ON THE UNIFORM DISTRIBUTION OF THE POWER MATRICES WITH DEGREE 2

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1. In this paper let us take up a problem with respect to the distribution of $S(A)=\{A^m; m=1,2,3,\cdots\}$, where A is a real square matrix with degree 2. It is difficult to solve this problem for an arbitrary A, so that we deal with a specialized A as follows: $|A|=1$ and $-2<\sigma(A)<2$, where $|A|$ and $\sigma(A)$ are determinant of A and trace of A respectively.

The first question that we propose to deal is to represent four elements of A^m by means of m successfully. In case of degree 2 we can solve this question with the trigonometrical function of m. Namely, roughly speaking, the elements of A^m move with the fluctuation of m presenting an aspect of simple oscillation. Furthermore these four simple oscillations have a common angular velocity, and on the principal diagonal line they have different amplitudes and phase differences, but on the subordinate diagonal line they have the same phase difference. In order to draw those conclusions, characteristics of Gegenbauer's polynomials are effectively used.

On the basis of those facts we propose to describe the uniformity of distribution of A^m, after some notions of uniformity are suitably defined in the last two sections.

2. Let $\alpha_m^{(1)}(X, Y)$, $\alpha_m^{(2)}(X, Y)$ be systems of polynomials with variables X and Y over the ring of rational integers Z, which are defined by

(1) $\alpha_{m+1}^{(1)}(X, Y)=X\alpha_m^{(1)}(X, Y)-\alpha_m^{(2)}(X, Y)$,

(2) $\alpha_{m+1}^{(2)}(X, Y)=Y\alpha_m^{(1)}(X, Y)$,

where we adopt initial conditions as follows:

(3) $\alpha_0^{(1)}(X, Y)=0, \qquad \alpha_0^{(2)}(X, Y)=-1.$

Now for an arbitrary matrix A with degree 2 over the real number field R, we denote its determinant and its trace by $|A|$ and $\sigma(A)$ respectively. Then we obtain the following proposition 1 by the induction on m without difficulty.

Proposition 1.

(4) $A^m=\alpha_m^{(1)}(\sigma(A), |A|)A-\alpha_m^{(2)}(\sigma(A), |A|)E_2$,

where E_2 is the unit matrix of degree 2.

Here we wish to decide the systems of the polynomials $\alpha_m^{(1)}(X, Y)$, $\alpha_m^{(2)}(X, Y)$ explicitly. The following proposition 2 answers this question, as by (2) we have only to consider the $\alpha_m^{(1)}(X, Y)$.

Proposition 2.

$$(5) \qquad \alpha_m^{(1)}(X, Y)=\sum_{i=0}^{\left[\frac{m-1}{2}\right]}(-1)^i\binom{m-i-1}{i}X^{m-2i-1}Y^i$$

for all $m=1,2,3,\cdots$.

Proof. We carry out the proof by the induction on m. In case of $m=1$, (5) is obvious. Now let us assume that the conclusion of (5) is right for $m=1,2,3,\cdots,k$, $(1\leq k)$. Then we obtain by (1) and (2)

$$\alpha_{k+1}^{(1)}(X, Y)=X\alpha_k^{(1)}(X, Y)-\alpha_k^{(2)}(X, Y)$$
$$=X\alpha_k^{(1)}(X, Y)-Y\alpha_{k-1}^{(1)}(X, Y)$$
$$=\sum_{i=0}^{\left[\frac{k-1}{2}\right]}(-1)^i\binom{k-i-1}{i}X^{k-2i}Y^i-\sum_{i=0}^{\left[\frac{k-2}{2}\right]}(-1)^i\binom{k-i-2}{i}X^{k-2i-2}Y^{i+1}. \qquad (*)$$

At the last summation of the above relation, we adopt $j-1$ for i, and because an equality $\left[\frac{k-2}{2}\right]+1=\left[\frac{k}{2}\right]$ is obvious, so that the summation turns into

$$\sum_{j=1}^{\left[\frac{k}{2}\right]}(-1)^{j-1}\binom{k-j-1}{j-1}X^{k-2j}Y^j.$$

But of course we know

$$\left[\frac{k}{2}\right]-\left[\frac{k-1}{2}\right]=\begin{cases}1, & \text{if } k\equiv 0 \pmod 2\\ 0, & \text{if } k\equiv 1 \pmod 2\end{cases}$$

and therefore we must proceed on separating the computation into two cases such that $k\equiv 0$ (mod 2) and $k\equiv 1$ (mod 2).

In the first case, let us assume $k\equiv 0$ (mod 2).

$$(*)=(-1)^0\binom{k-0-1}{0}X^{k-2\cdot 0}Y^0$$
$$+\sum_{i=1}^{\left[\frac{k-1}{2}\right]}\left\{(-1)^i\binom{k-i-1}{i}-(-1)^{i-1}\binom{k-i-1}{i-1}\right\}X^{k-2i}Y^i$$
$$-(-1)^{\left[\frac{k}{2}\right]-1}\binom{k-\left[\frac{k}{2}\right]-1}{\left[\frac{k}{2}\right]-1}X^{k-2\left[\frac{k}{2}\right]}Y^{\left[\frac{k}{2}\right]}$$
$$=(-1)^0\binom{k+1-0-1}{0}X^{k+1-2\cdot 0-1}Y^0$$
$$+\sum_{i=1}^{\left[\frac{k}{2}\right]-1}(-1)^i\left\{\binom{k-i-1}{i}+\binom{k-i-1}{i-1}\right\}X^{k+1-2i-1}Y^i$$
$$+(-1)^{\left[\frac{k}{2}\right]}\binom{k-\left[\frac{k}{2}\right]-1}{\left[\frac{k}{2}\right]-1}X^{k+1-2\left[\frac{k}{2}\right]-1}Y^{\left[\frac{k}{2}\right]}. \qquad (**)$$

Now clearly

$$\binom{k-i-1}{i}+\binom{k-i-1}{i-1}=\binom{k+1-i-1}{i},$$

and because of the assumption $k\equiv 0$ (mod 2)

$$\binom{k-\left[\frac{k}{2}\right]-1}{\left[\frac{k}{2}\right]-1}=\binom{k+1-\left[\frac{k}{2}\right]-1}{\left[\frac{k}{2}\right]}$$

are obvious. Accordingly

$$(**)=(-1)^0\binom{k+1-0-1}{0}X^{k+1-2\cdot 0-1}Y^0$$

$$+\sum_{i=1}^{\left[\frac{(k+1)-1}{2}\right]-1}(-1)^i\binom{k+1-i-1}{i}X^{k+1-2i-1}Y^i$$

$$+(-1)^{\left[\frac{(k+1)-1}{2}\right]}\binom{k+1-\left[\frac{(k+1)-1}{2}\right]-1}{\left[\frac{(k+1)-1}{2}\right]}X^{k+1-2\left[\frac{(k+1)-1}{2}\right]-1}Y^{\left[\frac{(k+1)-1}{2}\right]}$$

$$=\sum_{i=0}^{\left[\frac{(k+1)-1}{2}\right]}(-1)^i\binom{(k+1)-i-1}{i}X^{(k+1)-2i-1}Y^i.$$

This consequence is obtained in case of the transformation $m=k+1$ in (5).

In the second case, let us assume $k\equiv 1 \pmod 2$. Taking consideration of $\left[\frac{k}{2}\right]=\left[\frac{k-1}{2}\right]$, we assert

$$(*)=\sum_{i=0}^{\left[\frac{k-1}{2}\right]}(-1)^i\binom{k-i-1}{i}X^{k-2i}Y^i-\sum_{j=1}^{\left[\frac{k}{2}\right]}(-1)^{j-1}\binom{k-j-1}{j-1}X^{k-2j}Y^j$$

$$=(-1)^0\binom{k-0-1}{0}X^{k-2\cdot 0}Y^0$$

$$+\sum_{i=1}^{\left[\frac{k}{2}\right]}\left\{(-1)^i\binom{k-i-1}{i}-(-1)^{i-1}\binom{k-i-1}{i-1}\right\}X^{k-2i}Y^i$$

$$=(-1)^0\binom{(k+1)-0-1}{0}X^{(k+1)-2\cdot 0-1}Y^0$$

$$+\sum_{i=1}^{\left[\frac{(k+1)-1}{2}\right]}(-1)^i\left\{\binom{k-i-1}{i}+\binom{k-i-1}{i-1}\right\}X^{(k+1)-2i-1}Y^i$$

$$=(-1)^0\binom{(k+1)-0-1}{0}X^{(k+1)-2\cdot 0-1}Y^0$$

$$+\sum_{i=1}^{\left[\frac{(k+1)-1}{2}\right]}(-1)^i\binom{(k+1)-i-1}{i}X^{(k+1)-2i-1}Y^i$$

$$=\sum_{i=0}^{\left[\frac{(k+1)-1}{2}\right]}(-1)^i\binom{(k+1)-i-1}{i}X^{(k+1)-2i-1}Y^i.$$

This consequence is obtained in case of the transformation $m=k+1$ in (5). Consequently we have known that the formula (5) is proved by the induction on m.

Accordingly by (2) we also obtain

$$\alpha_1^{(2)}(X,Y)=0$$

$$\alpha_m^{(2)}(X,Y)=\sum_{i=0}^{\left[\frac{m-2}{2}\right]}(-1)^i\binom{m-i-2}{i}X^{m-2i-2}Y^{i+1}$$

for all $m=2,3,4,\cdots$.

3. In this section generating functions of $\alpha_m^{(1)}(X,Y)$, $\alpha_m^{(2)}(X,Y)$ come into question. We begin with the identity

$$(6)\qquad \frac{U^{m+1}-V^{m+1}}{U-V}=U^m+U^{m-1}V+\cdots+UV^{m-1}+V^m.$$

Since the right side of (6) is a symmetric polynomial with respect to U and V, we can find such polynomial $P_{m+1}(X,Y)$ with variables X and Y over the ring of rational integers Z that the identity

$$\frac{U^{m+1}-V^{m+1}}{U-V}=P_{m+1}(U+V,U\cdot V),\qquad m=0,1,2,\cdots$$

is obtained.

Now we describe the following proposition.

Proposition 3. If the real matrix A of degree 2 is semi-simple and $|A|\neq 0$, then

$$(7)\qquad \frac{1}{\varphi_A(x)}=\sum_{m=0}^{\infty}\frac{P_{m+1}(\sigma(A),|A|)}{|A|^{m+1}}x^m,$$

where $\varphi_A(x)$ is a characteristic polynomial of A.

Proof. (7) denotes

$$(8)\qquad \frac{1}{m!}\left[\frac{d^m}{dx^m}\left\{\frac{1}{\varphi_A(x)}\right\}\right]_{x=0}=\frac{P_{m+1}(\sigma(A),|A|)}{|A|^{m+1}}$$

for all $m=0,1,2,\cdots$. Since A is semi-simple, there exist θ_1 and θ_2 in the complex number field such that

$$\varphi_A(x)=(x-\theta_1)(x-\theta_2),\qquad \theta_1\neq\theta_2,\qquad \theta_1\theta_2\neq 0.$$

Accordingly

[*the left side of* (8)]

$$=\frac{1}{m!}\left[\frac{1}{\theta_2-\theta_1}\frac{d^m}{dx^m}\left(\frac{1}{x-\theta_2}-\frac{1}{x-\theta_1}\right)\right]_{x=0}$$

$$=\frac{1}{m!}\left[\frac{(-1)^m m!}{\theta_2-\theta_1}\left\{\frac{1}{(x-\theta_2)^{m+1}}-\frac{1}{(x-\theta_1)^{m+1}}\right\}\right]_{x=0}$$

$$=-\frac{1}{\theta_2-\theta_1}\left(\frac{1}{\theta_2^{m+1}}-\frac{1}{\theta_1^{m+1}}\right)$$

$$=\frac{\theta_1^{m+1}-\theta_2^{m+1}}{\theta_1-\theta_2}\,\frac{1}{(\theta_1\theta_2)^{m+1}}$$

$$=\frac{P_{m+1}(\theta_1+\theta_2,\theta_1\theta_2)}{(\theta_1\theta_2)^{m+1}}$$

$$=\frac{P_{m+1}(\sigma(A),|A|)}{|A|^{m+1}}.$$

For the sake of convenience we define that the polynomial $P_0(X,Y)$ denotes 0. This promise is employed in the next section.

4. In this section we must verify

Proposition 4.

$$(9)\qquad \alpha_m^{(1)}(X,Y)=P_m(X,Y)\qquad m=0,1,2,\cdots.$$

Accordingly by (2)

$$\alpha_m^{(2)}(X,Y)=YP_{m-1}(X,Y),\qquad m=1,2,3,\cdots.$$

Before giving the proof of this proposition, we need the following lemma.

Lemma. For an arbitrary natural number n and an arbitrary natural number r such that $1\leq r\leq\left[\frac{n}{2}\right]$,

$$(10)\qquad \sum_{k=0}^{r}(-1)^k\binom{n-k}{r}\binom{r}{k}=1.$$

Proof. [*the left side of* (10)]

$$=\frac{1}{r!}\sum_{k=0}^{r}(-1)^k\binom{r}{k}(n-k)(n-k-1)\cdot\cdots\cdot(n-k-r+1).$$

Now for each k $(0\leq k\leq r)$ and r $\left(1\leq r\leq\left[\frac{n}{2}\right]\right)$ we put

$$f_{k,r}(x)=(x-k)(x-k-1)\cdot\cdots\cdot(x-k-r+1),$$

$$F_r(x)=\sum_{k=0}^{r}(-1)^k\binom{r}{k}f_{k,r}(x).$$

Then for each r $\left(1\leq r\leq\left[\frac{n}{2}\right]\right)$, the roots of $f_{k,r}(x)=0$ are $k, k+1, \cdots, k+r-1$. Therefore for an arbitrary r, $\left(1\leq r\leq\left[\frac{n}{2}\right]\right)$ we obtain

$$f_{1,r}(r)=f_{2,r}(r)=\cdots=f_{r,r}(r)=0.$$

Now considering the fact

$$F_r(r)=r\,!,$$

if we can prove that $F_r(x)$ $\left(1\leq r\leq\left[\frac{n}{2}\right]\right)$ is a constant function, then we obtain the conclusion

$$\sum_{k=0}^{r}(-1)^k\binom{n-k}{r}\binom{r}{k}=\frac{1}{r\,!}F_r(n)=1,$$

namely the proof of this lemma is completed.

A reason of constantness of $F_r(x)$ $\left(1\leq r\leq\left[\frac{n}{2}\right]\right)$ is as follows.

By simple computation we have

$$\begin{aligned}&f_{k,r}(x)-f_{k,r}(x-1)\\&=rf_{k,r-1}(x-1)\\&=rf_{k-1,r-1}(x-2).\end{aligned}$$

Therefore for an arbitrary r $\left(1\leq r\leq\left[\frac{n}{2}\right]\right)$ we obtain the following relation.

$$\begin{aligned}&F_r(x)-F_r(x-1)\\
&=\sum_{k=0}^{r}(-1)^k\binom{r}{k}f_{k,r}(x)-\sum_{k=0}^{r}(-1)^k\binom{r}{k}f_{k,r}(x-1)\\
&=\sum_{k=0}^{r}(-1)^k\binom{r}{k}\{f_{k,r}(x)-f_{k,r}(x-1)\}\\
&=\sum_{k=0}^{r}(-1)^k\binom{r}{k}rf_{k,r-1}(x-1)\\
&=(-1)^0\binom{r}{0}rf_{0,r-1}(x-1)\\
&+\sum_{k=1}^{r-1}(-1)^k\left\{\binom{r-1}{k}+\binom{r-1}{k-1}\right\}rf_{k,r-1}(x-1)\\
&+(-1)^r\binom{r}{r}rf_{r,r-1}(x-1)\\
&=rf_{0,r-1}(x-1)\\
&+r\left\{F_{r-1}(x-1)-(-1)^0\binom{r-1}{0}f_{0,r-1}(x-1)\right\}\\
&-r\left\{F_{r-1}(x-2)-(-1)^{r-1}\binom{r-1}{r-1}f_{r-1,r-1}(x-2)\right\}\\
&+(-1)^r rf_{r,r-1}(x-1)\\
&=r\{F_{r-1}(x-1)-F_{r-1}(x-2)\}\\
&+(-1)^r r(x-1-r)(x-1-(r+1))\cdot\cdots\cdot(x-1-(r+r-2))\\
&+(-1)^{r-1}r(x-2-(r-1))(x-2-r)\cdot\cdots\cdot(x-2-(r+r-3))\end{aligned}$$

$=r\{F_{r-1}(x-1)-F_{r-1}(x-2)\}$;

i. e. $\quad F_r(x)-F_r(x-1)=r\{F_{r-1}(x-1)-F_{r-1}(x-2)\}$.

Since $F_1(x)=1$, by the induction on r we reach the conclusion that $F_r(x)$ is a periodic function with period 1. Here since the degree of $F_r(x)$ is not over r the proof of this lemma is completed.

Considering a case $n=m-1$, we suppose, when m is even, $r=0,1,2,\cdots,\frac{m}{2}-1\left(\leq\left[\frac{n}{2}\right]\right)$ and, when m is odd, $r=0,1,2,\cdots,\frac{m-1}{2}\ \left(\leq\left[\frac{n}{2}\right]\right)$ respectively. Then we obtain the following corollary.

Corollary.

$$(11) \qquad \sum_{i=0}^{j}(-1)^i\binom{m-i-1}{j}\binom{j}{i}=1$$

is established, if $m\equiv0 \pmod 2$ for all $j=0,1,2,\cdots,\frac{m}{2}-1$, and if $m\equiv1 \pmod 2$ for all $j=0,1,2,\cdots,\frac{m-1}{2}-1,\frac{m-1}{2}$.

Now we return to the verification of proposition 4.

Proof of proposition 4. For the purpose of verifying (9), it is enough to take consideration of the following identity for all $m=1,2,3,\cdots$, because

$$\alpha_0^{(1)}(X, Y)=P_0(X, Y), \qquad \alpha_1^{(1)}(X, Y)=P_1(X, Y)$$

are clear.

$$(12) \qquad \alpha_m^{(1)}(U+V, U\cdot V)=U^{m-1}+U^{m-2}V+\cdots+UV^{m-2}+V^{m-1}.$$

Now we make use of (5) in proposition 2. Then

[*the left side of* (12)]

$$=\sum_{i=0}^{\left[\frac{m-1}{2}\right]}(-1)^i\binom{m-i-1}{i}(U+V)^{m-2i-1}(U\cdot V)^i$$

$$=\sum_{i=0}^{\left[\frac{m-1}{2}\right]}(-1)^i\binom{m-i-1}{i}\sum_{j=0}^{m-2i-1}\binom{m-2i-1}{j}U^{m-i-j-1}V^{i+j}$$

$$=\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\sum_{j=0}^{m-2i-1}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j}U^{m-i-j-1}V^{i+j}. \qquad (\sharp)$$

To simplify the computation we carry on computating about two cases. As the first case we assume $m\equiv0 \pmod 2$. Then

$$(\sharp)=\sum_{j=0}^{\frac{m}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}U^{m-j-1}V^j$$

$$+\sum_{j=\frac{m}{2}}^{m-1}\left\{\sum_{i=0}^{m-j-1}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}U^{m-j-1}V^j$$

$$=\sum_{j=0}^{\frac{m}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}U^{m-j-1}V^j$$

$$+\sum_{j=0}^{\frac{m}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{m-i-j-1}\right\}U^jV^{m-j-1}$$

$$=\sum_{j=0}^{\frac{m}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}\{U^{m-j-1}V^j+U^jV^{m-j-1}\}. \qquad (\flat)$$

Now by simple computation we obtain

$$(13)\qquad \binom{m-i-1}{i}\binom{m-2i-1}{j-i}=\binom{m-i-1}{j}\binom{j}{i},$$

so that by making use of the foregoing corollary (11) the above computation is able to be continued as follows:

$$(\flat)=\sum_{j=0}^{\frac{m}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{j}\binom{j}{i}\right\}(U^{m-j-1}V^j+U^jV^{m-j-1})$$

$$=\sum_{j=0}^{\frac{m}{2}-1}(U^{m-j-1}V^j+U^jV^{m-j-1})$$

$=$[*the right side of* (12)].

In the next place as the second case we assume $m\equiv 1 \pmod 2$. Then

$$(\sharp)=\sum_{j=0}^{\frac{m-1}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}U^{m-j-1}V^j$$

$$+\left\{\sum_{j=0}^{\frac{m-1}{2}}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{\frac{m-1}{2}-i}\right\}U^{\frac{m-1}{2}}V^{\frac{m-1}{2}}$$

$$+\sum_{j=\frac{m-1}{2}+1}^{m-1}\left\{\sum_{i=0}^{m-j-1}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}U^{m-j-1}V^j. \qquad (\flat)$$

Now the above third summation is able to turn into

$$\sum_{j=0}^{\frac{m-1}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}U^jV^{m-j-1}.$$

On the other hand considering the foregoing formula (13) and the next formula which can be easily verified, namely

$$\binom{m-i-1}{i}\binom{m-2i-1}{\frac{m-1}{2}-i}=\binom{m-i-1}{\frac{m-1}{2}}\binom{\frac{m-1}{2}}{i},$$

we obtain the conclusion as follows:

$$(\flat)=\sum_{j=0}^{\frac{m-1}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{j-i}\right\}(U^{m-j-1}V^j+U^jV^{m-j-1})$$

$$+\left\{\sum_{i=0}^{\frac{m-1}{2}}(-1)^i\binom{m-i-1}{i}\binom{m-2i-1}{\frac{m-1}{2}-i}\right\}U^{\frac{m-1}{2}}V^{\frac{m-1}{2}}$$

$$=\sum_{j=0}^{\frac{m-1}{2}-1}\left\{\sum_{i=0}^{j}(-1)^i\binom{m-i-1}{j}\binom{j}{i}\right\}(U^{m-j-1}V^j+U^jV^{m-j-1})$$

$$+\left\{\sum_{i=0}^{\frac{m-1}{2}}(-1)^i\binom{m-i-1}{\frac{m-1}{2}}\binom{\frac{m-1}{2}}{i}\right\}U^{\frac{m-1}{2}}V^{\frac{m-1}{2}}$$

$$=\sum_{j=0}^{\frac{m-1}{2}-1}(U^{m-j-1}V^j+U^jV^{m-j-1})+U^{\frac{m-1}{2}}V^{\frac{m-1}{2}}$$

$=$[*the right side of* (12)].

Here the reason of the equality before the last one is based upon the foregoing corollary (11) Now we have come to the end of the proof of proposition 4.

5. In this section we deal with generating functions of $\alpha_m^{(1)}(X, Y)$ and $\alpha_m^{(2)}(X, Y)$ attached to A, which is a real matrix with degree 2. To represent $\alpha_m^{(1)}(X, Y)$ and $\alpha_m^{(2)}(X, Y)$ by the generating function is important for the description of our last conclusion.

By the foregoing propositions 3 and 4, we have the following formula, namely

$$\frac{1}{\varphi_A(x)}=\sum_{m=0}^{\infty}\frac{\alpha_{m+1}^{(1)}(\sigma(A), |A|)}{|A|^{m+1}}x^m, \tag{14}$$

where A is a real regular semi-simple matrix with degree 2 and $\varphi_A(x)$ is its characteristic polynomial. Therefore by (2)

$$\frac{1}{\varphi_A(x)}=\sum_{m=0}^{\infty}\frac{\alpha_{m+2}^{(2)}(\sigma(A), |A|)}{|A|^{m+2}}x^m$$

are very clearly established. Here roughly speaking we can say that two generating functions of $\alpha_m^{(1)}(X, Y)$ and $\alpha_m^{(2)}(X, Y)$ attached to A, are the same function, namely $\frac{1}{\varphi_A(x)}$. We can arrange those circumstances in the following proposition 5.

Proposition 5. If A is a real regular semi-simple matrix with degree 2, then generating functions of $\alpha_m^{(1)}(X, Y)$ and $\alpha_m^{(2)}(X, Y)$ at A, which are defined by generating functions of $\alpha_m^{(1)}(\sigma(A), |A|)$ and $\alpha_m^{(2)}(\sigma(A), |A|)$ respectively, are $\frac{x}{\varphi_A(x)}$ and $\frac{\sigma(A)x-|A|}{\varphi_A(x)}$ respectively.

To say more precisely, the following formulas are able to be verified,

$$\frac{x}{\varphi_A(x)}=\sum_{m=0}^{\infty}\alpha_m^{(1)}(\sigma(A), |A|)\left(\frac{x}{|A|}\right)^m, \tag{15}$$

$$\frac{\sigma(A)x-|A|}{\varphi_A(x)}=\sum_{m=0}^{\infty}\alpha_m^{(2)}(\sigma(A), |A|)\left(\frac{x}{|A|}\right)^m$$

for every A that satisfies the foregoing conditions.

Proof. It is clear by regarding the initial conditions (3) and propositions 3 and 4.

6. From now on we restrict our standpoint; namely we deal with all the matrices that satisfy the following condition.

$$|A|=1, \quad -2<\sigma(A)<2. \tag{16}$$

Then we obtain the following obvious result.

Proposition 6. If a real matrix A of degree 2 satisfies the conditions (16), then A is semi-simple.

Proof. If A is not semi-simple, the characteristic equation of A is not separable; namely $\varphi_A(x)=0$ and $\varphi'_A(x)=0$ have a common root. By the way

$$\varphi_A(x)=x^2-\sigma(A)x+1,$$
$$\varphi'_A(x)=2x-\sigma(A),$$

therefore we must obtain

$$\varphi_A\left(\frac{\sigma(A)}{2}\right)=0$$

which means $\sigma(A)=\pm 2$. It is a contradiction.

By this proposition 6 there is no necessity for the condition of semi-simplicity. Accordingly we can freely make use of the conclusion in proposition 5, on condition that A satisfies (16).

Since $|A|=1$, we simply denote $\alpha_m^{(1)}(X, 1)$ and $\alpha_m^{(2)}(X, 1)$ by $\alpha_m^{(1)}(X)$ and $\alpha_m^{(2)}(X)$ respectively.

Now we put $A=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, and then on the (x_1, x_4) plane we classify the domain which is characterized by $-2<\sigma(A)<2$, into twelve parts.

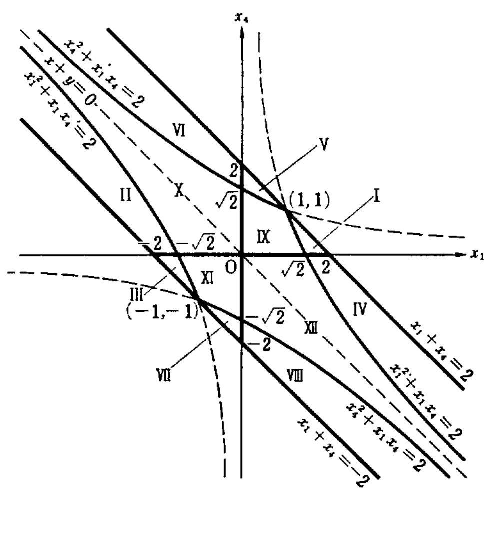
First of all, we define ξ and η by

$$\xi=x_1^2+x_1x_4-2, \qquad \eta=x_4^2+x_1x_4-2$$

and consider (x_1, x_4) which satisfies $-2<x_1+x_4<2$, and define the domains from I to XII as follows:

$$
(17)\quad
\begin{aligned}
\mathrm{I}&=\{(x_1, x_4);\ \xi>0, x_1>0, x_4>0\}\\
\mathrm{II}&=\{(x_1, x_4);\ \xi>0, x_1<0, x_4>0\}\\
\mathrm{III}&=\{(x_1, x_4);\ \xi>0, x_1<0, x_4<0\}\\
\mathrm{IV}&=\{(x_1, x_4);\ \xi>0, x_1>0, x_4<0\}\\
\mathrm{V}&=\{(x_1, x_4);\ \eta>0, x_1>0, x_4>0\}\\
\mathrm{VI}&=\{(x_1, x_4);\ \eta>0, x_1<0, x_4>0\}\\
\mathrm{VII}&=\{(x_1, x_4);\ \eta>0, x_1<0, x_4<0\}\\
\mathrm{VIII}&=\{(x_1, x_4);\ \eta>0, x_1>0, x_4<0\}\\
\mathrm{IX}&=\{(x_1, x_4);\ \xi<0, \eta<0, x_1>0, x_4>0\}\\
\mathrm{X}&=\{(x_1, x_4);\ \xi<0, \eta<0, x_1<0, x_4>0\}\\
\mathrm{XI}&=\{(x_1, x_4);\ \xi<0, \eta<0, x_1<0, x_4<0\}\\
\mathrm{XII}&=\{(x_1, x_4);\ \xi<0, \eta<0, x_1>0, x_4<0\}.
\end{aligned}
$$

This classification is able to be described by the geometrical method as follows.



Now using the above classification, we claim the following proposition 7.

Proposition 7. If A satisfies (16), then for an arbitrary $m=1, 2, 3, \cdots$,

$$A^m=\begin{bmatrix} \varepsilon_1\dfrac{(x_1^2-2x_1\cos\theta+1)^{\frac{1}{2}}}{\sin\theta}\sin\{(m-1)\theta+\varepsilon_1'\varphi_1\} & \dfrac{x_2}{\sin\theta}\sin m\theta \\ \dfrac{x_3}{\sin\theta}\sin m\theta & \varepsilon_4\dfrac{(x_4^2-2x_4\cos\theta+1)^{\frac{1}{2}}}{\sin\theta}\sin\{(m-1)\theta+\varepsilon_4'\varphi_4\} \end{bmatrix}$$

where

$$\theta=\cos^{-1}\frac{\sigma(A)}{2},\qquad 0<\theta<\pi;$$

$$\varphi_i=\tan^{-1}\left|\frac{x_i\sin\theta}{x_i\cos\theta-1}\right|,\qquad 0\leq\varphi_i\leq\frac{\pi}{2};\qquad (i=1,4)$$

and $\varepsilon_1, \varepsilon_1', \varepsilon_4, \varepsilon_4'$ which are either 1 or -1, are defined by the following table in accordance with the domain in which x_1 and x_4 exist.

	ε_1	ε_1'	ε_4	ε_4'
I:	1	1	-1	-1
II:	1	-1	-1	-1
III:	1	-1	-1	1
IV:	1	1	-1	1
V:	-1	-1	1	1
VI:	-1	1	1	1
VII:	-1	1	1	-1
VIII:	-1	-1	1	-1
IX:	-1	-1	-1	-1
X:	-1	1	-1	-1
XI:	-1	1	-1	1
XII:	-1	-1	-1	1

Proof. Before giving the proof of this proposition, we need some results with respect to Gegenbauer's polynomial. It is a system of polynomials $C_n^\nu(x)$ which are defined by the following relation; namely

$$C_n^\nu(x)=\frac{(-1)^n}{2^n}\frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(n+2\nu)}{\Gamma(2\nu)\Gamma\left(n+\nu+\frac{1}{2}\right)}\frac{(1-x^2)^{\frac{1}{2}-\nu}}{n!}\frac{d^n}{dx^n}\{(1-x^2)^{n+\nu-\frac{1}{2}}\}$$
$$=\frac{\Gamma(n+2\nu)}{n!\,\Gamma(2\nu)}F\left(-n, n+2\nu, \nu+\frac{1}{2};\ \frac{1-x}{2}\right)$$

where $\Gamma(x)$ is gamma function and $F(\alpha, \beta, \gamma; z)$ is Gauss' hypergeometric function. It is well known that $C_n^\nu(x)$ has such generating function as follows:

$$(18)\qquad (1-2xt+x^2)^{-\nu}=\sum_{m=0}^{\infty}C_m^\nu(t)x^m;\quad -1<t<1, |x|<1.$$

It is the above formula in case of $\nu=1$ that we need now, namely

$$(19)\qquad \frac{1}{1-2tx+x^2}=\sum_{m=0}^{\infty}C_m^1(t)x^m;\quad -1<t<1, |x|<1.$$

Now for this special value of ν, the next formula (20) is a well known one.

$$(20)\qquad C_m^1(\cos\theta)=\frac{\sin(m+1)\theta}{\sin\theta}.$$

We need (19) and (20) for the proof of proposition 7. Now back to the proof, by the fact $|A|=1$, (14) is

$$\frac{1}{1-2\frac{\sigma(A)}{2}x+x^2}=\sum_{m=0}^{\infty}\alpha_{m+1}^{(1)}(\sigma(A))x^m.$$

Therefore in comparison with (16) and (18) we obtain

$$(21)\qquad C_m^1\left(\frac{\sigma(A)}{2}\right)=\alpha_{m+1}^{(1)}(\sigma(A)),\qquad m=0,1,2,\cdots.$$

Then the formula (20) and the definition of θ in the statement of propositio 7 give us the following formula:

$$(22)\qquad \alpha_m^{(1)}(\sigma(A))=\frac{\sin m\theta}{\sin\theta},\qquad m=0,1,2,\cdots.$$

Accordingly

$$(23)\qquad \alpha_m^{(2)}(\sigma(A))=\frac{\sin(m-1)\theta}{\sin\theta},\qquad m=1,2,3,\cdots.$$

Here proposition 1 and (21) and (22) lead to

$$(24)\qquad A^m=\begin{bmatrix}\dfrac{x_1\sin m\theta-\sin(m-1)\theta}{\sin\theta} & \dfrac{x_2\sin m\theta}{\sin\theta}\\ \dfrac{x_3\sin m\theta}{\sin\theta} & \dfrac{x_4\sin m\theta-\sin(m-1)\theta}{\sin\theta}\end{bmatrix}$$

for all $m=1,2,3,\cdots$. We continue the computation about the diagonal elements of the right side of (24). By some elementary methods we obtain

$$\begin{aligned}&x_1\sin m\theta-\sin(m-1)\theta\\ &=x_1\sin\{(m-1)\theta+\theta\}-\sin(m-1)\theta\\ &=(x_1\cos\theta-1)\sin(m-1)\theta+x_1\sin\theta\cos(m-1)\theta.\end{aligned}$$

Here we must consider whether $x_1\cos\theta-1$ and $x_1\sin\theta$ is positive or negative.

$$x_1\cos\theta-1=\frac{1}{2}(x_1^2+x_1x_4-2)=\frac{1}{2}\xi$$

is clear. Since $0<\theta<\pi$, $\sin\theta>0$, so the sign of $x_1\sin\theta$ depends on only the sign of x_1. In the same way we obtain

$$\begin{aligned}&x_4\sin m\theta-\sin(m-1)\theta\\ &=(x_4\cos\theta-1)\sin(m-1)\theta+x_4\sin\theta\cos(m-1)\theta\end{aligned}$$

and

$$x_4\cos\theta-1=\frac{1}{2}(x_4^2+x_1x_4-2)=\frac{1}{2}\eta,$$

and the sign of $x_4\sin\theta$ depends on only the sign of x_4.

In accordance with the classification (17), we obtain the following table.

		$x_1\cos\theta-1$,	$x_1\sin\theta$,	$\dfrac{x_1\sin m\theta-\sin(m-1)\theta}{\sin\theta}$
I ∪ IV	:	$+$	$+$	$L_1\sin\{(m-1)\theta+\varphi_1\}$
V ∪ VIII ∪ IX ∪ XII	:	$-$	$+$	$-L_1\sin\{(m-1)\theta+\varphi_1\}$
VI ∪ VII ∪ X ∪ XI	:	$-$	$-$	$-L_1\sin\{(m-1)\theta-\varphi_1\}$
II ∪ III	:	$+$	$-$	$L_1\sin\{(m-1)\theta-\varphi_1\}$

where $L_1=\dfrac{(x_1^2-2x_1\cos\theta+1)^{\frac{1}{2}}}{\sin\theta}$,

		$x_4\cos\theta-1,$	$x_4\sin\theta,$	$\dfrac{x_4\sin m\theta-\sin(m-1)\theta}{\sin\theta}$
V∪VI	:	+	+	$L_4\sin\{(m-1)\theta+\varphi_4\}$
I∪II∪IX∪X	:	−	+	$-L_4\sin\{(m-1)\theta+\varphi_4\}$
III∪IV∪XI∪XII	:	−	−	$-L_4\sin\{(m-1)\theta-\varphi_4\}$
VII∪VIII	:	+	−	$L_4\sin\{(m-1)\theta-\varphi_4\}$

where $L_4=\dfrac{(x_4^2-2x_4\cos\theta+1)^{\frac{1}{2}}}{\sin\theta}$.

Therefore we have proved proposition 7 except when $\xi\eta=0$ and $x_1x_4=0$. But it is so easily verified that in cases of $\xi\eta=0$ and $x_1x_4=0$ proposition 7 is also valid that we omit the rest of proof.

Corollary 1. If a real matrix A of degree 2 satisfies the condition (16), and if we denote A^m by $\begin{pmatrix} y_1^{(m)} & y_2^{(m)} \\ y_3^{(m)} & y_4^{(m)} \end{pmatrix}$ for all $m=1,2,3,\cdots$, then

$$|y_1^{(m)}|\leq\frac{(x_1^2-2x_1\cos\theta+1)^{\frac{1}{2}}}{\sin\theta},\qquad |y_2^{(m)}|\leq\frac{x_2}{\sin\theta}$$

$$|y_3^{(m)}|\leq\frac{x_3}{\sin\theta},\qquad |y_4^{(m)}|\leq\frac{(x_4^2-2x_4\cos\theta+1)^{\frac{1}{2}}}{\sin\theta}.$$

Corollary 2. If a real matrix A of degree 2 satisfies the condition (16) and if $2\pi/\theta$, namely $2\pi/\cos^{-1}\dfrac{\sigma(A)}{2}$, is a rational number, then there exists a certain natural number m such that A^m=diagonal matrix.

7. Here for the sake of description of the last theorem in this paper, we establish some definitions about a uniform distribution of matrix with degree 2. We denote the ring of rational integer and real number field by Z and R respectively. In this section we mainly treat with real numbers modulo 2π, namely elements in $R/2\pi Z$. Let a sequence $\{q_m^{(1)}\}$ in $R/2\pi Z$ be uniformly distributed on the interval $[0,2\pi)$, namely on the unit circle. We define that the sequence $\{q_m^{(2)}\}$ in $R/2\pi Z$ is *similar* to $\{q_m^{(1)}\}$ with the *phase difference* β, if and only if there exist $\alpha\in R$ and $\beta\in R/2\pi Z$, which are independent from m, such that for all $m=0,1,2,\cdots$

$$q_m^{(2)}=\alpha q_m^{(1)}+\beta \quad \text{in } R/2\pi Z.$$

Since $\{q_m^{(1)}\}$ is uniformly distributed on the unit circle, it is the same with this $\{q_m^{(2)}\}$.

We proceed on the second definition. Let $\left\{B_m=\begin{pmatrix} b_1^{(m)} & b_2^{(m)} \\ b_3^{(m)} & b_4^{(m)} \end{pmatrix};\ m=0,1,2,\cdots\right\}$ be a sequence of the square matrices with degree 2 over $R/2\pi Z$. We define that the sequence of the matrices $\{B_m\}$ in $R/2\pi Z$ *is uniformly distributed* with the phase difference (β_1,β_2) on the interval $[0,2\pi)$, namely on the unit circle, if and only if $\{b_1^{(m)}\}$ and $\{b_2^{(m)}\}$ are uniformly distributed on the interval $[0,2\pi)$, namely on the unit circle, and there exist $\alpha_1,\alpha_2\in R$ and $\beta_1,\beta_2\in R/2\pi Z$, which are independent from m, such that for all $m=0,1,2,\cdots$

$$b_4^{(m)}=\alpha_1 b_1^{(m)}+\beta_1 \quad \text{in } R/2\pi Z$$
$$b_3^{(m)}=\alpha_2 b_2^{(m)}+\beta_2 \quad \text{in } R/2\pi Z,$$

namely $\{b_4^{(m)}\}$ and $\{b_3^{(m)}\}$ are similar to $\{b_1^{(m)}\}$ and $\{b_2^{(m)}\}$ with the phase difference β_1 and β_2 respectively.

Now we proceed on the third definition. Let $\left\{W_m=\begin{pmatrix} w_1^{(m)} & w_2^{(m)} \\ w_3^{(m)} & w_4^{(m)} \end{pmatrix};\ m=0,1,2,\cdots\right\}$ be a sequence of any complex square matrices with degree 2. We define that this sequence $\{W_m\}$ is *circumferential*, if and only if the absolute value $\{W_j^{(m)}\}$ for an arbitrary $j=1,2,3,4$ is independent from m. Now let the sequence of any complex square matrices with degree 2, $\left\{W_m=\begin{pmatrix} w_1^{(m)} & w_2^{(m)} \\ w_3^{(m)} & w_4^{(m)} \end{pmatrix}\right\}$ be circumferential, and at that time for this sequence we define the second sequence of square matrices with degree 2 over $R/2\pi Z$ by $\left\{\begin{pmatrix} q_1^{(m)} & q_2^{(m)} \\ q_3^{(m)} & q_4^{(m)} \end{pmatrix};\ m=0,1,2,\cdots\right\}$, where for an arbitrary $j=1,2,3,4$, $q_j^{(m)}\in R/2\pi Z$ is defined by

$$w_j^{(m)}=|w_j^{(m)}|\exp(iq_j^{(m)});\ i^2=-1.$$

We denote $\begin{pmatrix} q_1^{(m)} & q_2^{(m)} \\ q_3^{(m)} & q_4^{(m)} \end{pmatrix}$ by *arg* W_m and we call this matrix the *argument* of W_m.

Now the last definition is described as follows. Let $\{W_m;\ m=0,1,2,\cdots\}$ be a sequence of complex square matrices with degree 2 and be circumferential.

We define that $\{W_m\}$ *is uniformly distributed* with the phase difference (β_1,β_2), if and only if $\{arg\ W_m;\ m=0,1,2,\cdots\}$ is the same.

8. On the basis of the above definitions we describe the last theorem.

Theorem. Let A be a real matrix with degree 2 such that $|A|=1$ and $-2<\sigma(A)<2$. Let us denote $\{A^m;\ m=1,2,3,\cdots\}$ by $S(A)$. Then the distribution of $S(A)$ in four dimensional Euclidean space R^4 is described as follows,

(a) if $2\pi/\cos^{-1}\dfrac{\sigma(A)}{2}$ is a rational number, then $S(A)$ is a finite set.

(b) if $2\pi/\cos^{-1}\dfrac{\sigma(A)}{2}$ is an irrational number, then there exists a sequence $\{W_m;\ m=1,2,3,\cdots\}$ of complex square matrices with degree 2 which is uniformly distributed with the phase differences, which are defined by the following table, such that

$$A^m=Im(W_m),$$

namely $S(A)$ is the imaginary part of the sequence $\{W_m;\ m=1,2,3,\cdots\}$ which is uniformly distributed:

$(\varphi_1+\varphi_4,0)$	if	A	exists in	I ∪ VII ∪ X
$(-\varphi_1+\varphi_4,0)$	if	A	exists in	II ∪ VIII ∪ IX
$(-\varphi_1-\varphi_4,0)$	if	A	exists in	III ∪ V ∪ XII
$(\varphi_1-\varphi_4,0)$	if	A	exists in	IV ∪ VI ∪ XI,

where $\varphi_i=\tan^{-1}\left|\dfrac{x_i\sin\theta}{x_i\cos\theta-1}\right|$, $(i=1,4)$ and $\theta=\cos^{-1}\dfrac{\sigma(A)}{2}$, $(0<\theta<\pi)$.

Proof. (a). In general, let

$$\left\{T_m=\begin{pmatrix} a_1\sin(m\theta_1+\zeta_1) & a_2\sin(m\theta_2+\zeta_2) \\ a_3\sin(m\theta_3+\zeta_3) & a_4\sin(m\theta_4+\zeta_4) \end{pmatrix}\right\}$$

be a sequence of real square matrices with degree 2 which has twelve real parameters a_j, θ_j (>0), ζ_j $(j=1,2,3,4)$. Let us assume that $2\pi/\theta_j$ $(j=1,2,3,4)$ are all rational numbers, say $2\pi/\theta_j=s_j/r_j$.

Then if $m\equiv m'$ (mod $s_1\cdot s_2\cdot s_3\cdot s_4$), we obtain $a_j\sin(m\theta_j+\zeta_j)=a_j\sin(m'\theta_j+\zeta_j)$, $j=1,2,3,4$ immediately. Therefore $\{T_m;\ m=1,2,3,\cdots\}$ is a finite set. Accordingly (a) is clear.

(b). We denote A by $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. Let us define W_m by

$$\begin{bmatrix} \dfrac{(x_1^2-2x_1\cos\theta+1)^{\frac{1}{2}}}{\sin\theta}\exp i\{(m-1)\theta+\varepsilon_1'\varphi_1\} & \dfrac{x_2}{\sin\theta}\exp im\theta \\ \dfrac{x_3}{\sin\theta}\exp im\theta & \dfrac{(x_4^2-2x_4\cos\theta+1)^{\frac{1}{2}}}{\sin\theta}\exp i\{(m-1)\theta+\varepsilon_4'\varphi_4\} \end{bmatrix}$$

where $\theta=\cos^{-1}\dfrac{\sigma(A)}{2}$, $0<\theta<\pi$ and $\varphi_i=\tan^{-1}\left|\dfrac{x_i\sin\theta}{x_i\cos\theta-1}\right|$, $(i=1, 4)$ and the values of ε_1' and ε_4' are determined as the foregoing table in proposition 7. Then our conclusion is obvious.