Y})\tilde{C}_0 + (\bar{X}/\bar{Y})\tilde{X}_0, \tag{46}$$

$$\mathrm{E}_{-1}\tilde{Y}_0 = \tilde{K}_{-1} - \frac{1-\alpha}{\alpha}\mathrm{E}_{-1}\tilde{w}_0, \tag{47}$$

$$\tilde{w}_0 = \tilde{C}_0 + \left(\zeta - 1 + \bar{w}\bar{L}/\bar{C}\right)\tilde{L}_0, \tag{48}$$

$$0 = \frac{1-\alpha}{\alpha}\mathrm{E}_{-1}\tilde{w}_0 + \frac{\bar{R}}{\bar{R}-1+\delta}\mathrm{E}_{-1}\tilde{R}_0, \tag{49}$$

$$\tilde{R}_0 = \sigma(\tilde{C}_0 - \tilde{C}_{-1}) - (\sigma-1)(\bar{w}\bar{L}/\bar{C})(\tilde{L}_0 - \tilde{L}_{-1}), \tag{50}$$

$$\tilde{X}_0 = \mathrm{E}_{-1}\tilde{X}_0 + \epsilon_0, \tag{51}$$

$$\tilde{C}_0 = \mathrm{E}_{-1}\tilde{C}_0 + \eta_0^C, \quad \tilde{L}_0 = \mathrm{E}_{-1}\tilde{L}_0 + \eta_0^L, \tag{52}$$

$$\tilde{Y}_0 = \mathrm{E}_{-1}\tilde{Y}_0 + \eta_0^Y, \quad \tilde{w}_0 = \mathrm{E}_{-1}\tilde{w}_0 + \eta_0^w, \tag{53}$$

where $(\eta_0^C, \eta_0^L, \eta_0^Y, \eta_0^w)$ are endogenous expectation errors caused by the expectation error in investment, ϵ_0.

The labor share $\bar{w}\bar{L}/\bar{Y}$ is constant at c_L. I set the following definitions: $a_c \equiv \bar{C}/\bar{Y}$, $a_x \equiv \bar{X}/\bar{Y}$, and $a_R \equiv \bar{R}/(\bar{R} - 1 + \delta)$. I also denote the marginal utility of consumption as $\mu_t \equiv C_t^{-\sigma}(1 - \psi L_t^{\zeta})^{1-\sigma}$. Then,

$$\tilde{\mu}_0 = -\sigma \tilde{C}_0 + (\sigma - 1)(c_L/a_c)\tilde{L}_0. \tag{54}$$

Thus, (48) and (54) yield the compensated labor supply function $\tilde{L}_0 = \eta_L(\tilde{w}_0 + \tilde{\mu}_0/\sigma)$, where $\eta_L \equiv ((2 - 1/\sigma)(c_L/a_c) + \zeta - 1)^{-1}$ is Frisch elasticity. Combining this with (49, 50), I obtain

$$\left(\frac{\alpha a_R}{1-\alpha} + \frac{1}{\sigma}\right) \mathrm{E}_{-1}\tilde{\mu}_0 = \frac{\alpha a_R}{1-\alpha}\tilde{\mu}_{-1} + \eta_L^{-1}\mathrm{E}_{-1}\tilde{L}_0. \tag{55}$$

Combining (54) with (45, 47, 48), I obtain

$$-\frac{1}{\sigma}\mathrm{E}_{-1}\tilde{\mu}_0 = \alpha\tilde{K}_{-1} - (\alpha + \eta_L^{-1})\mathrm{E}_{-1}\tilde{L}_0. \tag{56}$$

Substituting $\mathrm{E}_{-1}\tilde{L}_0$ out from (55, 56), I obtain

$$\left(\frac{\eta_L}{\sigma(1+\alpha\eta_L)} + \frac{a_R}{1-\alpha}\right) \mathrm{E}_{-1}\tilde{\mu}_0 = \frac{1}{1+\alpha\eta_L}K_{-1} + \frac{a_R}{1-\alpha}\tilde{\mu}_{-1}. \tag{57}$$

Combining (54) with the capital accumulation process (44, 45, 46), I obtain $A_1\mathrm{E}_{-1}\tilde{L}_0 = -(a_c/\sigma)\mathrm{E}_{-1}\tilde{\mu}_0 + (a_x/\delta)\mathrm{E}_{-1}\tilde{K}_0 - (\alpha + a_x(1-\delta)/\delta)\tilde{K}_{-1}$, where $A_1 \equiv 1 - \alpha - c_L(\sigma - 1)/\sigma$. Substituting $\mathrm{E}_{-1}\tilde{L}_0$ by using (56), I obtain

$$\frac{a_x}{\delta}\mathrm{E}_{-1}\tilde{K}_0 - \left(\frac{a_c}{\sigma} + \frac{A_1}{\sigma(\alpha + \eta_L^{-1})}\right)\mathrm{E}_{-1}\tilde{\mu}_0 = \left(\alpha + \frac{a_x(1-\delta)}{\delta} + \frac{A_1\alpha}{\alpha + \eta_L^{-1}}\right)\tilde{K}_{-1}. \tag{58}$$

Note that (58) and (57) represent the expectation system and are stacked in a matrix form:

$$B \begin{bmatrix} \mathrm{E}_{-1}\tilde{K}_0 \\ \mathrm{E}_{-1}\tilde{\mu}_0 \end{bmatrix} = D \begin{bmatrix} \tilde{K}_{-1} \\ \tilde{\mu}_{-1} \end{bmatrix}. \tag{59}$$

I note that $B_{21} = D_{12} = 0$, where the subscript ij denotes (i,j)-th coordinate of the matrices B and D. Using this property, I obtain $\det(B^{-1}D) = D_{11}D_{22}/(B_{11}B_{22}) > 0$. Similarly, for a two-by-two identity matrix I, I obtain

$$\begin{aligned} \det(B^{-1}D - I) &= \det(B^{-1}D) + 1 - \det(B)^{-1}(B_{22}D_{11} - B_{12}D_{21} + B_{11}D_{22}) \\ &= \frac{(B_{11} - D_{11})(B_{22} - D_{22}) + B_{12}D_{21}}{B_{11}B_{22}}. \end{aligned} \tag{60}$$

I have $B_{22} - D_{22} > 0$, $B_{12}D_{21} < 0$, and $B_{11}B_{22} > 0$, while $B_{11} - D_{11} \leq 0$ if $a_x \leq \alpha$. Hence, $\det(B^{-1}D - I) < 0$ holds if the investment-to-output ratio is less than α at the steady state.

Let $\varphi(\xi)$ denote the determinant of $B^{-1}D - \xi I$. From the earlier results, I determine that $\varphi(0) = \det(B^{-1}D) > 0$ and $\varphi(1) = \det(B^{-1}D - I) < 0$. Thus, $\varphi(\xi)$ is a convex quadratic function that has a strictly positive intercept at $\xi = 0$ and takes a strictly negative value at $\xi = 1$. Therefore, $\varphi(\xi)$ has two real roots ξ_1, ξ_2 such that $0 < \xi_1 < 1 < \xi_2$, and hence, the dynamics of $(\tilde{K}, \tilde{\mu})$ have a unique saddle point path in the neighborhood of the steady state if $a_x < \alpha$.

Now, the capital accumulation is driven by aggregate investment shock as $\tilde{K}_0 = \mathrm{E}_{-1}\tilde{K}_0 + \delta\epsilon_0$. Thus, $[\tilde{K}_0\ \mathrm{E}_{-1}\tilde{\mu}_0]' = B^{-1}D[\tilde{K}_{-1}\ \tilde{\mu}_{-1}]' + [\delta\ 0]'\epsilon_0$. As shown earlier, $B^{-1}D$ has one eigenvalue for each inside and outside of a unit circle. Hence, the expectation system is determinate.

APPENDIX B: DERIVATIONS IN SECTION 3

Derivation of Equation (16)

Since $p_{i,t+1}$ is predetermined in period t, $l_{i,t+1}$ is determined passively by a production function and a demand function as

$$l_{i,t+1} = \left(\frac{p_{i,t+1}^{-\eta} Y_{t+1}}{a_{i,t+1} k_{i,t+1}^{\alpha}} \right)^{1/(1-\alpha)} .$$

By using these relations, I can write the firm's objective in period t as

$$\mathrm{E}_t \left[R_{t+1}^{-1} \left(p_{i,t+1}^{1-\eta} Y_{t+1} - w_{t+1} \left(\frac{p_{i,t+1}^{-\eta} Y_{t+1}}{a_{i,t+1} k_{i,t+1}^{\alpha}} \right)^{1/(1-\alpha)} + (1-\delta) k_{i,t+1} \right) \right] - k_{i,t+1}.$$

The first order condition with respect to $p_{i,t+1}$ yields

$$p_{i,t+1}^{1-\eta+\eta/(1-\alpha)} = (a_{i,t+1} k_{i,t+1}^{\alpha})^{-1/(1-\alpha)} \mathrm{E}_t[(w_{t+1}/c_L) Y_{t+1}^{1/(1-\alpha)}/R_{t+1}]/\mathrm{E}_t[Y_{t+1}/R_{t+1}],$$

where $c_L \equiv (1-1/\eta)(1-\alpha)$. Substituting this into the normalization condition $P_t = 1$ and using $K_t \equiv \left(\sum_{i=1}^{N} \left(a_{i,t}^{1/\alpha} k_{i,t} \right)^{\rho} /N \right)^{1/\rho}$, I obtain (16).

Derivation of Equation (28)

The threshold capital $k_{i,t+1}^*$ can be translated to the threshold gap $s_{i,t}^*$, where firms with $s_{i,t} \in [0, s_{i,t}^*)$ invest in period t. Because $a_{i,t+1}$ is known to i in period t, $s_{i,t+1} = 0$ holds when $s_{i,t} = s_{i,t}^*$. Thus, the threshold is obtained from (13) as

$$s_{i,t}^* = \frac{\log k_{i,t+1}^* - \log k_{i,t}^*}{\log \lambda_i} - \frac{\log(1-\delta)}{\log \lambda_i}. \tag{61}$$

Because of the assumption of the bounded increment of $\log a_{i,t}$, the gap $s_{i,t}$ always decreases over time unless there is an increase by 1.

Aggregate gross investment under the stationary uniform distribution of $s_{i,t}$ is calculated as follows.

$$\begin{aligned}
X_t &= \int \int_0^{s_{i,t}^*} (\lambda_i - 1)(1-\delta) k_{i,t} \, ds_{i,t} \, di \\
&= (1-\delta) \int \int_0^{s_{i,t}^*} (\lambda_i - 1) \lambda_i^{s_{i,t}} k_{i,t}^* \, ds_{i,t} \, di \\
&= (1-\delta) \int \frac{(\lambda_i - 1)(\lambda_i^{s_{i,t}^*} - 1)}{\log \lambda_i} k_{i,t}^* \, di \\
&= (1-\delta) \int \frac{\lambda_i - 1}{\log \lambda_i} ((1-\delta)^{-1} k_{i,t+1}^* - k_{i,t}^*) di \\
&= \int \frac{\lambda_i - 1}{\log \lambda_i} b_{i,t+1} di \, K_{t+1} - \int \frac{\lambda_i - 1}{\log \lambda_i} b_{i,t} di \, (1-\delta) K_t \\
&= \rho A^{1-\rho} (K_{t+1} - (1-\delta) K_t) \qquad (62)
\end{aligned}$$

Thus, (28) is obtained.

APPENDIX C: DETAILS ON CALIBRATION AND COMPUTATION

The firms' markup rate $1/(\eta - 1)$ is set at 10%. The capital intensity α is set such that the labor share $\bar{w}\bar{L}/\bar{Y}$ is equal to 0.67. The annual rate of depreciation is set at 8%, and the annual risk-free rate is at 4%. Disutility from labor is specified as a quadratic function. Indivisibility parameter λ_i is a random variable drawn in period 0 and fixed for later periods. Note that λ_i is set to be drawn from a normal distribution with mean 1.028 and standard deviation 0.004 truncated at two standard deviations. I choose this specification to match the 2.8% plant Herfindahl index estimated by Ellison and Glaeser (1997). Plant Herfindahl measures the representative share of a plant's employment in an industry. When capital size is adjusted by changing the number of plants, the plant Herfindahl can be interpreted as a lower bound of capital indivisibility, which coincides with firm-level capital indivisibility if the industry is a monopoly. These parameters and steady-state values for the benchmark specification are summarized in Table I.

The number of firms N is set at 110,000 to match the number of firms with 100 employees or more in the US Census data in 2008. The logarithm of the idiosyncratic productivity $\log a_{i,t}$ is assumed to follow a normal distribution with standard deviation 0.05%. The mean productivity is set such that the mean of $a_{i,t}^{\rho/(\alpha(1-\rho))}$ (which appears in the threshold rule (17)) is normalized to 1. In the initial period, $s_{i,0}$ is randomly drawn from a uniform distribution, and in each period, productivity $a_{i,t}$ is drawn independently. The equilibrium path is simulated for 1,100 periods, from which the first 100 periods are discarded. The reported moments in Table II are averages of 10 simulated runs. Figures in parentheses report standard errors for each averaged moment.

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