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DEMAND FUNCTIONS WITH INFERIOR GOODS: THE IMPLICIT FUNCTION APPROACH

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Abstract

In this article, we propose a numerically computable utility function that can apply to inferior goods. The implicit function and its optimization technique are fully used. Since the implicit function is carefully formulated, it works well as a standard utility function. This technique ensures tractability and extendability. We propose the following: (1) a simple utility function of an inferior good which contains only two parameters; (2) a total cost function and its extension to the Cobb-Douglas production function with an inferior input; (3) a generalized utility function whose Engel-curve always stems from the origin.

Keywords: inferior goods, utility function, production function, implicit function
JEL Classification Codes: D01, D11, D21

I. *Introduction*

Inferior goods are widely observed in an economy. It is natural for consumers to upgrade or downgrade their purchases according to their budget. However, a numerically computable utility function for inferior goods has not been fully developed yet and has been studied by researchers. Epstein and Spiegel (2000) found a simple production function for an inferior input, which has only two parameters in the minimum setting, although it does not explain why the input is inferior or how to extend their two-variable model to n-variables. Moffatt (2002) uses a hyperbola to avoid intersection of the indifference curves and succeeds in dealing with Giffen goods. Doi, Iwasa, and Shinomura (2006) use the logarithmic function in their formulation.

In this study, we propose the implicit function approach to deal with inferior goods.[1] The utility function, $u(x, y)$, is given by $U(u, x, y)=0$, which is unsolvable for u. At first glance, this function appears awkward and difficult to use or incompatible with the economic theory.

[1] We focus on the two-goods case in this study. However, this two-goods model is extended to the n-goods case. The n-goods case will be presented in another paper.

However, we find that our implicit utility function gives all types of tractable functions such as the compensated demand, normal demand, and total cost functions. These are compatible with the economic theory.

The remainder of the paper consists of three parts. First, Sections 2 and 3 illustrate the structure of our implicit function using the simplest model, which contains only two parameters, a and c. Second, Section 4 discusses the production function which contains full parameters.[2] Furthermore, we show the addition of an inferior good to the Cobb-Douglas production function. Third, the generalized utility function is presented in Section 5, where both goods are normal when income is low; however, either can change from a normal to an inferior good as income increases.

II. *Illustration*

Let us denote utility by u, and the quantities of the two goods by x and y. If the utility function is

$$u(x, y)=x^a+y, \ (0<a<1),$$

then x would never be an inferior good. For x to be an inferior good, its marginal utility must gradually reduce with an increase in overall utility u, since x is less favored by high-utility individuals. Therefore, we place a decreasing function, $\frac{1}{u^c}$, $(c>0)$ before x^a, and reformulate $u=x^a+y$ into

$$u=\frac{1}{u^c}x^a+y. \tag{1}$$

Thus, $u(x, y)$ is given by the implicit function,

$$U(u, x, y)\equiv\frac{1}{u^c}x^a+y-u=0.$$

One of the solvable cases for u is when $c=1$. When $c=1$, (1) is given by

$$u=\frac{1}{u}x^a+y,$$

which gives $u^2-uy-x^a=0$. Therefore, this can be solved as

$$u(x, y)=\frac{y+\sqrt{y^2+4x^a}}{2}. \tag{2}$$

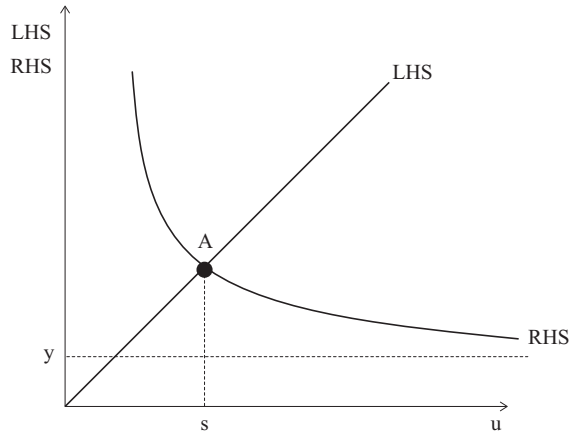
Equation (2) resembles the quadratic formula in mathematics.[3] If we solve the problem of minimizing the expenditure $px+y$, where p denotes the relative price, subject to

[2] Sections 4 and 5 are independent of the earlier sections. The reader can begin to read from Section 4 or 5.

[3] The production function of Epstein and Spiegel (2000) resembles (2). It can be expressed as $u=(y+x)^{\alpha}-\beta x^{\nu}$, $(0<\alpha<1, \beta<1$ and $\nu>0)$.

FIGURE 1. UTILITY LEVEL

The LHS curve is on the left-hand side of $u=u^{-c}x^a+y$ and the RHS curve on the right-hand side. They always have a single intersection point, point A. The projection of point A on the x-axis represents the real value s that we use as the utility level. An increase in x or y shifts the RHS curve upward. Therefore, marginal utility is always positive.



$u_0-(y+\sqrt{y^2+4x^a})/2=0$ for a fixed u_0, then we get $x(p, u_0)=(a/(u_0p))^{\frac{1}{1-a}}$ as the compensated demand function, which is decreasing in u. Thus, we can confirm that (2) is the utility function of an inferior good.

We must note that although (1) is unsolvable for u in general, it can give the utility level. This is explained as follows. Let us decompose (1) into two functions: $\mathrm{LHS}(u)=u$ and $\mathrm{RHS}(u)=u^{-c}x^a+y$. The former is the 45-degree line in Figure 1. The latter is the downward sloping curve. These always intersect, in this case at point A, where $LHS(u)=RHS(u)$. Then, (1) always gives u as a real and unique number.

Let us consider the case where $c=1$. The indifference curves of $u=u^{-1}x^a+y$ are illustrated in Figure 2(i). The indifference curves for $u=1$ and $u=2$ are given by

$$1=x^a+y, \text{ and}$$

$$2=(1/2)x^a+y,$$

respectively, where the latter equation has the larger constant term and smaller coefficient of x than the former. Therefore, the two indifference curves never intersect for non-negative x and y. The simplest case is that of perfect substitutes (a=1),

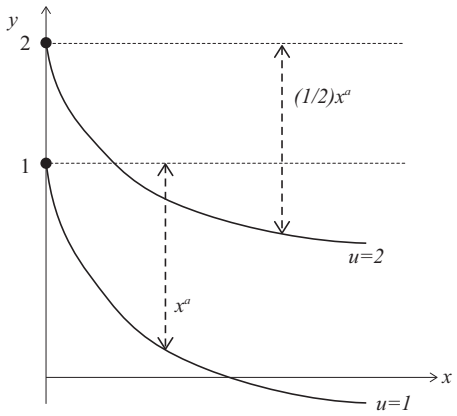
$$u=u^{-1}x+y.$$

The indifference curves for $u=1, 2, 3,...,$ are given by $1=x+y$, $2=(1/2)x+y$, $3=(1/3)x+y,...,$ respectively. These equations are illustrated in Figure 2 (ii). The gradient of the indifference curves decreases with an increase in u.

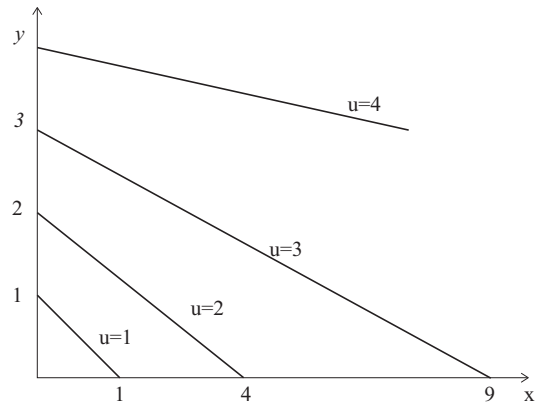
FIGURE 2. INDIFFERENCE CURVES

(i): The indifference curve (IC) of u = 1 on (x, y) is given by $1=x^a+y$. If u increases to u = 2, the IC of u = 2 becomes $2=(1/2)x^a+y$. Therefore, the two ICs never intersect each other. This IC of u = 1 is represented by the x^a curve, being given by $y=1-x^a$. The IC of u = 2 is represented by $(1/2)x^a$. (ii): In the perfect substitute case, the ICs are straight lines.

(i) Substitutes Case



(ii) Perfect Substitutes Case



III. *Consumer's Demand Function with Inferior Goods*

Let $m=px+y$ be the budget constraint, where m is an income level, and p is the relative price. For a given m, the problem faced by a consumer is

$$\max_{x,y} u$$

$$\text{s.t } u^{-c}x^a+y-u=0, \text{ and } m-px-y=0.$$

To solve this problem, we use the Lagrange function[4]

$$L=u+\mu(u^{-c}x^a+y-u)+\lambda(m-px-y)\ .$$

From this, we obtain

$$L_x=\mu a u^{-c}x^{a-1}-\lambda p=0 \tag{3}$$

$$L_y=\mu-\lambda=0 \tag{4}$$

$$L_u=1+\mu(-cu^{-c-1}x^a)=0$$

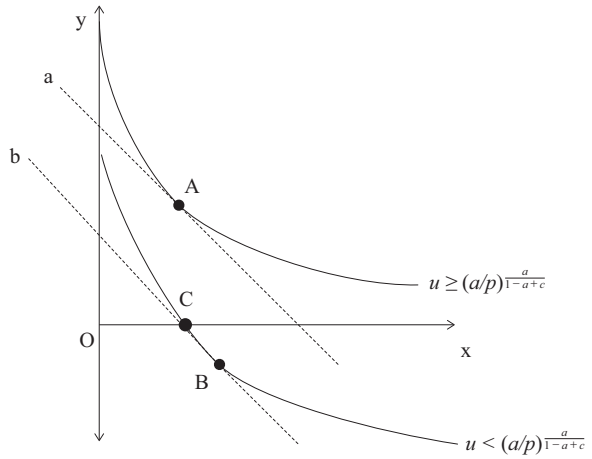
$$L_\lambda=m-px-y=0 \tag{5}$$

$$L_\mu=u^{-c}x^a+y-u=0. \tag{6}$$

[4] An alternative method for solving this problem is available in Appendix A. In Appendix A, we use the optimization technique for an implicit function to derive (7) and (12).

FIGURE 3. INTERIOR AND EXTERIOR SOLUTIONS

Point A is the interior solution. However, the IC of our function intersects the x-axis. For a utility level below $(a/p)^{\frac{a}{1-a+c}}$, the FOC yields point B. In this case, point C is the optimal point to avoid a negative demand for y.



From (3) and (4) we obtain "the law of weighted equi-marginal utility" $(au^{-c}x^{a-1})/p=1$, from which we can also derive the compensated demands, $x(p, u)$ and $y(p, u)$. Then, prior to deriving the normal demand, $x(p, m)$ and $y(p, m)$, we derive $x(p, u)$ and $y(p, u)$.

From (3) and (4), we obtain

$$x(p, u)=\left(\frac{a}{u^c p}\right)^{\frac{1}{1-a}}. \tag{7}$$

From (6) and (7), we obtain

$$y(p, u)=u-\frac{1}{u^c}\left(\frac{a}{u^c p}\right)^{\frac{a}{1-a}} \text{ for } u\geq\left(\frac{a}{p}\right)^{\frac{a}{1-a+c}}, \tag{8}$$

$$y(p, u)=0 \qquad \text{for } u\leq\left(\frac{a}{p}\right)^{\frac{a}{1-a+c}}. \tag{9}$$

From (6) and (9), we obtain

$$x(p, u)=u^{\frac{1+c}{a}} \quad \text{for } u\leq\left(\frac{a}{p}\right)^{\frac{a}{1-a+c}}. \tag{10}$$

The interior case given by (7) and (8) is shown as point A in Figure 3. The exterior case given by (9) and (10) is shown as point C in Figure 3. Although (7) gives point B, it is negative in y, therefore, point C given by (10) and (9) is the optimal point.
Summing (7), (8), (9), and (10), we obtain Proposition 1.

Proposition 1 (Compensated Demand Function)

Let $\bar{u}$ denote the threshold of u, which is

$$\bar{u}(p) \equiv \left(\frac{a}{p}\right)^{\frac{a}{1-a+c}}.$$

(i) For $u \geq \bar{u}$, *the compensated demands are*

$$x(p, u) = \left(\frac{a}{u^c p}\right)^{\frac{1}{1-a}},$$

$$y(p, u) = u - \frac{1}{u^c}\left(\frac{a}{u^c p}\right)^{\frac{a}{1-a}}.$$

where the utility and price elasticity of $x(p, u)$ *are constant*

$$\frac{\partial x}{\partial u}\frac{u}{x} = \frac{-c}{1-a} < 0,$$

$$-\frac{\partial x}{\partial p}\frac{p}{x} = \frac{1}{1-a} > 1.$$

(iii) For $0 \leq u \leq \bar{u}$, *the compensated demands are*

$$x(p, u) = u^{\frac{1+c}{a}}.$$

$$y(p, u) = 0.$$

Proposition 1(i) gives $x(p, u)$ and $y(p, u)$ for the interior solution. x's elasticity with respect to u is a negative constant (i.e., $-c/(1-a) < 0$). Thus, we confirm that x is an inferior good. However, the price elasticity is constant and larger than 1 (i.e., $1/(1-a) > 1$). This means that our inferior good is very elastic in price. Proposition 1(ii) shows the corner solution, where the demand for y is zero.

Next, we derive the normal demand function. From (3) and (4), we obtain

$$u = \left(\frac{a}{px^{1-a}}\right)^{\frac{1}{c}} \text{ for } u \geq \left(\frac{a}{p}\right)^{\frac{a}{1-a+c}}. \tag{11}$$

From (5), (6), and (11), we obtain

$$m = \left(\frac{a}{px^{1-a}}\right)^{\frac{1}{c}} - \left(\frac{1-a}{a}\right)px \text{ for } u \geq \left(\frac{a}{p}\right)^{\frac{a}{1-a+c}}. \tag{12}$$

When $u = (a/p)^{\frac{a}{1-a+c}}$, the demand functions are $x = m/p$ and $y((a/p)^{\frac{a}{1-a+c}}) = 0$. Therefore, equation (1) with $u = (a/p)^{\frac{a}{1-a+c}}$ is given by $u = u^{-c}(m/p)^a + 0$, which gives

$$u = \left(\frac{m}{p}\right)^{\frac{a}{c+1}}. \tag{13}$$

From (12) and (13), we obtain

$$m=\left(\frac{a}{px^{1-a}}\right)^{\frac{1}{c}}-\left(\frac{1-a}{a}\right)px \text{ for } m\geq\frac{1}{p}\left(\frac{a}{p}\right)^{\frac{1+c}{1+c-a}}. \tag{15}$$

From (15), we obtain Proposition 2.

Proposition 2 (Demand Function)

Let $\bar{m}$ denote the threshold of m, which is

$$\bar{m}(p)\equiv\frac{1}{p}\left(\frac{a}{p}\right)^{\frac{1+c}{1+c-a}}.$$

(i) For $m\geq\bar{m}(p)$, *the normal demand* $x(p, m)$ *is given implicitly by*

$$m=\left(\frac{a}{px^{1-a}}\right)^{\frac{1}{c}}-\left(\frac{1-a}{a}\right)px.$$

(ii) For $0<m\leq\bar{m}(p)$, *the normal demand* $x(p, m)$ *is given by*

$$x(p, m)=\frac{m}{p}.$$

(iii) For $m\geq\bar{m}(p)$, $\frac{dx}{dm}<0$

From Propositions 2(i), and 2(iii), $dx/dm<0$ since $0<a<1$ and $c>0$. Therefore, our utility function successfully exhibits the property of an inferior good. However, despite the existence of an inferior good, it yields $dx/dp<0$. Therefore, our utility function does not exhibit the property of a Giffen good, even if the parameter c is very large.

The income-consumption curve obtained from Propositions 2(i) and 2(ii) is illustrated in Figure 4. Line AB is the income-consumption path for $m\geq\bar{m}$, while line OA is that for $m\leq\bar{m}$.

IV. *Factor Demand Functions with Inferior Inputs*

1. The Total Cost Function

In this section, we examine the total cost function. We use Q instead of u to discuss the cost function. Let $C=p_x x+p_y y$ be the total cost, where p_x and p_y denote the prices of x and y, respectively. Input x is inferior, such as a compact machine, which is convenient and handy for small-scale production, but is unsuitable for large-scale production.

The production function $Q=f(x, y)$ is given by

$$Q=Q^{-c}x^{a}+y.$$

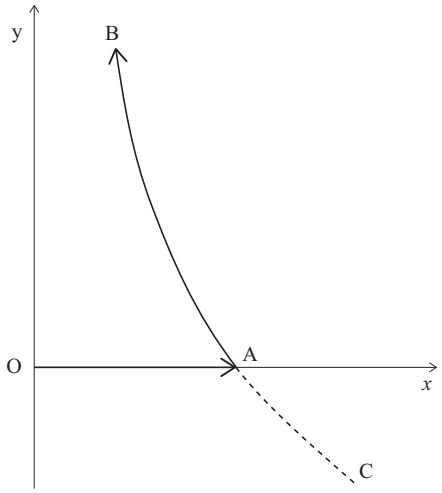
which is the same as (1). The firm solves the minimization problem with a given Q,

$$\min_{x, y} p_x x+p_y y$$

$$\text{s.t. } Q^{-c}x^{a}+y-Q=0.$$

FIGURE 4. INCOME-CONSUMPTION CURVE

Consumption (x, y) moves from point O to point A and then point B when increasing in m or u. OA is given by Proposition 1(iii) and AB by Proposition 1(i).



The Lagrange function is

$$\mathcal{L}=p_x x+p_y y+\lambda(Q^{-c}x^a+y-Q).$$

Let $x(p_x, p_y, Q)$ and $y(p_x, p_y, Q)$ denote the factor demand functions (with a given Q) obtained by solving this problem. We do not discuss factor demand in detail, since these are the same as the compensated demands shown in Proposition 1. If we replace (p, u) in Proposition 1 with (p_x/p_y, Q), then we obtain the factor demand functions $x(p_x, p_y, Q)$ and $y(p_x, p_y, Q)$.[5]

The total cost function is given by $C(Q)=p_x x(p_x, p_y, Q)+p_y y(p_x, p_y, Q)$. By using these, we obtain Proposition 3 for the total cost function.

Proposition 3 (Total Cost Function)

Let $\overline{Q}$ denote the threshold of Q, which is

$$\overline{Q}(p_x, p_y)\equiv\left(\frac{ap_y}{p_x}\right)^{\frac{a}{1-a+c}}.$$

(i) For $\overline{Q}\leq Q$, both x and y are used in production and $C(Q)$ is given by

$$C(Q)=p_y Q-(a^{\frac{a}{1-a}}-a^{\frac{1}{1-a}})\cdot p_x^{\frac{-a}{1-a}}\cdot p_y^{\frac{1}{1-a}}\cdot Q^{\frac{-c}{1-a}}. \tag{17}$$

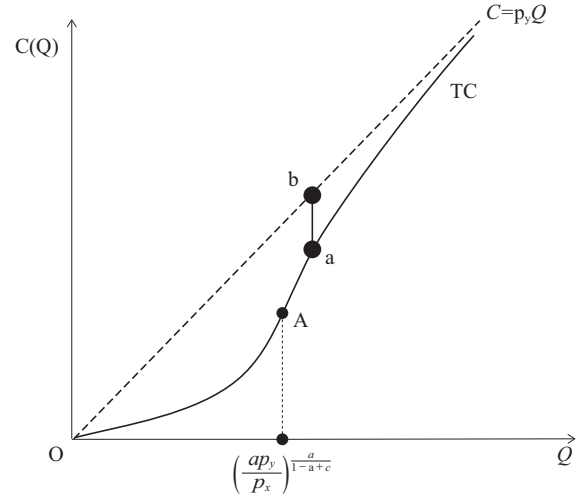
(ii) For $0\leq Q\leq\overline{Q}$, the factor demand $y(p_x, p_y, Q)$ is zero. Then $C(Q)$ is given by

$$C(Q)=p_x Q^{\frac{1+c}{a}}. \tag{16}$$

[5] The full and self-contained proof of Proposition 3 is available in Appendix B. In Appendix B, a general function $Q=Q^{-c}bx^a+\beta y$, ($b>0$, $\beta>0$) is discussed for further extensions.

FIGURE 5. TOTAL COST CURVE

The dotted straight line represents the TC curve when the inferior input is unavailable, and the spoon-shape TC curve represents total cost when the inferior input is available. By using the inferior input, cost is reduced for a small production level. The length of ab represents this cost saving.



(iii) The limit of the second term of (17) is zero, since

$$\lim_{u\to\infty} -(a^{\frac{a}{1-a}}-a^{\frac{1}{1-a}})\cdot p_x^{\frac{-a}{1-a}}\cdot p_y^{\frac{1}{1-a}}\cdot Q^{\frac{-c}{1-a}}\to 0.$$

Then, the total cost curve converges to the straight line $C=p_yQ$ *as q increases. Therefore, the total cost curve is spoon-shaped and shown as* TC *in Figure 5.*

Proposition 2(iii) means that although marginal cost is not constant, it converges to the constant p_y. In this sense, $C(Q)$ is a quasi-constant marginal cost. The reason is that when Q is sufficiently large, the need for an inferior good almost vanishes. Production is carried out using normal goods. In $Q=Q^{-c}x^a+y$, the term y denotes the normal good and is the constant marginal cost. The inferior good is used only when its marginal product per price is higher than or equal to that of the normal good. If the inferior good is not available, we must only use the normal good, which gives the total cost curve $C=p_yQ$ in Figure 5. Therefore, the depth of the spoon represents the savings from the inferior good.

2. Extension to the Cobb-Douglas Production Function

In $Q=Q^{-c}x^a+y$, the variable y is the simplest homothetic of degree 1 functions. Then, we can replace y with $y=\beta K^zL^{1-z}$, where K is capital, and L is labor. Furthermore, we add a parameter, b. The production function $Q(x, y)$ is given by

$$Q=\frac{b}{Q^c}x^a+\beta K^zL^{1-z} \tag{18}$$

where $0<b$, $0<\beta$, and $0<z<1$. [6]

Total cost is $p_x x+rK+wL$, where r is the rental rate, and w is wage. The firm solves the minimization problem:

$$\min_{x,\,K,\,L}\ p_x x+rK+wL$$
$$\text{s.t } Q^{-c}bx^{a}+\beta Y^{z}L^{1-z}-Q=0.$$

The Lagrange function is

$$\mathcal{L}=p_x x+rK+wL-\lambda\left(\frac{b}{Q^{c}}x^{a}+\beta K^{z}L^{1-z}-Q\right).$$

Then, the first order conditions (FOC) are:

$$\mathcal{L}_x=p_x-\lambda a\frac{b}{Q^{c}}x^{a-1}=0$$
$$\mathcal{L}_K=r-\lambda\beta zK^{z-1}=0$$
$$\mathcal{L}_L=w-\lambda\beta(1-z)L^{-z}=0$$
$$\mathcal{L}_\lambda=\frac{b}{Q^{c}}x^{a}+\beta K^{z}L^{1-z}-q=0.$$

From these FOCs, we obtain Proposition 4 and Proposition 5.[7]

Proposition 4 (Factor Demand)

Let $\overline{Q}$ denote the threshold of Q, which is given by

$$\overline{Q}=\overline{Q}(p_x,\,r,\,w)\equiv\left\{\frac{ab^{\frac{1}{a}}}{\beta p_x}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right\}^{\frac{a}{1-a+c}}. \tag{19}$$

(i) For $\overline{Q}\leq Q$, the factor demand functions are given by

$$x(p_x,\,r,\,w,\,Q)=\left\{\frac{ab}{\beta p_x Q^{c}}\cdot\left(\frac{r}{z}\right)^{z}\cdot\left(\frac{w}{1-z}\right)^{1-z}\right\}^{\frac{1}{1-a}}, \tag{20}$$

$$K(p_x,\,r,\,w,\,Q)=\frac{1}{\beta}\left(\frac{z}{r}\,\frac{w}{1-z}\right)^{1-z}\left[Q-\frac{b}{Q^{c}}\left\{\frac{ab}{\beta p_x Q^{c}}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right\}^{\frac{a}{1-a}}\right],\ \textit{and} \tag{21}$$

$$L(p_x,\,r,\,w,\,Q)=\frac{1}{\beta}\left(\frac{1-z}{w}\,\frac{r}{z}\right)^{z}\left[Q-\frac{b}{Q^{c}}\left\{\frac{ab}{\beta p_x Q^{c}}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right\}^{\frac{a}{1-a}}\right]. \tag{22}$$

[6] An inferior input is a problem of organization, technology, and strategy, rather than the physical property of the input. For example, labor is an inferior input if it is used in the old production system that is harmful to the body, or that is slavish. These old systems work well in the poor circumstance. However, these dangerous systems lose their efficiency in large-scale production.

[7] The proofs of Proposition 4 and Proposition 5 are given in Appendix C.

(ii) For $0 \leq Q \leq \overline{Q}$, *the factor demand functions are given by*

$$x(p_x, r, w, Q) = b^{\frac{-1}{a}} Q^{\frac{1+c}{a}},$$

$$K(p_x, r, w, Q) = 0, \text{ and}$$

$$L(p_x, r, w, Q) = 0.$$

Proposition 4(i) shows the interior solution. Proposition 4(ii) shows the corner solution. From (20), we obtain $\partial x/\partial Q < 0$. Therefore, we can confirm that x is an inferior input. From (20)-(22), we also obtain derivatives such as $\partial x/\partial r > 0$, $\partial x/\partial w > 0$, $\partial L/\partial w < 0$, *and* $\partial K/\partial r < 0$, which are standard results due to a price change. However, $\partial K/\partial w$ and $\partial L/\partial r$ vary according to Q.

Proposition 5 (Cross-Price Effect)

(i) For $\overline{Q} \leq Q < \left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}} \cdot \overline{Q}$,

$$\frac{\partial K(p_x, r, w, Q)}{\partial w} < 0 \text{ and } \frac{\partial L(p_x, r, w, Q)}{\partial r} < 0$$

(ii) For $\left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}} \cdot \overline{Q} \leq Q$,

$$\frac{\partial K(p_x, r, w, Q)}{\partial w} \geq 0 \text{ and } \frac{\partial L(p_x, r, w, Q)}{\partial r} \geq 0.$$

Proposition 5(i) implies that if Q is so small that $\overline{Q} \leq Q \leq \left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}} \cdot \overline{Q}$, then capital demand is decreasing in wage. Similarly, labor demand is decreasing in rental rate. The inferior input is more suitable for production than the production system using K and L.

Yet, Proposition 5(ii) means that if Q is so large that $(1/(1-a))^{\frac{1-a}{1-a+c}} \cdot \overline{Q} \leq Q$, then the cross factor-price effects are negative and labor and capital are substitutes.

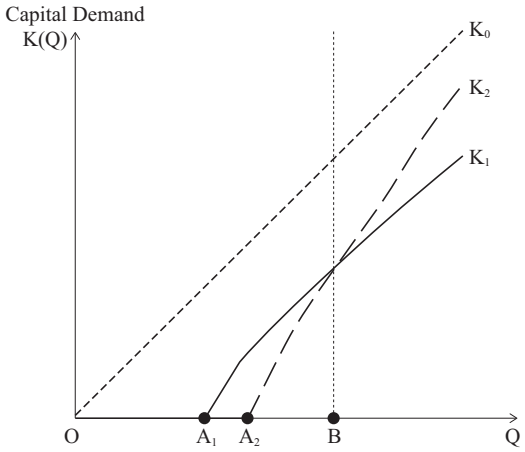
Figure 6 shows the Q-K graph of $K(p_x, r, w, Q)$. Let w_1 and w_2, ($w_1 < w_2$), denote wages. K_1 shows the graph of $K(p_x, r, \underline{w}, Q)$. Point A_1 represents $\overline{Q}(p_x, r, w_1)$, and point B represents $(1/(1-a))^{\frac{1-a}{1-a+c}} \cdot \overline{Q}(p_x, r, w_1)$.

If wage rises from w_1 to w_2, then A_1 moves to A_2, which represents $\overline{Q}(p_x, r, w_2)$. Point A_1 is located to the left of A_2 since $\partial \overline{Q}(p_x, r, w)/\partial w > 0$.

Proposition 5(i) means that K_2 is drawn below K_1 and to the left of point B. Proposition 5 (ii) means that K_2 is drawn above K_1 to the right of point B. K_0 represents $K(\cdot, r, w, Q)$, the original Cobb-Douglas function, where the inferior input is not available to the firm.

FIGURE 6. CROSS-PRICE EFFECT TO K

The K_0 straight line represents the factor demand curve with K (FDK) when the inferior input is not available for the firm. If the firm uses the inferior input, then FDK shifts downward from K_0 to K_1. If the wage increases, K_1 shifts to K_2. When Q is small, both K and L are substituted by x, rather than K substituting L. Therefore, FDK shift downward to K_2 for production range A_1-B.



V. *General Model*

1. Formulation of the General Case

In this section, we examine the general model. Utility $u(x, y)$ is now given by

$$u=b_1u^{-c_1}x_1^{a_1}+b_2u^{-c_2}x_2^{a_2}, \tag{23}$$

with the budget constraint $m=px_1+x_2$, where $0<a_i<1$, $0<b_i$, $0<c_i$, $0<p$, and $m>0$, (i=1, 2). The net-utility function is

$$g(x_1, u, m)=b_1u^{-c_1}x_1^{a_1}+b_2u^{-c_2}(m-px_1)^{a_2}-u=0. \tag{24}$$

Let u be a constant. The partial derivative, $\partial g/\partial x_1$, is

$$g_1(x_1, u, m)=a_1b_1u^{-c_1}x_1^{a_1-1}+a_2b_2u^{-c_2}(m-px_1)^{a_2-1}(-p)=0 \tag{25}$$

for $0\leq x_1\leq m/p$. In addition, we can confirm that

$$g_u(x_1, u, m)=-b_1c_1u^{-c_1-1}x_1^{a_1}-b_2c_2u^{-c_2-1}(m-px_1)^{a_2}-1<0,$$

$$g_m(x_1, u, m)=a_2b_2u^{-c_2}(m-px_1)^{a_2-1}>0, \textit{ and}$$

$$g_{11}(x_1, u, m)=a_1(a_1-1)b_1u^{-c_1}x_1^{a_1-2}+a_2(a_2-1)b_2u^{-c_2}(m-px_1)^{a_2-2}(-\mathrm{p})^2<0.$$

2. Compensated Demand

Let x_1 minimize the expenditure m subject to $g(x_1, u, m)=0$ with a fixed u. Since u is a constant, the two equations

$$\begin{cases} g(x_1, u, m)=0 \\ g_1(x_1, u, m)=0. \end{cases}$$

give the solution $(x_1^*(u), m^*(u))$. Mathematically, if $g_m>0$, then the solution $x_1^*(u)$ minimizes m, and the other solution $m^*(u)$ is its minimized value. Therefore, from (24) and (25), we obtain the compensated demand function, $x_1(p, u)$.[8]

Proposition 6 (Compensated demand)
(i) The compensated demand, $x_1(p, u)$, is given implicitly by

$$F(x_1, u)\equiv b_1u^{-c_1}x_1^{a_1}+b_2\left(\frac{a_2b_2p}{a_1b_1}\right)^{\frac{a_2}{1-a_2}}u^{\frac{c_1a_2-c_2}{1-a_2}}x_1^{(1-a_1)\frac{a_2}{1-a_2}}-u=0. \tag{26}$$

(ii) $x_1(p, u)>0$ for $p>0$ and $u>0$.
(iii) $\lim_{u\to 0} x_1(p, u)\to 0$.
(iv) If $\frac{c_1a_2-c_2}{1-a_2}>1$, then $\lim_{u\to\infty} x_1(p, u)\to 0$.
(v) If $\frac{c_1a_2-c_2}{1-a_2}\leq 1$, then $\lim_{u\to\infty} x_1(p, u)\to\infty$.

Property (ii) means that the solution is always interior for $p>0$ and $u>0$. Thus, our compensated demand function $F(x_i, u)=0$ never yields a corner solution. Property (iii) means that the income-consumption curve always stems from the origin. Property (iv) states that a good with a large c_1 is inferior when u is sufficiently large, but (iii) implies that it is a normal good when u is small. The OA curve in Figure 7 is the income-consumption curve of case (iv). Property (v) states that a good with a small c_1 is a normal good for all levels of u. The OB curve in Figure 7 is the income-consumption curve of case (v).

3. Normal Demand

Let x_1 maximize u with a given m. Since m is a constant, (24) and (25),

$$\begin{cases} g(x_1, u, m)=0 \\ g_1(x_1, u, m)=0, \end{cases}$$

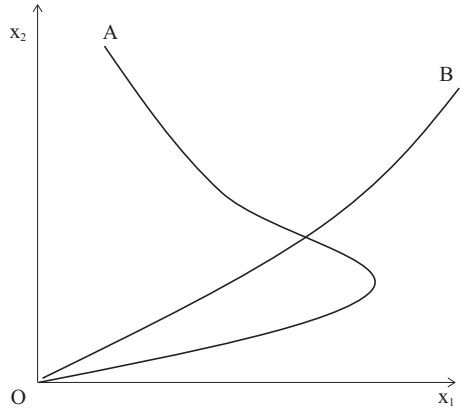
gives the solution (x_1^*, u^*). Mathematically, if $g_u<0$ holds, then the solution x_1^* maximizes u, and the other solution u^* is its maximized value. Therefore, from (24) and (25), we obtain the normal demand function $x_1(p, m)$.

Proposition 7 (Normal demand)

[8] We can also obtain (26) by using the Lagrange function $L=px_1+x_2+\lambda(b_1u^{-c_1}x_1^{a_1}+b_2u^{-c_2}x_2^{a_2}-u)$, instead of using the net-utility function (24).

FIGURE 7. INCOME-CONSUMPTION CURVE

The income-consumption Curve (ICC) is always positive both for x1 and x_2. Additionally, it always starts from origin point O. The OA curve is the ICC of Proposition 6 (iv). Both x_1 and x_2 are normal goods when income is low. However, x_1 changes into an inferior good as income increases. Curve OB is the ICC of Proposition 6(v). This shape of the ICC is that of the normal model.



(i) Normal demand $x_1(p, m)$ *is given implicitly by*

$$D(x_1, m) \equiv$$

$$b_1\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{-c_1}{-c_1+c_2}}x_1^{a_1}+b_2\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{-c_2}{-c_1+c_2}}(m-px_1)^{a_2}$$

$$-\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{1}{-c_1+c_2}}=0. \tag{27}$$

(ii) $x_1(p, m)>0$ *for* $p>0$, *and* $m>0$.

Proposition 7*(ii)* means that our function always yields an interior solution.

4. A numerical Example for the Two-goods Model

In this subsection, we provide a numerical example for the two-good model.

Case. 1: Normal good case

Consider the following parameters:

$$c_1=0.2,\ c_2=0.1,\ a_1=a_2=0.9,\ \textit{and}\ b_1=b_2=p=1.$$

The utility equation and budget constraint are

$$u=u^{-0.2}x_1^{0.9}+u^{-0.1}x_2^{0.9} \text{ and} \tag{28}$$

$$m=x_1+x_2,$$

respectively. From (26), the compensated demand equation is given by:

TABLE 1. THE TWO-GOODS MODEL (NORMAL GOODS)

m	0.3	0.8	1	2	3	5	10	20
x_1	0.22	0.44	0.49	0.72	0.85	1.05	1.31	1.58

$$F(x_1, u)=u^{-0.2}x_1^{0.9}+u^{0.8}x_1^{0.9}-u=0,$$

which is solvable for u. The compensated demand for the good 1 is

$$x_1(1, u)=\left(\frac{u}{u^{-0.2}+u^{0.9}}\right)^{\frac{1}{0.9}},$$

which is a normal good because $\frac{dx_1(u)}{du}>0$.

From (27), we obtain the normal demand, $x_1(1, m)$, as

$$\left(\frac{x_1}{m-x_1}\right)^{0.2}x_1^{0.9}+\left(\frac{x_1}{m-x_1}\right)^{0.1}(m-x_1)^{0.9}-\left(\frac{x_1}{m-x_1}\right)^{-1}=0. \tag{29}$$

Table 1 shows the values of $x_1(1, m)$ calculated using a computer.[9] For m = 5 and m = 10 in Table 1, the income elasticity is $\frac{\Delta x}{\Delta m}\frac{m}{x}=\frac{1.31-1.05}{10-5}\frac{5}{1.05}\doteqdot 0.25<1$. Thus, x_1 is a necessary good.

Case 2: Inferior good case

Consider the following parameters:

$$c_1=0.3,\ c_2=0.1,\ a_1=a_2=0.9, \text{ and } b_1=b_2=p=1.$$

The utility equation and the budget constraint are

$$u=u^{-0.3}x_1^{0.9}+u^{-0.1}x_2^{0.9} \text{ and} \tag{30}$$

$$m-x_1-x_2=0,$$

respectively. From (26), we have

$$F(x_1, u)=u^{-0.3}x_1^{0.9}+u^{1.7}x_1^{0.9}-u=0. \tag{31}$$

Here, we can solve (31) for x_1. Thus, we derive the compensated demand function as

$$x_1(1, u)=\left(\frac{u}{u^{-0.3}+u^{1.7}}\right)^{\frac{1}{0.9}}. \tag{33}$$

From (33), we have $lim_{u\to\infty}x_1(1, u)\to 0$. Therefore, x_1 is an inferior good for large a u, as discussed in Proposition 6(iv).

From (27), the normal demand function, $x_1(1, m)$ is implicitly given by

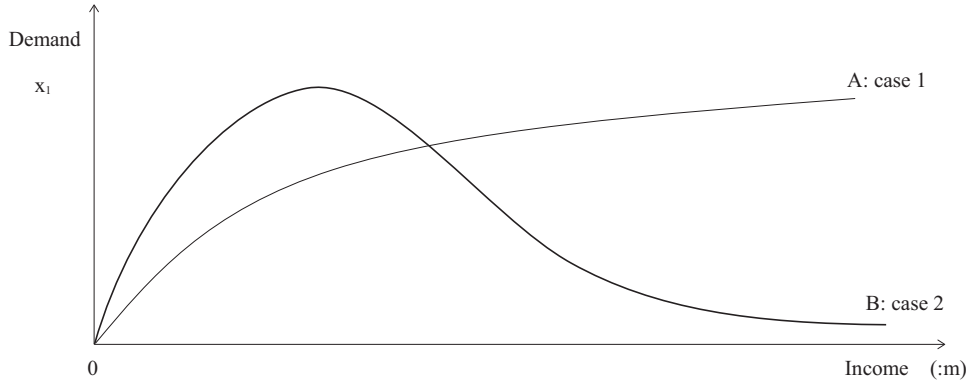
[9] The program code for Table 1 and Table 2 is attached as "055Eq (27) Table1and2.xlsx" for Excel, or "055Eq27CalculatorTwoGoods.m" for GNU octave. These programs are available upon a request.

TABLE 2. THE TWO-GOODS MODEL (INFERIOR GOODS)

m	0.3	0.8	1	2	3	5	10	20
x_1	0.26	0.45	0.49	0.5	0.43	0.34	0.23	0.14

FIGURE 8. ENGEL CURVE

Curve A is the Engel curve of the necessary good given by Eq (29), and curve B the Engel curve of an inferior good given by Eq (34).



$$D(x_1, m)=\left(\frac{x_1}{m-x_1}\right)^{0.15} x_1^{0.9}+\left(\frac{x_1}{m-x_1}\right)^{0.05}(m-x_1)^{0.9}-\left(\frac{x_1}{m-x_1}\right)^{-0.5}=0. \tag{34}$$

Table 5 shows the values of $x_1(m)$ calculated from (34).
From Table 2, we draw the Engel curve shown in Figure 8.

Figure 8 shows that good 1 is a normal good when m is small. However, as income increases, demand stagnates, eventually decreasing to almost zero.

VI. *Conclusion*

Our function is so flexible that it can generate very various income-consumption curves as shown in Figure 7. This flexibility is useful for analyzing the consumer's data. This paper concentrates on the two-variable model. However, this function can be extended to the multi-variable model. Using fully the implicit function may improve the researches in this fields including Giffen Case in future, since the approach we use in this paper make the function tractable and simple.

APPENDIX

A. Alternative Derivation of (7) and (15)

We derive $x(p, u)$ and $x(p, m)$ using the optimization technique of the implicit function. This

technique is an alternative to the Lagrange method for optimization of the implicit function.
Let the utility function and the budget constraint be[10]

$$u=bu^{-c}x^{a}+\beta y, \tag{A.1}$$

$$m=px+y. \tag{A.2}$$

From (A.1) and (A.2), by removing y, we obtain the in-budget utility as

$$g(x, u, m)\equiv bu^{-c}x^{a}+\beta(m-px)-u=0. \tag{A.3}$$

Derivation of (7)

The expenditure $m(x, \bar{u})$ is implicitly given by

$$g(x, \bar{u}, m(x, \bar{u}))=0$$

for a constant $\bar{u}$.
Mathematically, $m(x, \bar{u})$ is minimized if $g(x, \bar{u}, m)=0$, $g_x=0$, and $g_m\neq 0(>0)$. That is, the condition for the minimization is:

$$g(x, \bar{u}, m)=b\bar{u}^{-c}x^{a}+\beta(m-px)-\bar{u}=0, \tag{A.4}$$

$$g_x=ab\bar{u}^{-c}x^{a-1}-\beta p=0, \tag{A.5}$$

$$g_m=\beta\neq 0(>0). \tag{A.6}$$

From (A.4), we successfully obtain

$$x(\bar{u})=\left(\frac{ab}{\beta p\bar{u}^{c}}\right)^{1-a}. \tag{A.7}$$

By removing the bar from u and setting β, $b=1$, (A.7) becomes (7).

Q.E.D.

Derivation of (15)

The utility $u(x, \bar{m})$ is implicitly given by

$$g(x, u(x, \bar{m}), \bar{m})=0$$

for a constant $\bar{m}$.
Mathematically, the utility $u(x, \bar{m})$ is maximized if $g(x, u, \bar{m})=0$, $g_x=0$, and $g_u\neq 0(<0)$. That is, the condition for the maximization is:

$$g(x, u, \bar{m})=bu^{-c}x^{a}+\beta(\bar{m}-px)-u=0, \tag{A.8}$$

$$g_x=abu^{-c}x^{a-1}-\beta p=0, \tag{A.9}$$

$$g_u=-bcu^{-c-1}x^{a}-1\neq 0\ (<0). \tag{A.10}$$

[10] In Appendix A, we use $\bar{u}$ and $\bar{m}$ to represent "a constant value," although, in Appendix B and C, $\bar{u}$ and $\bar{m}$ are reused as "the threshold value." This duplication is only to reduce the variable.

From (A.9), we obtain

$$u=\left(\frac{ab}{\beta px^{1-a}}\right)^{\frac{1}{c}}. \tag{A.11}$$

From (A.8) and (A.11), we obtain

$$b\left(\left(\frac{ab}{\beta px^{1-a}}\right)^{\frac{1}{c}}\right)^{-c}x^a+\beta(\bar{m}-px)-\left(\frac{ab}{\beta px^{1-a}}\right)^{\frac{1}{c}}=0$$

$$\rightarrow\bar{m}+\left(\frac{1}{a}-1\right)px-\left(\frac{ab}{\beta px^{1-a}}\right)^{\frac{1}{c}}=0. \tag{A.12}$$

By removing the bar from m and setting β, $b=1$, (A.12) becomes (12).

Q.E.D.

B. Proof of Proposition 3

Proof of Proposition 3(i) and (ii)

(a) Factor demand function

The firm's optimization problem is

$$\min_{x,\,y}\ p_xx+p_yy$$

$$\text{s.t } Q^{-c}bx^a+\beta y-Q=0.$$

The Lagrange function is

$$\mathcal{L}=p_xx+p_yy-\lambda\left(\frac{b}{Q^c}x^a+\beta y-Q\right). \tag{B.1}$$

The first order condition is as follows:

$$\mathcal{L}_x=p_x-\lambda a\frac{b}{Q^c}x^{a-1}=0 \tag{B.2}$$

$$\mathcal{L}_y=p_y-\lambda\beta=0 \tag{B.3}$$

$$\mathcal{L}_\lambda=Q-\frac{b}{Q^c}x^a+\beta y=0. \tag{B.4}$$

From (B.2) and (B.3), we obtain

$$\frac{abQ^{-c}x^{a-1}}{p_x}=\frac{\beta}{p_y}$$

$$\rightarrow x^{a-1}=\frac{\beta}{abQ^{-c}}\cdot\frac{p_x}{p_y}$$

$$\rightarrow x=\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{1}{1-a}}. \tag{B.5}$$

From (B.4) and (B.5), the demand for y is

$$y=\frac{1}{\beta}\left\{Q-\frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\right\}. \tag{B.6}$$

In (B.6), $y(Q)$ is positive only if

$$\begin{aligned}
&Q-\frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\geq 0\\
\rightarrow &Q\geq\frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\\
\rightarrow &Q\cdot Q^c\cdot Q^{c\left(\frac{a}{1-a}\right)}\geq b\left(\frac{ab}{\beta}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\\
\rightarrow &Q^{\frac{1-a+c}{1-a}}\geq b\left(\frac{ab}{\beta}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\\
\rightarrow &Q^{\frac{1-a+c}{1-a}}\geq\left(\frac{ab^{\frac{1}{a}}}{\beta}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\\
\rightarrow &Q\geq\left(\frac{ab}{\beta}\cdot\frac{p_y}{p_x}\right)^{\frac{1-a}{1-a+c}}\equiv\overline{Q}(p_x,\,p_y).
\end{aligned} \tag{B.7}$$

The formula $Q\geq\overline{Q}(p_x,\,p_y)$ is the interior condition for y. From (B.5), (B.6), and (B.7), if $Q\leq\overline{Q}(p_x,\,p_y)$, then

$$y=0. \tag{B.8}$$

Therefore, from (B.4) and (B.8), we obtain

$$\begin{aligned}
&Q-bQ^{-c}x^a+\beta\cdot 0=0\\
\rightarrow &x=b^{\frac{-1}{a}}Q^{\frac{c+1}{a}}.
\end{aligned} \tag{B.9}$$

Summing up (B.5) and (B.9), we obtain the full statement of the factor demand function of $Q=Q^{-c}bx^a+\beta y$ as follows.

Factor Demand Function

The threshold level for the interior solution is

$$\overline{Q}(p_x,\,p_y)\equiv\left(\frac{ab}{\beta}\cdot\frac{p_y}{p_x}\right)^{\frac{1-a}{1-a+c}}. \tag{B.10}$$

For $Q\geq\overline{Q}(p_x,\,p_y)$,

$$x(p_x,\,p_y,Q)=\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{1}{1-a}}, \tag{B.11}$$

$$y(p_x, p_y, Q) = \frac{1}{\beta}\left\{Q - \frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\right\}. \tag{B.12}$$

For $Q \leq \overline{Q}(p_x, p_y)$,

$$x(p_x, p_y, Q) = b^{\frac{-1}{a}} Q^{\frac{c+1}{a}}, \tag{B.13}$$

$$y(p_x, p_y, Q) = 0. \tag{B.14}$$

(b) Total cost function

The total cost function for $Q \geq \overline{Q}(p_x, p_y)$ is given by

$$C = p_x \cdot x(p_x, p_y, Q) + p_y \cdot y(p_x, p_y, Q)$$

$$\rightarrow C = p_x\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{1}{1-a}} + p_y\frac{1}{\beta}\left\{Q - \frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\right\}$$

$$\rightarrow C = p_x\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{1}{1-a}} + p_y\frac{1}{\beta}Q - p_y\frac{1}{\beta}\left\{\frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}\right\}$$

$$\rightarrow C = p_x a^{\frac{1}{1-a}}\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-1}{1-a}} + p_y\frac{1}{\beta}Q - p_y\frac{1}{\beta}\frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\cdot\frac{p_y}{p_x}\right)^{\frac{a}{1-a}}$$

$$\rightarrow C = p_x a^{\frac{1}{1-a}}\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-1}{1-a}} + p_y\frac{1}{\beta}Q - p_y\frac{1}{\beta}\frac{b}{Q^c}\left(\frac{ab}{\beta Q^c}\right)^{\frac{a}{1-a}} p_y^{\frac{a}{1-a}} p_x^{\frac{-a}{1-a}}$$

$$\rightarrow C = a^{\frac{1}{1-a}}\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}} + p_y\frac{1}{\beta}Q - \left(\frac{1}{\beta}\frac{b}{Q^c}\right)\left(\frac{ab}{\beta Q^c}\right)^{\frac{a}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}}$$

$$\rightarrow C = a^{\frac{1}{1-a}}\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}} + p_y\frac{1}{\beta}Q - a^{\frac{a}{1-a}}\left(\frac{1}{\beta}\frac{b}{Q^c}\right)\left(\frac{b}{\beta Q^c}\right)^{\frac{a}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}}$$

$$\rightarrow C = a^{\frac{1}{1-a}}\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}} + p_y\frac{1}{\beta}Q - a^{\frac{a}{1-a}}\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}}$$

$$\rightarrow C = p_y\frac{1}{\beta}Q - \left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}} \tag{B.15}$$

The total cost function for $Q \leq \overline{Q}(p_x, p_y)$ is given by

$$C = p_x \cdot x(p_x, p_y, Q) + p_y \cdot y(p_x, p_y, Q)$$

$$\rightarrow C = p_x b^{\frac{-1}{a}} Q^{\frac{c+1}{a}} + p_y \cdot 0$$

$$\rightarrow C = p_x b^{\frac{-1}{a}} Q^{\frac{c+1}{a}}. \tag{B.16}$$

Summing up (B.15) and (B.16), we obtain the full description of $C(Q)$ as $Q = Q^{-c} b x^a + \beta y$, as follows.

Total Cost Function

For $Q \geq \overline{Q}(p_x, p_y)$,

$$C(Q, p_x, p_y) = p_y \frac{1}{\beta} Q - \left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)\left(\frac{b}{\beta Q^c}\right)^{\frac{1}{1-a}} p_y^{\frac{1}{1-a}} p_x^{\frac{-a}{1-a}}. \tag{B.17}$$

For $Q \geq \overline{Q}(p_x, p_y)$,

$$C(Q, p_x, p_y) = p_x b^{\frac{-1}{a}} Q^{\frac{c+1}{a}}. \tag{B.18}$$

If b = 1 and $\beta = 1$, then (B.17) and (B.18) give (16) and (17).

Q.E.D.

Proof of Proposition 3(iii)

The term $\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) \cdot p_x^{\frac{-a}{1-a}} \cdot p_y^{\frac{1}{1-a}} \cdot Q^{\frac{-c}{1-a}}$ in (A6) is positive since $0 < a < 1$.

The limit of this term is

$$\lim_{u \to \infty}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) \cdot p_x^{\frac{-a}{1-a}} \cdot p_y^{\frac{1}{1-a}} \cdot Q^{\frac{-c}{1-a}} \to 0.$$

Then, the total cost curve must be approaching the straight line $C = p_y Q$ as Q increases. Thus, the TC curve is drawn below the dotted straight line labelled $C = p_y Q$ in Figure 5. The TC curve O-A-TC in Figure 5 takes a *spoon* shape.

Q.E.D.

C. Proofs of Proposition 4 and Proposition 5

Proofs of Proposition 4(i)–(iii)

The Lagrange function is

$$\mathcal{L} = p_x x + rK + wL - \lambda\left(\frac{b}{Q^c} x^a + \beta K^z L^{1-z} - Q\right). \tag{C.1}$$

The FOC is

$$\mathcal{L}_x = p_x - \lambda a \frac{b}{Q^c} x^{a-1} = 0, \tag{C.2}$$

$$\mathcal{L}_K = r - \lambda \beta z K^{z-1} = 0, \tag{C.3}$$

$$\mathcal{L}_L = w - \lambda \beta (1-z) L^{-z} = 0, \tag{C.4}$$

$$\mathcal{L}_\lambda = Q - \frac{b}{Q^c} x^a + \beta K^z L^{1-z} = 0. \tag{C.5}$$

From (C.3) and (C.4), we obtain

$$K = \frac{z}{r} \frac{w}{1-z} L \tag{C.6a}$$

or

$$L=\frac{r}{z}\frac{1-z}{w}K. \tag{C.6b}$$

From (C.2) and (C.3) we obtain

$$\frac{ab}{p_xQ^c}x^{a-1}=\frac{(1-z)}{w}\beta K^zL^{-z}. \tag{C.7}$$

From (C.6a) and (C.7), we obtain

$$\frac{ab}{p_xQ^c}x^{a-1}=\frac{(1-z)}{w}\beta\left(\frac{z}{r}\frac{w}{1-z}L\right)^zL^{-z}. \tag{C.8}$$

By solving (C.8) for x, we obtain

$$x(p_x, r, w, Q)=\left[\frac{ab}{\beta p_xQ^c}\cdot\left(\frac{r}{z}\right)^z\cdot\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{1}{1-a}}. \tag{C.9}$$

From (C.5), (C.6b), and (C.9), we obtain

$$K(p_x, r, w, Q)=\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}\left\{Q-\frac{b}{Q^c}\left[\frac{ab}{\beta p_xQ^c}\left(\frac{r}{z}\right)^z\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}\right\}. \tag{C.10}$$

From (C.5), (D6a), and (C.9),

$$L(p_x, r, w, Q)=\frac{1}{\beta}\left(\frac{1-z}{w}\frac{r}{z}\right)^z\left\{Q-\frac{b}{Q^c}\left[\frac{ab}{\beta p_xQ^c}\left(\frac{r}{z}\right)^z\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}\right\}. \tag{C.11}$$

(C.9)–(C.11) give (20)–(22), respectively.
The value of $K(p_x, r, w, Q)$ and $L(p_x, r, w, Q)$ shown in (C.10) and (C.11) must be non-negative for $Q\geq 0$. From (C.10), $K(p_x, r, w, Q)\geq 0$ if

$$\begin{aligned}
&0\leq Q-\frac{b}{Q^c}\left[\frac{ab}{\beta p_xQ^c}\left(\frac{r}{z}\right)^z\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}\\
&\rightarrow b^{\frac{1-a}{a}}\cdot\left[\frac{ab}{\beta p_x}\left(\frac{r}{z}\right)^z\left(\frac{w}{1-z}\right)^{1-z}\right]\leq Q^{(c+1)\frac{1-a}{a}+c}\\
&\rightarrow\overline{Q}(p_x, r, w)\equiv\left[\frac{ab^{\frac{1}{a}}}{\beta p_x}\left(\frac{r}{z}\right)^z\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{c}{1-a+c}}\leq Q.
\end{aligned} \tag{C.12}$$

The LHS of (C.12) is denoted as $\overline{Q}(p_x, r, w)$ in (19). Therefore, we obtain the interior condition of K as

$$\overline{Q}(p_x, r, w)\leq Q. \tag{C.13}$$

Similar calculation to (C11)–(C.13) gives $L(p_x, r, w, Q)\geq 0$ for (C.13).
(C.13) is used in Proposition 4(i).

Q.E.D.

Proof of Proposition 4(ii)

If $0 \leq Q \leq \overline{Q}$, $K(p_x, r, w, Q)=0$, and $L(p_x, r, w, Q)=0$, then the production function is simply the single variable function $Q=bQ^{-c}x^{a}$. Solving this for u, we obtain $x(p_x, r, w, Q)=b^{\frac{-1}{a}}Q^{\frac{1+c}{a}}$. Therefore, we obtain (ii).

Q.E.D.

Proof of Proposition 5(i)

By removing the curly braces from (C.10), for $\overline{Q}<Q$, we obtain

$$K(p_x, r, w, Q)=\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}Q-\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}\cdot\frac{b}{Q^{c}}\left[\frac{ab}{\beta p_x Q^{c}}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}. \tag{C.14}$$

The differentiation of (C.14) for w is negative if

$$\frac{\partial K}{\partial w}=\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}Q(1-z)w^{-1}-\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}\frac{b}{Q^{c}}\left[\frac{ab}{\beta p_x Q^{c}}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}(1-z)\frac{1}{1-a}w^{-1}<0,$$

which is rewritten as

$$\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}Q(1-z)w^{-1}<\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}\frac{b}{Y^{c}}\left[\frac{ab}{\beta p_x Q^{c}}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}(1-z)\frac{1}{1-a}w^{-1}. \tag{C.15}$$

In the variables of (C.15), three terms $\frac{1}{\beta}\left(\frac{z}{r}\frac{w}{1-z}\right)^{1-z}$, $(1-z)$, and w^{-1} appear on both the LHS and RHS. Thus, we can remove these from (C.15). Therefore, (C.15) is now

$$Q<\frac{b}{Q^{c}}\left[\frac{ab}{\beta p_x Q^{c}}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}\frac{1}{1-a} \tag{C.16}$$

$$\rightarrow Q<\frac{1}{Q^{c}}\left(\frac{1}{Q^{c}}\right)^{\frac{a}{1-a}}b\left[\frac{ab}{\beta p_x}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}\frac{1}{1-a}$$

$$\rightarrow Q^{1+c+\frac{ca}{1-a}}<b\left[\frac{ab}{\beta p_x}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a}}\frac{1}{1-a}$$

$$\rightarrow Q<\left[\frac{ab^{\frac{1}{a}}}{\beta p_x}\left(\frac{r}{z}\right)^{z}\left(\frac{w}{1-z}\right)^{1-z}\right]^{\frac{a}{1-a+c}}\left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}}$$

$$\rightarrow Q<\overline{Q}\cdot\left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}}. \tag{C.17}$$

From (C.14)–(C.17), we successfully obtain

$$\frac{\partial K(p_x, r, w, Q)}{\partial w}<0 \text{ for } \overline{Q}\leq Q<\left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}}\cdot\overline{Q}. \tag{C.18}$$

A similar calculation to (C.14)–(C.17) gives

$$\frac{\partial L(p_x, r, w, Q)}{\partial \mathrm{r}} < 0 \text{ for } \overline{Q} \leq Q < \left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}} \cdot \overline{Q}. \tag{C.19}$$

Q.E.D.

Proof of Proposition 5(ii)

By changing the inequality "$<$" of (C.14)–(C.17) by "$\geq$," we obtain

$$\frac{\partial K(p_x, r, w, Q)}{\partial w} \geq 0 \text{ for } \left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}} \cdot \overline{Q} \leq Q.$$

A similar calculation to (C.14)–(C.17) gives

$$\frac{\partial L(p_x, r, w, Q)}{\partial r} \geq 0 \text{ for } \left(\frac{1}{1-a}\right)^{\frac{1-a}{1-a+c}} \cdot \overline{Q} \leq Q.$$

Q.E.D.

D. Proof of Proposition 6

Proof of Proposition 6(i)

From (25), we obtain

$$\begin{aligned} & g_1(x_1, u, m) = a_1 b_1 u^{-c_1} x_1^{a_1-1} + a_2 b_2 u^{-c_2} (m - p x_1)^{a_2-1} (-p) = 0 \\ \rightarrow & a_1 b_1 u^{-c_1} x_1^{a_1-1} + a_2 b_2 u^{-c_2} (m - p x_1)^{a_2-1} (-p) = 0 \\ \rightarrow & m - p x_1 = \left(\frac{a_1 b_1 u^{-c_1}}{a_2 b_2 p u^{-c_2}} x_1^{a_1-1}\right)^{\frac{1}{a_2-1}}. \end{aligned} \tag{D.1}$$

From (24) and (D.1), we obtain

$$\begin{aligned} & g(x_1, u, m) = b_1 u^{-c_1} x_1^{a_1} + b_2 u^{-c_2} (m - p x_1)^{a_2} - u = 0 \\ \rightarrow & b_1 u^{-c_1} x_1^{a_1} + b_2 u^{-c_2} \left(\frac{a_1 b_1 u^{-c_1} x_1^{a_1-1}}{a_2 b_2 p u^{-c_2}}\right)^{\frac{a_2}{a_2-1}} - u = 0 \\ \rightarrow & F(x_1, u) \equiv b_1 u^{-c_1} x_1^{a_1} + b_2 \left(\frac{a_2 b_2 p}{a_1 b_1}\right)^{\frac{a_2}{1-a_2}} u^{\frac{c_1 a_2 - c_2}{1-a_2}} x_1^{(1-a_1)\frac{a_2}{1-a_2}} - u = 0. \end{aligned} \tag{D.2}$$

(D.2) is (26).

Proof of Proposition 6(ii)

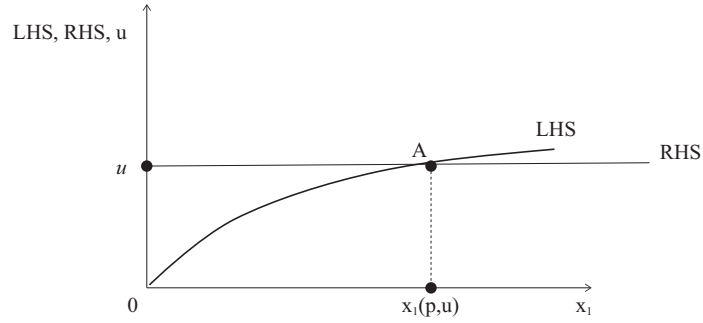
We must prove that $x_1(p, u) > 0$ for $p > 0$ and $u > 0$.

Let us decompose (D.2) into two functions, LHS and RHS, where

$$LHS(x_1, u) = b_1 u^{-c_1} x_1^{a_1} + b_2 \left(\frac{a_2 b_2 p}{a_1 b_1}\right)^{\frac{a_2}{1-a_2}} u^{\frac{c_1 a_2 - c_2}{1-a_2}} x_1^{(1-a_1)\frac{a_2}{1-a_2}}, \text{ and}$$

FIGURE 9. EXISTENCE OF $x_1(p,u)$

The compensated demand function given by (D.2) is also unsolvable for x. However, (D.2) gives the unique and real demand value. The RHS curve on (x1, u) is given by the right-hand side of (D.2), which is always monotonically increasing in x_1. RHS is a horizontal line, since RHS = u. Therefore, the two curves have a unique intersection (point A). Point A gives the compensated demand $x_1(p,u)$.



$$RHS(x_1, u) = u.$$

For a given fixed u, the graph of LHS and RHS is shown in Figure 8. The curve of LHS is upward-sloping with the origin zero.

In Figure 9, two curves always have a single intersection (Point A). This point gives $b_1 u^{-c_1} x_1^{a_1} + b_2\left(\frac{a_2 b_2 p}{a_1 b_1}\right)^{\frac{a_2}{1-a_2}} u^{\frac{c_1 a_2 - c_2}{1-a_2}} x_1^{(1-a_1)\frac{a_2}{1-a_2}} - u = 0$. Point A gives the compensated demand $x_1(p, u)$. Q.E.D.

Proof of Proposition 6(iii)

For three terms that contain u in (D.2), the limit of these are

$$\lim_{u\to 0} u^{-c_1} \to \infty,$$

$$\lim_{u\to 0} u^{\frac{c_1 a_2 - c_2}{1-a_2}} \to 0, \text{ and}$$

$$\lim_{u\to 0} u \to 0.$$

Therefore, from these and (D.2), we obtain $lim_{u\to 0}\, x_1(p, u) \to 0$. Q.E.D.

Proof of Proposition 6(iv)

We must show that, if $\frac{c_1 a_2 - c_2}{1-a_2} > 1$, then $lim_{u\to\infty}\, x_1(p, u) \to 0$.

Dividing (D.2) by u, we rewrite (D.2) as

$$b_1 u^{-c_1 - 1} x_1^{a_1} + b_2\left(\frac{a_2 b_2 p}{a_1 b_1}\right)^{\frac{a_2}{1-a_2}} u^{\frac{c_1 a_2 - c_2}{1-a_2} - 1} x_1^{(1-a_1)\frac{a_2}{1-a_2}} - 1 = 0. \quad \text{(D.3)}$$

For two terms that contain u in (D.3), we obtain

$$\lim_{u\to\infty} u^{-c_1 - 1} \to 0, \text{ and}$$

$$\lim_{u\to\infty} u^{\frac{c_1a_2-c_2}{1-a_2}-1}\to\infty \quad \left(\because \frac{c_1a_2-c_2}{1-a_2}>1\right).$$

Therefore, from these and (D.3), we obtain $lim_{u\to\infty} x_1(p, u)\to 0$ Q.E.D.

Proof of Proposition 6(v)

We must show that, if $\frac{c_1a_2-c_2}{1-a_2}\leq 1$, then $lim_{u\to\infty} x_1(p, u)\to\infty$.

The compensated demand x_1 must hold (D.3). For the two terms that contain u in (D.3), we obtain

$$\lim_{u\to\infty} u^{-c_1-1}\to 0, \text{ and}$$

$$\lim_{u\to\infty} u^{\frac{c_1a_2-c_2}{1-a_2}-1}\to 0 \quad \left(\because \frac{c_1a_2-c_2}{1-a_2}<1\right).$$

Therefore, from these and (D.3), x_1 must be an infinite. Then, we obtain $lim_{u\to\infty} x_i(p, u)\to\infty$. Q.E.D.

E. Proof of Proposition 7

Proof of Proposition 7(i)

From (25), for $c_1\neq c_2$, we obtain

$$\left(\frac{a_ib_i}{a_2b_2p}u^{-c_i+c_2}x_1^{a_i-1}\right)^{\frac{1}{a_2-1}}=m-px_1 \tag{E.1}$$

$$\to u=\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{1}{-c_1+c_2}}. \tag{E.2}$$

By substituting (E.2) into (24), we obtain

$$b_1\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{-c_1}{-c_1+c_2}}x_1^{a_1}+b_2\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{-c_2}{-c_1+c_2}}(m-px_1)^{a_2}-\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{1}{-c_1+c_2}}=0 \tag{E.3}$$

Equation (E.3) is (27). Q.E.D.

Proof of Proposition 7(ii)

We must show that (E.3) always gives $x_1(p, m)$ as a positive real number.

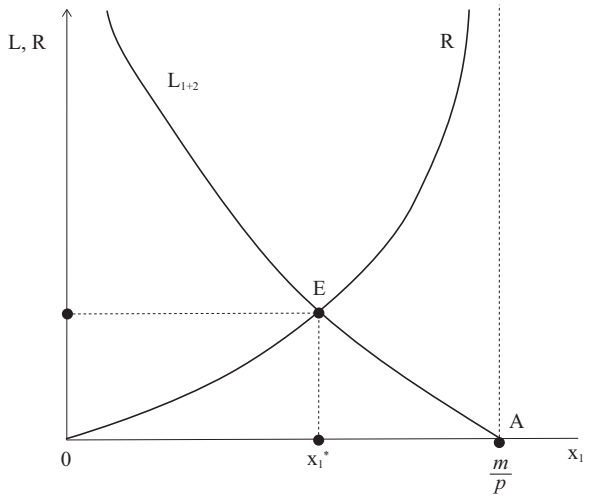
We rewrite (E.3) as

$$b_1\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{-c_1}{-c_1+c_2}}x_1^{a_1}+b_2\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{-c_2}{-c_1+c_2}}(m-px_1)^{a_2}=\left(\frac{x_1^{1-a_1}}{(m-px_1)^{1-a_2}}\frac{a_2b_2p}{a_1b_1}\right)^{\frac{1}{-c_1+c_2}}. \tag{E.4}$$

Figure 10 shows the case of $c_1<c_2$. This implies $\frac{-c_1}{-c_1+c_2}<0$, $\frac{-c_2}{-c_1+c_2}<0$, and $\frac{1}{-c_1+c_2}>0$. In Figure 10, the R curve represents the RHS of (E.4). On the other hand, the L_{1+2} curve represents the LHS of (E.4).[11] At point E, the RHS is equal to the LHS. Then, both x_1^* and u^* are given by E. If m increases, then point A moves to the right. Thus, point E moves either left or right. Q.E.D.

FIGURE 10. NORMAL DEMAND

The L_{1+2} curve represents the sum of the first and second terms of the left-hand side of (E.4), whose value is [0, ∞). The R curve represents the right-hand side of (E.4), whose value is [0, ∞) and is increasing in x1. Point E gives the demand for x_1. If m increases, then point A shifts to the right.



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[11] The L_2 curve in Figure 10 represents the second term of the LHS. Although L_2 is drawn as a concave curve, it is not always like this. Depending on a_1, a_2, c_1, and c_2, the L_2 curve can be an upward-sloping curve. However, regardless of whether it is concave or upward-sloping, L_{1+2} remains an upward-sloping curve.