A THEOREM ON AN INTERMEDIATE PREDICATE LOGIC

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Umezawa [4] formulated various intermediate predicate logics and investigated the relations between them. One of his intermediate logics is the system *LD* obtained from the intuitionistic predicate logic by adding the axiom schema

$$\forall x(A \vee B(x)) \supset A \vee \forall x B(x).$$

On the other hand, Gabbay [1] introduced an intermediate logic *CD* as the logic determined semantically by constant-domain Kripke structures. Klemke, Görnemann and the author independently proved that *CD* is equivalent to *LD*. Motohashi established certain faithful interpretations of *LJ* and *LD* in a two-sorted logic**. The two-sorted system, which we shall call "*tree logic T*", is a syntactic counterpart of Kripke structure.

For any formula A and a unary predicate P not occurring in A, let A^P denote the formula obtained from A by relativizing every occurrence of quantifier to P. For any formula A whose free variables are $a_1, \dots, a_n$, the *P-relativization* of A is the formula

$$\exists x P(x) \wedge P(a_1) \wedge \dots \wedge P(a_n) \supset A^P.$$

The author [3] conjectured that a formula is provable in *LJ* if and only if its *P*-relativization is provable in *LD*. Semantically, this is evident. We shall prove this statement by finitary methods. The proof uses Motohashi's faithful interpretations.

We suppose that the logics *LJ* and *LD* are formulated in a first-order language *L*. Now we introduce the tree logic *T*. The language of *T* is two-sorted. In addition to the free and the bound *L*-variables, *T* has the free tree variables $\alpha, \beta, \dots$ and the bound tree variables $\xi, \eta, \dots$. For any n-ary *L*-predicate P, there corresponds a $(1, n)$-ary *T*-predicate P'. Moreover, *T* has a (1, 1)-ary predicate U and a (2, 0)-ary predicate $\leq$. If Q is an (m, n)-ary *T*-predicate, an expression of the form

$$Q(\alpha_1, \dots, \alpha_m, a_1, \dots, a_n)$$

is an atomic *T*-formula. We write $\alpha \leq \beta$ instead of $\leq(\alpha, \beta)$. If A and B are *T*-formulae, then $\neg A$, $A \wedge B$, $A \vee B$ and $A \supset B$ are *T*-formulae. If $F(a)$ is a *T*-formula, then $\forall x F(x)$ and $\exists x F(x)$ are *T*-formulae. If $G(\alpha)$ is a *T*-formula, then $\forall \xi G(\xi)$ and $\exists \xi G(\xi)$ are *T*-formulae. The axioms and the inference rules of *T* are those of the classical two-sorted first-order predicate logic and the additional axioms:

$$\forall \xi (\xi \leq \xi)$$

and

$$\forall \xi \forall \eta \forall \zeta (\xi \leq \eta \wedge \eta \leq \zeta \supset \xi \leq \zeta).$$

For any free tree variable α, mappings f_α and g_α (from the set of *L*-formulae into the set of *T*-formulae) are defined recursively as follows***:

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** Motohashi calls it "two-sorted classical predicate logic *LK*".
*** [2, § 1]. Modifications due to the present author.

$$f_\alpha(P(a_1, \dots, a_n)) = P'(\alpha, a_1, \dots, a_n),$$
$$g_\alpha(P(a_1, \dots, a_n)) = P'(\alpha, a_1, \dots, a_n),$$
$$f_\alpha(\neg A) = \forall \xi(\alpha \leq \xi \supset \neg f_\xi(A)),$$
$$g_\alpha(\neg A) = \forall \xi(\alpha \leq \xi \supset \neg g_\xi(A)),$$
$$f_\alpha(A \wedge B) = f_\alpha(A) \wedge f_\alpha(B),$$
$$g_\alpha(A \wedge B) = g_\alpha(A) \wedge g_\alpha(B),$$
$$f_\alpha(A \vee B) = f_\alpha(A) \vee f_\alpha(B),$$
$$g_\alpha(A \vee B) = g_\alpha(A) \vee g_\alpha(B),$$
$$f_\alpha(A \supset B) = \forall \xi(\alpha \leq \xi \supset (f_\xi(A) \supset f_\xi(B))),$$
$$g_\alpha(A \supset B) = \forall \xi(\alpha \leq \xi \supset (g_\xi(A) \supset g_\xi(B))),$$
$$f_\alpha(\forall x A(x)) = \forall \xi \forall x(\alpha \leq \xi \supset (U(\xi, x) \supset f_\xi(A(x)))),$$
$$g_\alpha(\forall x A(x)) = \forall \xi \forall x(\alpha \leq \xi \supset g_\xi(A(x))),$$
$$f_\alpha(\exists x A(x)) = \exists x(U(\alpha, x) \wedge f_\alpha(A(x))),$$
$$g_\alpha(\exists x A(x)) = \exists x g_\alpha(A(x)).$$

For any L-formula A, let $U_\alpha(A)$ be the conjunction of formulae $U(\alpha, a)$ for all the free variables a occurring in A, let Ag be the conjunction of sentences

$$\forall \xi \forall \eta \forall x_1 \dots \forall x_n(\xi \leq \eta \supset (P'(\xi, x_1, \dots, x_n) \supset P'(\eta, x_1, \dots, x_n)))$$

for all the predicates P occurring in A, and let Af be the conjunction of the sentences

$$\forall \xi \forall \eta \forall x(\xi \leq \eta \supset (U(\xi, x) \supset U(\eta, x))),$$
$$\forall \xi \exists y U(\xi, y)$$

and Ag. Mappings f and g from the set of L-formulae into the set of T-formulae are defined as follows:

$$f(A) = Af \supset \forall \xi(U_\xi(A) \supset f_\xi(A)),$$
$$g(A) = Ag \supset \forall \xi g_\xi(A).$$

FAITHFUL INTERPRETATION THEOREM (Motohashi [2]). *For any L-formula A, $f(A)$ is provable in T if and only if A is provable in the intuitionistic predicate logic LJ. For any L-formula A, $g(A)$ is provable in T if and only if A is provable in the intermediate predicate logic LD.*

For any L-formula A and any tree variable α, let $h_\alpha(A)$ be the result of substitution of U for P' in $g_\alpha(A^P)$ where P is a unary L-predicate not occurring in A.

LEMMA. *For any L-formula A and any free tree variable α, $h_\alpha(A) \sim f_\alpha(A)$ is provable in T.*

Proof. By induction. Fix a unary L-predicate P and let F^* denote the formula obtained by substituting U for P' in a T-formula F. Then $h_\alpha(A)$ is $g_\alpha(A^P)^*$. For any atomic L-formula A,

$$h_\alpha(A) = g_\alpha(A^P)^* = g_\alpha(A)^* = g_\alpha(A) = f_\alpha(A).$$

Induction step is proved by dividing cases according to the outermost logical symbol occurrence of A.

Case 1. A is $B \wedge C$. Then $f_\alpha(A) = f_\alpha(B) \wedge f_\alpha(C)$ and

$$h_\alpha(A) = g_\alpha((B \wedge C)^P)^* = g_\alpha(B^P \wedge C^P)^* = (g_\alpha(B^P) \wedge g_\alpha(C^P))^* = h_\alpha(B) \wedge h_\alpha(C),$$

hence $h_\alpha(A) \sim f_\alpha(A)$ is deducible from the induction hypotheses.

Case 2. A is $B \supset C$. Then

$$h_\alpha(A) = g_\alpha((B \supset C)^P)^* = (\forall \xi(\alpha \leq \xi \supset (g_\xi(B^P) \supset g_\xi(C^P))))^* = \forall \xi(\alpha \leq \xi \supset (h_\xi(B) \supset h_\xi(C)))$$

and

$$f_\alpha(A) = \forall \xi(\alpha \leq \xi \supset (f_\xi(B) \supset f_\xi(C))),$$

hence $h_\alpha(A) \sim f_\alpha(A)$ follows from the induction hypotheses.

Case 3. A is $\forall x F(x)$. Then

$$h_\alpha(A) = g_\alpha(\forall x F(x)^P)^* = g_\alpha(\forall x(P(x) \supset F^P(x)))^*$$
$$= (\forall \xi \forall x(\alpha \leq \xi \supset g_\xi(P(x) \supset F^P(x))))^*$$
$$= (\forall \xi \forall x(\alpha \leq \xi \supset \forall \eta(\xi \leq \eta \supset (P'(\eta, x) \supset g_\eta(F^P(x))))))^*.$$

hence $h_\alpha(A) \sim k_\alpha(A)$ is deducible from reflexiveness and transitivity of $\leq$ where

$$k_\alpha(A) = (\forall \xi \forall x(\alpha \leq \xi \supset (P'(\xi, x) \supset g_\xi(F^P(x)))))^* = \forall \xi \forall x(\alpha \leq \xi \supset (U(\xi, x) \supset h_\xi(F(x)))).$$

Since $f_\alpha(A) = \forall \xi \forall x(\alpha \leq \xi \supset (U(\xi, x) \supset f_\xi(F(x))))$, $h_\alpha(A) \sim f_\alpha(A)$ is deducible from the induction hypothesis.

Case 4. A is $\exists x F(x)$. Then $h_\alpha(A) \sim f_\alpha(A)$ follows from

$$h_\alpha(A) = \exists x(U(\alpha, x) \wedge h_\alpha(F^P(x))),$$
$$f_\alpha(A) = \exists x(U(\alpha, x) \wedge f_\alpha(F(x)))$$

and the induction hypothesis.

The other cases are treated similarly.

THEOREM. *Let A be a formula, P be a unary predicate not occurring in A, and a_1, ..., a_n be the list of all free variables occurring in A. Then A is provable in LJ if and only if*

$$\exists x P(x) \wedge P(a_1) \wedge \ldots \wedge P(a_n) \supset A^P$$

is provable in LD.

Proof. If A is provable in *LJ*, then $\exists x P(x) \wedge P(a_1) \wedge \ldots \wedge P(a_n) \supset A^P$ is provable in *LJ*, hence provable in *LD*. Conversely, assume that

$$\exists x P(x) \wedge P(a_1) \wedge \ldots \wedge P(a_n) \supset A^P$$

is provable in *LD*. Let $P_\alpha(A)$ be the conjunction of $P'(\alpha, a_i)$ for all i and let $M(Q)$ denote $\forall \xi \forall \eta \forall x(\xi \leq \eta \supset (Q(\xi, x) \supset Q(\eta, x)))$. By Motohashi's Faithful Interpretation Theorem,

$$A^P g \supset \forall \xi \forall \eta(\xi \leq \eta \supset (\exists x P'(\eta, x) \wedge P_\eta(A) \supset g_\eta(A^P))$$

is provable in T. Hence

$$Ag \wedge M(P') \wedge \exists x P'(\alpha, x) \wedge P_\alpha(A) \supset g_\alpha(A^P)$$

is provable in T. Substituting U for P', we obtain

$$Ag \wedge M(U) \wedge \exists x U(\alpha, x) \wedge U_\alpha(A) \supset h_\alpha(A),$$

i.e. $Af \wedge U_\alpha(A) \supset h_\alpha(A)$. By Lemma, this is equivalent to

$$Af \wedge U_\alpha(A) \supset f_\alpha(A).$$

Hence $f(A)$ is provable in T. *LJ*-provability of A follows by Motohashi's Faithful Interpretation Theorem.

COROLLARY. *Let A be a formula, P be a unary predicate not occurring in A, and $a_1, \ldots, a_n$ be the list of all free variables occurring in A. Then the P-relativization*

$$\exists x P(x) \wedge P(a_1) \wedge \ldots \wedge P(a_n) \supset A^P$$

of A is provable in LD if and only if it is provable in LJ.

Corrections of [3]. Page 53. For "(1)", read "[1]". For "$2^T = \{T, F\}$", read "$2^T \cup 2^D \cup 2^{D \times D} \cup 2^{D \times D \times D} \cup \ldots$". Page 55. In proof of Theorem 2, for "$\exists y F y$" read "$\exists x F x$". Page 56. In proof of Theorem 3, the treatment of the free variables in Case 1 is incorrect. The sequence of free variables exhibited in the antecedent of the left-hand uppermost sequent should be the sequence of the free variables occurring in D but not occurring in Γ, Θ. Similarly, the sequence in the right-hand sequent should be that of the free variables occurring in D but not occurring in Δ, Λ.

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