AN INTERMEDIATE PREDICATE LOGIC

By TAKASHI NAGASHIMA*

This paper is a preliminary report on the intermediate predicate logic called *CD* by Gabbay. Gabbay (1) defined *CD* as the logic determined semantically by Kripke structures with constant domains, and he mentioned that the intuitionistically unprovable formula

$$\forall x(F(x) \vee G(x)) \supset \exists xF(x) \vee \forall xG(x)$$

is valid in *CD*. Kripke remarked that the formula

$$\forall x(A \vee F(x)) \supset A \vee \forall xF(x)$$

is valid in any structure with constant domain. Gabbay raised the problem that whether *CD* is axiomatizable or not. In this paper, we propose two axiomatizations of *CD*: the one is a sequent calculus intermediate between Gentzen's *LJ* and *LK*, the other is intuitionistic predicate logic with Gabbay's formula as the additional axiom schema. Further axiomatizations of *CD* will be published later on. Independently of the author, Görnemann [3] established an axiomatization of *CD*. She adopts Kripke's formula as the additional axiom schema to the intuitionistic predicate logic. Besides axiomatizations, some considerations on the logic *CD* is given in this paper. Except the unprovability results, proofs are carried out by using syntactical methods. Gentzen's Hauptsatz does not hold for our sequent calculus of *CD*. This fact causes some difficulties in applying syntactical methods.

An *atomic formula* is an expression of the form $Pa_1 \ldots a_n$ where P is an n-adic predicate symbol and $a_1, \ldots, a_n$ are free variables. *Formulae* are constructed from atomic formulae according to the usual formation rules. Propositional variables are regarded as 0-adic predicate symbols. A sentence is a formula containing no free variables. As in [2], we use different letters for free and bound variables. The set of all free variables is denoted $\mathfrak{V}$. The set of all predicate symbols is denoted $\mathfrak{P}$. We assume that the reader is familiar with Gentzen sequent calculi and Kripke models.

A *CD-structure* is defined to be (g, K, R, D) where K and D are nonempty sets, $g \in K$, and R is a reflexive and transitive binary relation on K. Let $\mathbf{T}$ stand for the one-element set consisting of the empty sequence, and $\mathbf{F}$ stand for the empty set ϕ. We suppose $D^n = \mathbf{T}$ when $n=0$. A *CD-model* on a *CD*-structure (g, K, R, D) is (g, K, R, D, φ) where φ is a function from $\mathfrak{P} \times K$ into $2^{\mathbf{T}} = \{\mathbf{T}, \mathbf{F}\}$ such that (i) if P is an n-adic predicate symbol and $i \in K$ then $\varphi(P, i) \subset D^n$, and (ii) if $i, j \in K$ and iRj then $\varphi(P, i) \subset \varphi(P, j)$. Let $\mathfrak{A}_D$ denote the set of all assignments $\alpha: \mathfrak{V} \to D$. For $\alpha \in \mathfrak{A}_D$ and $a \in \mathfrak{V}$, $\mathfrak{A}_D(\alpha, a)$ is defined to be the set of all $\beta \in \mathfrak{A}_D$ such that $\alpha(b) = \beta(b)$ for all $b \in \mathfrak{V} - \{a\}$. For $\alpha \in \mathfrak{A}_D$, $a \in \mathfrak{V}$ and any formula A, we shall define $\varphi(A, \alpha, i)$ and $\varphi(A, i)$. The function $\varphi(A, \alpha, i)$ is defined inductively as follows:

(1) If P is an n-adic predicate symbol ($n \geq 0$) then

* Assistant Professor (*Jokyōju*) in Mathematics.

$$\varphi(Pa_1 \ldots a_n, \alpha, i) = \begin{cases} \mathbf{T} & \text{if } (\alpha(a_1), \ldots, \alpha(a_n)) \in \varphi(P, i), \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

(2) $\varphi(A \wedge B, \alpha, i) = \varphi(A, \alpha, i) \cap \varphi(B, \alpha, i)$.

(3) $\varphi(A \vee B, \alpha, i) = \varphi(A, \alpha, i) \cup \varphi(B, \alpha, i)$.

(4) $\varphi(A \supset B, \alpha, i) = \mathbf{T} - \cup \{\varphi(A, \alpha, j) - \varphi(B, \alpha, j) \mid j \in K, iRj\}$.

(5) $\varphi(\neg A, \alpha, i) = \mathbf{T} - \cup \{\varphi(A, \alpha, j) \mid j \in K, iRj\}$.

(6) $\varphi(\forall x F(x), \alpha, i) = \cap \{\varphi(F(a), \beta, i) \mid \beta \in \mathfrak{A}_D(\alpha, a)\}$, where a is a free variable not occurring in $\forall x F(x)$.

(7) $\varphi(\exists x F(x), \alpha, i) = \cup \{\varphi(F(a), \beta, i) \mid \beta \in \mathfrak{A}_D(\alpha, a)\}$, where a is a free variable not occurring in $\exists x F(x)$.

As mentioned in [5], if iRj then $\varphi(A, \alpha, i) \subset \varphi(A, \alpha, j)$. Hence $\varphi(\forall x F(x), \alpha, i) = \mathbf{T}$ if and only if for all $\beta \in \mathfrak{A}_D(\alpha, a)$ and for all $j \in K$ such that iRj, $\varphi(F(a), \beta, j) = \mathbf{T}$ where a is a free variable not occurring in $\forall x F(x)$. Next we define $\varphi(A, i) = \cap \{\varphi(A, \alpha, i) \mid \alpha \in \mathfrak{A}_D\}$. $(g, K, R, D, \varphi) \models A$ is defined as $\varphi(A, g) = \mathbf{T}$. A formula A is *CD-valid* if and only if for all CD-model $\mathfrak{M} = (g, K, R, D, \varphi)$, $\mathfrak{M} \models A$. A sequent $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$ is *CD-valid* if and only if the formula $A_1 \wedge \ldots \wedge A_m \supset B_1 \vee \ldots \vee B_n$ is CD-valid.

Now we set up three predicate calculi $C1$, $C2$ and $C3$. $C2$ is equivalent to Görnemann's system. $C1$ is a sequent calculus lying between Gentzen's LJ and LK. $C1$ is LK with the restriction for inference rules $\rightarrow\supset$ and $\rightarrow\neg$: *one and only one formula* (*i. e. the principal formula*) *occurs in the succedent of the lower sequent.* It should be compared with Maehara's system which we shall call LJ'. LJ' is LK with the above restriction for rules $\rightarrow\supset$, $\rightarrow\neg$ and $\rightarrow\forall$. Maehara [6, 7] proved that LJ' is equivalent to LJ. $C2$ is intuitionistic predicate calculus with the additional axiom schema

$$\forall x(A \vee F(x)) \supset A \vee \forall x F(x).$$

$C3$ is intuitionistic predicate calculus with the additional axiom schema

$$\forall x(F(x) \vee G(x)) \supset \exists x F(x) \vee \forall x G(x).$$

THEOREM 1. *For any formula A, the following are equivalent:*

(a) *A is CD-valid;*

(b) *A is $C1$-provable;*

(c) *A is $C2$-provable;*

(d) *A is $C3$-provable.*

PROOF. We omit the proof of the fact that (a) implies either (b), (c) or (d). A proof of completeness is published by Görnemann.

(1) Implication of (a) by (b) is evident because CD-validity is preserved by $C1$-inferences.

(2) Implication of (c) by (b). It suffices to deduce

$$\Gamma \rightarrow \Theta, \forall x F(x)$$

from $C2$-axiom and

$$\Gamma \rightarrow \Theta, F(a)$$

in LJ' under the assumption that a does not occur in Γ, Θ, $\forall x F(x)$. If Θ is empty then it is clear. If Θ is a sequence $A_1, \ldots, A_n$ $(n \geq 1)$ then it is shown as follows: Let A be the formula $A_1 \vee \ldots \vee A_n$, then

$$\forall x(A\vee F(x)) \to \Theta,\ \forall xF(x)$$

is LJ'-deducible from an axiom sequent

$$\to \forall x(A\vee F(x))\supset A\vee \forall xF(x),$$

thence we have

$$\dfrac{\dfrac{\dfrac{\Gamma\to\Theta,\ F(a)}{\Gamma\to A\vee F(a)}}{\Gamma\to\forall x(A\vee F(x))}\qquad \forall x(A\vee F(x))\to\Theta,\ \forall xF(x)}{\Gamma\to\Theta,\ \forall xF(x).}$$

(3) Any formula of the form $\forall x(A\vee F(x))\supset A\vee\forall xF(x)$ follows immediately from a $C3$-axiom. Hence (c) implies (d).
(4) As shown below, any formula of the form $\forall x(F(x)\vee G(x))\supset\exists xF(x)\vee\forall xG(x)$ is $C1$-provable. Hence (d) implies (b).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a)\to F(a)\qquad G(a)\to G(a)}{F(a)\vee G(a)\to F(a),\ G(a)}}{\forall x(F(x)\vee G(x))\to F(a),\ G(a)}}{\forall x(F(x)\vee G(x))\to\exists xF(x),\ G(a)}}{\forall x(F(x)\vee G(x))\to\exists xF(x),\ \forall xG(x)}}{\forall x(F(x)\vee G(x))\to\exists xF(x)\vee\forall xG(x).}$$

A formula is said *CD-provable* if it is provable in $C1$, $C2$ or $C3$.

THEOREM 2. *Gentzen's Hauptsatz fails for* $C1$.

PROOF. Consider the following proof in $C1$, where F is a monadic predicate symbol.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fa\to Fa}{Fa\to\exists yFy,\ \neg Fa}\qquad\dfrac{\neg Fa\to\neg Fa}{\neg Fa\to\exists xFx,\ \neg Fa}}{Fa\vee\neg Fa\to\exists xFx,\ \neg Fa}}{\forall x(Fx\vee\neg Fx)\to\exists xFx,\ \neg Fa}}{\forall x(Fx\vee\neg Fx)\to\exists xFx,\ \forall x\neg Fx}\qquad\dfrac{\dfrac{\dfrac{\dfrac{Fa\to Fa}{\neg Fa,\ Fa\to}}{Fa,\ \forall x\neg Fx\to}}{\exists xFx,\ \forall x\neg Fx\to}}{\forall x\neg Fx\to\neg\exists xFx}}{\forall x(Fx\vee\neg Fx)\to\exists xFx,\ \neg\exists xFx}$$

The cut can not be eliminated from this proof. For, suppose there were cut-free proof of $\forall x(Fx\vee\neg Fx)\to\exists xFx,\ \neg\exists xFx$. By Subformula Property, no inference $\to\forall$ occurs in this proof since there is no positive occurrence of $\forall$ in the endsequent. Hence this would be an LJ'-proof. By Maehara's theorem,

$$\forall x(Fx\vee\neg Fx)\supset\exists xFx\vee\neg\exists xFx$$

would be LJ-provable, which is a contradiction. Q. E. D.

Let P be a monadic predicate symbol. The *P-relativization* A^P of an arbitrary formula A is defined by induction as follows:
(1) If no quantifiers occur in A, then A^P is A.
(2) $(A\wedge B)^P$ is $A^P\wedge B^P$.

(3) $(A \vee B)^P$ is $A^P \vee B^P$.
(4) $(A \supset B)^P$ is $A^P \supset B^P$.
(5) $(\neg A)^P$ is $\neg A^P$.
(6) $(\forall x F(x))^P$ is $\forall x(Px \supset F^P(x))$, where $F^P(a)$ is $(F(a))^P$.
(7) $(\exists x F(x))^P$ is $\exists x(Px \wedge F^P(x))$, where $F^P(a)$ is $(F(a))^P$.
If Ξ stands for a sequence $A_1, \ldots, A_n$ of formulae then Ξ^P denotes the sequence $A_1{}^P, \ldots, A_n{}^P$. If $\mathfrak{a}$ denotes a sequence $a_1, \ldots, a_n$ of variables then $P\mathfrak{a}$ denotes the sequence $Pa_1, \ldots, Pa_n$ of formulae.

The well-known Relativization Theorem reads: For sentence A containing no P, A is provable in the classical (intuitionistic) predicate calculus if and only if $\exists x Px \supset A^P$ is provable in the classical (intuitionistic) predicate calculus.

THEOREM 3. *Let A be a sentence and P be a monadic predicate symbol not occurring in A. Then A is provable in CD if and only if $\forall x(Px \vee \neg Px) \supset (\exists x Px \supset A^P)$ is provable in CD.*

PROOF. For any sequence Ξ of formulae, let $P(\Xi)$ denote the sequence $Pa_1, Pa_2, \ldots, Pa_n$ where $a_1, a_2, \ldots, a_n$ are all of the free variables contained in Ξ. Let Π denote the sequence consisting of the formulae $\forall x(Px \vee \neg Px)$ and $\exists x Px$. Given proof H in $C1$ of a sequent $\Psi \rightarrow \Omega$ is transformed into a proof in $C1$ of

$$\Psi^P, P(\Psi, \Omega), \forall x(Px \vee \neg Px), \exists x Px \rightarrow \Omega^P$$

by induction on the length of H. We divide cases according to the lowermost inference S in H.

Case 1: S is a cut. Let S be of the from

$$\frac{\Gamma \rightarrow \Theta,\ D \qquad D,\ \Delta \rightarrow \Lambda}{\Gamma,\ \Delta \rightarrow \Theta,\ \Lambda,}$$

and let $\mathfrak{a}$ be the sequence of all the free variables contained in D but not in $\Gamma, \Theta, \Delta, \Lambda$. We transform S into

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^P, P\mathfrak{a}, P(\Gamma, \Theta), \Pi \rightarrow \Theta^P, D^P \qquad D^P, \Delta^P, P\mathfrak{a}, P(\Delta, \Lambda), \Pi \rightarrow \Lambda^P}{\Gamma^P, P\mathfrak{a}, P(\Gamma, \Theta), \Pi, \Delta^P, P\mathfrak{a}, P(\Delta, \Lambda), \Pi \rightarrow \Theta^P, \Lambda^P}}{P\mathfrak{a}, \Gamma^P, \Delta^P, P(\Gamma, \Delta, \Theta, \Lambda), \Pi \rightarrow \Theta^P, \Lambda^P}}{\exists x Px, \Gamma^P, \Delta^P, P(\Gamma, \Delta, \Theta, \Lambda), \Pi \rightarrow \Theta^P, \Lambda^P}}{\Gamma^P, \Delta^P, P(\Gamma, \Delta, \Theta, \Lambda), \Pi \rightarrow \Theta^P, \Lambda^P.}$$

Case 2: S is an $\rightarrow\supset$. Suppose S runs as follows:

$$\frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B.}$$

Then S is transformed into

$$\frac{A^P, \Gamma^P, P(A, B, \Gamma), \Pi \rightarrow B^P}{\Gamma^P, P(A, B, \Gamma), \Pi \rightarrow A^P \supset B^P.}$$

Case 3: S is an $\rightarrow\forall$. Suppose S runs as

$$\frac{\Gamma \rightarrow \Theta,\ F(a)}{\Gamma \rightarrow \Theta,\ \forall x F(x),}$$

then S is transformed into

$$\dfrac{\dfrac{\Gamma^P, Pa, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, F^P(a) \qquad \dfrac{F^P(a) \to F^P(a)}{F^P(a) \to Pa \supset F^P(a)}}{\Gamma^P, Pa, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, Pa \supset F^P(a)}}{Pa, \Gamma^P, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, Pa \supset F^P(a)} \qquad \ldots(1)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(1) \qquad \dfrac{\dfrac{Pa \to Pa}{\neg Pa \to Pa \supset F^P(a)}}{Pa, \Gamma^P, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, Pa \supset F^P(a)}}{Pa \vee \neg Pa, \Gamma^P, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, Pa \supset F^P(a)}}{\forall x(Px \vee \neg Px), \Gamma^P, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, Pa \supset F^P(a)}}{\Gamma^P, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, Pa \supset F^P(a)}}{\Gamma^P, P(\Gamma, \Theta, \forall xF(x)), \Pi \to \Theta^P, \forall x(Px \supset F^P(x)).}$$

The other cases are treated similarly. Q. E. D.

Remark. We can not dispense with the formula $\forall x(Px \vee \neg Px)$ in the last theorem. Let F be a monadic predicate symbol and A be the CD-provable sentence

$$\forall x(Fx \vee \neg Fx) \supset \exists xFx \vee \neg \exists xFx,$$

then $\exists xPx \supset A^P$ is not CD-provable. For if it were provable then by substituting $\lambda x(Fx \vee \neg Fx)$ for P we could obtain

$$\exists x \neg Fx \supset \exists xFx \vee \neg \exists xFx$$

while, as shown later, this formula is not provable.

Theorem 3 depends on the fact that for any CD-model $\mathfrak{M}=(g, K, R, D, \varphi)$, $\mathfrak{M} \models \forall x(Px \vee \neg Px)$ if and only if $\varphi(P, i)=\varphi(P, j)$ for all $i, j \in k$.

Now let A be an LJ-provable sentence and P a monadic predicate symbol not occurring in A. Then $\exists xPx \supset A^P$ is CD-provable since it is LJ-provable by Relativization Theorem. The converse seems to hold:

CONJECTURE. *Let A be a sentence and P be a monadic predicate symbol not occurring in A. Then A is provable in LJ if and only if $\exists xPx \supset A^P$ is provable in CD.*

THEOREM 4. *If F is a monadic predicate symbol then the following formulae are not provable in CD:*

(1) $\exists xFx \wedge \exists x \neg Fx \supset \neg\neg \forall x(Fx \vee \neg Fx)$;

(2) $\neg\neg(\forall x \neg\neg Fx \supset \neg\neg \forall xFx)$;

(3) $\neg\neg \exists xFx \supset \exists x \neg\neg Fx$;

(4) $\exists x \neg Fx \supset \exists xFx \vee \neg \exists xFx$;

(5) $\exists x \forall y(Fy \supset Fx) \supset \exists xFx \vee \neg \exists xFx$;

(6) $\exists x \forall y(Fy \supset Fx)$.

PROOF. Let $\mathbf{N}$ be the set of finite ordinals (i. e. nonnegative integers).

(1) Countermodel $(0, \mathbf{N}, \leq, \mathbf{N}, \varphi)$ where $\varphi(F, i)=\{j \mid j \in \mathbf{N}, 0<j \leq i\}$ for $i \in \mathbf{N}$.

(2) Let G be $\lambda x(Fx \vee \neg Fx)$. If $\neg\neg(\forall x \neg\neg Fx \supset \neg\neg \forall x Fx)$ were provable then $\neg\neg\forall x G(x)$ would be provable (cf. [4], p. 491), contradicting (1).
(3) Countermodel $(\phi, K, \subset, \{0, 1\}, \varphi)$ where $K=\{\phi, \{0\}, \{1\}\}$ and $\varphi(F, i)=i$ for $i \in K$.
(4) Countermodel $(0, \{0, 1\}, \leq, \{0, 1\}, \varphi)$ where $\varphi(F, 0)=\phi$ and $\varphi(F, 1)=\{0\}$.
(5) Countermodel $(0, \mathrm{N}, \leq, \mathrm{N}, \varphi)$where $\varphi(F, i)=\{j | j \in \mathrm{N}, j<i\}$.
(6) Countermodel $(\omega, \mathrm{N} \cup \{\omega\}, \geq, \mathrm{N}, \varphi)$ where $\varphi(F, \omega)=\phi$ and $\varphi(F, i)=\{j | j \in \mathrm{N}, i \leq j\}$ for $i \in \mathrm{N}$. Another countermodel is the countermodel to (3) given above. Q. E. D.

Remark. The formula $\neg\neg\forall x(Fx \vee \neg Fx)$ is valid in every model (g, K, R, D, φ) with finite K.

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