The Appropriation of Surplus Labour: A Game-theoretic Point of View

by

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Abstract: This paper considers the process of the appropriation of surplus labour in a capitalist economy from a game-theoretic point of view. It presents a very simple model of a market economy with production. It is shown that, given the mobility of labour, the concentration of the means of production plays a crucial role in the process.

……this original state of things, in which the labourer enjoyed the whole produce of his own labour, could not last beyond the first introduction of the appropriation of land and the accumulation of stock.

Adam Smith

1. Introduction and Summary

The purpose of this paper is to consider the process of the appropriation of surplus labour in a capitalist economy from a game-theoretic point of view. By the appropriation of surplus labour we mean the fact that, in the face of the freedom of contract, some people are able to obtain more goods than can be produced with the amount of labour provided by them.(1) Various authors (e. g., [3], [4], [6]) have so far considered the process in the Marxian framework, expressing labour values and prices in terms of simultaneous equatious and thereby comparing simple commodity production[2] with capitalist commodity production. In their models, however, the fact mentioned above is a premise, rather than a result, of the analysis. In order to see how the appropriation of surplus labour *emerges* in a market economy, we present a very simple model where labour is the only binding factor of production. It must be noted that, unlike the usual argument concerning the income distribution, our argument does not rely on the scarcity of other factors of production.

Consider an economy where m products are produced out of one primary factor, i. e., the homogeneous labour, at constant returns to scale, with no joint production. The assumtion of constant returns to scale implies that the technical efficiency which is due to the division of labour in industry[3] is abstracted from. However, this will enable us to focus our attention on the effects of the division of labour upon the distribution of net output. The economy consists of n agents, who are each endowed with a certain amount of labour and processes embodying the means of production which he owns and his technical knowledge. The agents as such may be interpreted as labourers in the "original state of things" (see the above citation from [10]) or "communities which still substantially produce for use-value" ([2] vol. III, ch. XX). Each agent accommodates himself to the expansion of the market, by exchanging his products with others' products in the market. The assumption is not made *a priori* that products are exchanged according to their labour contents. (By the market we do not mean the well-organized Walrasian market but envisage direct trade carried on between the agents.) Thus the division of labour emerges in the economy. When the freedom of contract is guaranteed, the situation resulting from the division of labour can be represented by the core, in which, roughly speaking, the economy never splits into subeconomies. The following points are worth noticing here. First, "the division of labour is limited by the extent of the market" ([10] ch. 3). Second, the character of the division of labour depends crucially on whether the mobility of labour is established or not. In the absence of the mobility of labour, the economy is not very far from a pure exchange economy, though the division of labour emerges to the extent that the principle of comparative advantage holds. Therefore it is almost meaningless to compare the amount of labour supplied by an agent with that embodied in the products consumed by him. On the other hand, the introduction of the mobility of labour makes inefficient processes virtually unnecessary for production and thereby increases the bargaining power of a group which possesses the efficient processes. Thus the situation changes very considerably. In the following sections we shall therefore concentrate our attention on the case where the mobility of labour is established. It is shown, under the various assumptions, that if there exists a group which monopolizes the efficient processes, the members of the group as a whole never lose labour (i. e., the labour content of the products consumed by them is at least as great as the amount of labour provided by them) and are able to gain labour. It is also shown that if there exist at least two disjoint groups each of which possesses the efficient processes, none of the agents lose labour because of competition between such groups. Therefore it seems very likely that all the members belonging to such groups cooperate to increase their bargaining power. If they do so, the situation reduces itself to the monopoly case mentioned above. It is easily seen that the counteraction on the part of the agents who do not belong to such groups is of no effect.

All these results seem to suggest that, given the mobility of labour, the concentration of the means of production plays a crucial role in the process of the appropriation of surplus labour in a capitalist economy.

2. The Model

The set of all products is denoted M. The agent i is assumed to have a utility function $u^i(L;x)=u^i(L;(x_c)_{c\in M})$, where L is the amount of labour supplied by him and x_c the quantity of the product c consumed by him, with $0 \leqq L \leqq \overline{L}^i$, his labour endowment. He is assumed to possess processes $a^i=(a^i_c)_{C\in M}$, where a^i_c denotes the quantity of the product c that can be produced with one unit of labour input, independently of the intensities of operation of other processes. (a^i_c denotes net output. It is easy to introduce explicitly "production of commodities by means of commodities" into the model. This does not make any difference to the results, provided that labour is the only binding factor of production. For instance, we can construct a model on the basis of a spectrum of the Leontief-type techniques.) The set of all agents is denoted N. Nonnull subsets of N, *i. e.*, coalitions, are denoted S, T and so forth. Write $a^s_c=\max\{a^i_c \mid i\in S\}$ and $a^s=(a^s_c)_{C\in M}$, we assume:

Assumption 1. $\overline{L}^i>0$ for all $i\in N$.

Assumption 2. $a^N>0$.

Assumption 3. $u^i(L;x)$ is continuous and monotone in the sense that $L'<L$ or $x'\geq x$ implies $u^i(L';x')>u^i(L;x)$, for all $i \in N$.

The meanings of the above assumptions are self-explanatory. Note that the quasi-concavity of the utility function is not assumed.

4. Labour Allocations in the Core

The division of labour in a coalition S can be represented by a labour allocation $L^s = (L^{ik}_{jc})$, where L^{ik}_{jc} denotes the amount of labour which the agent $i \in S$ contributes for the agent $j \in S$ using the process a^k_c $(k \in S,\ c \in M)$. For simplicity, we write $L^i(S) = \sum_{j \in s} \sum_{k \in s} \sum_{C \in M} L^{ik}_{jc}$ and $L_i(S) = \sum_{j \in s} \sum_{k \in s} \sum_{C \in M} L^{jk}_{ic}$ for all $i \in S$. In the following, a labour allocation L^s is always supposed to be feasible in the sense that $L^i(S) \leqq \overline{L}^i$ for all $i \in S$. A labour allocation determines the quantities of products consumed by each agent as well as the amount of labour supplied by him and consequently his utility level. We may therefore write the resulting situation in terms of utility as follows:

$$u^s(L^s) = [u^i(L^i(S)); (\sum_{j \in s} \sum_{k \in s} a^k_c L^{jk}_{ic})_{C \in M})]_{i \in s}$$

Following Scarf [8], we then construct a game in characteristic form:

$$V^s = \{u \in R^s \mid u \leqq u^s(L^s) \text{ for some labour allocation } L^s\},$$

where R^s denotes the utility space of the coalition S, *i. e.*, the subspace of the Euclidean space R^n associated with the members of S. For all $S \subset N$ we define the core C^s as follows:

$$C^s = \{u \in R^s \mid u^T \notin \text{int } V^T \text{ for all } T \subset S\},$$

where u^T denotes the projection of u into R^T.

Wa shall now focus our attention on the labour allocations which sustain the core C^N. For convenience, let $v^i(a)$ denote the maximal utility level that the agent i can attain, using the processes a and his labour endowment. Write $V^s(a)=[v^i(a)]_{i\in s}$ for all $S\subset N$.

Proposition 1. Suppose that Assumptions 1, 2, and 3 hold. Then we have $v^N(a^N)\in C^N$. Moreover, if a labour allocation L^N sustains $u^N(a^N)$, then

$$L^i(N)=L_i(N) \qquad \text{for all } i\in N. \tag{1}$$

Conversely, if a Pareto-optimal point of V^N is sustained by a labour allocation which satisfies (1), the point is $V^N(a^N)$.

Proof. Suppose there is a labour allocation L^s such that

$$u^s(L^s)\geq v^s(a^N). \tag{2}$$

If there exists an agent $i\in S$ such that $L^i(S)>L_i(S)$, we have $u^i(L^s)<v^i(a^s)\leqq u^i(a^N)$, a contradiction to (2). Therefore $L^i(S)\leqq L_i(S)$ for all $i\in S$. In view of the fact that $\sum_{i\in S}(L^i(S)-L_i(S))=0$, we then obtain $L^i(S)=L_i(S)$ for all $i\in S$. This implies, however, that $u^s(L^s)\leqq v^s(a^s)\leqq v^s(a^N)$, a contradiction to (2). Therefore $v^s(a^N)\notin \text{int } V^s$ for all $S\subset N$. Hence $v^N(a^N)\in C^N$.

Suppose that L^N sustains $V^N(a^N)$ and that there is an agent i such that $L^i(N)>L_i(N)$. This implies that the agent i is better off if he is allowed to use the processes a^N, which contradicts the definition of $v^N(a^N)$. Next, suppose a Pareto-optimal point is sustained by a labour allocation which satisfies (1). Evidently, the point can be sustained by an autarkic labour allocation (*i. e.*, a labour allocation L^N such that $L^i(N)=\sum_{k\in N}\sum_{C\in M}L^{ik}_{ic}$ holds for all $i\in N$), if each agent is allowed to use the processes a^N. Since this point is Pareo-optimal, each agent uses a^N. It follows that this point is nothing but $v^N(a^N)$. q. e. d.

Note that we can prove $v^s(a^s)\in C^s$ $({}^{\forall}S\subset N)$ in the same way as the above proof. Therefore the core C^s is nonempty for all $S\subset N$.

This implies that the expansion of the market makes every agent better off, or at least does not make him worse off.(4)

Now we shall see whether the core C^N contains a point other than $v^N(a^N)$, or what amounts to the same thing, whether a point of the core C^N can be sustained by a labour allocation which does not satisfy (1). In order to do so, we have assume the following:

Assumption 4. (i) For each agent the processes a^N are useful in the sense that $v^i(a^N) > u^i(0;0)$.

(ii) Each agent always requires all products; more precisely, the maximal value of $u^i(L;x)$ is attained at a point belonging to the interior of the simplex which is spanned by m points $(0,\cdots 0, L'a_c^N, 0,\cdots 0)$ $(c \in M)$, where $0 \leqq L \leqq \overline{L^i}$ and $L' > 0$, for all $i \in N$.

Assumption 4(i) is fairly weak. In economic terms Assumption 4(ii) says that a certain amount of each product is necessary at the subsistence level. It will turn out to be convenient to write $\tilde{S} = \{S \neq N \mid a^N = a^s$ and $a^N \neq a^T$ for all proper subsets T of $S\}$.

Proposition 2. Suppose that Assumptions 1, 2, 3 and 4 hold and that $|\tilde{S}| = 0$. Corresponding to each $i \in N$ there exist labour allocations L^N and L'^N which satisfy $u^N(L^N) \in C^N$ and

$$L^i(N) < L_i(N), \tag{3}$$

$u^N(L'^N) \in C^N$ and $L'^i(N) > L'_i(N)$, respectively.

Proof. It suffices to see that $v^N(a^N)$ is in the interior of the core C^N. Note that $|\tilde{S}| = 0$ is equivalent to $a^N \neq a^s$ (for all $S \neq N$). Suppose there exists a coalition $S \neq N$ such that $v^s(a^N) \in V^s$. Obviously, we have $v^s(a^N) > v^s(a^s)$ from Assumption 4. Therefore $v^s(a^N) \in V^S$ implies $v^s(a^s) \in \text{int } V^s$. On the other hand, $v^s(a^s) \in C^s$. This is a contradiction. g.e.d.

The condition $|\tilde{S}| = 0$ is equivalent to the following: for each $i \in N$ there exists at least one product $c_{(i)}$ such that $a^N_{c(i)} = a^i_{c(i)} > a^{N/\{i\}}_{c(i)}$. This observation makes Proposition 2 intuitively obvious, Note that if $n > m$, the condition $|\tilde{S}| = 0$ does not hold; therefore this. case is of no significance when the number of the agents is great.

Proposition 3. Suppose that Assumptions 1, 2, 3 and 4 hold. Suppose further that $|\tilde{S}| = 1$ and let S^* be the sole element of $\tilde{S}$. Then there exists a labour allocation L^N which satisfies $u^N (L^N) \in C^N$ and (3) for all $i \in S^*$. On the other hand, there exists no labour allocation L^N which satisfies $u^N (L^N) \in C^N$ and $\sum_{i \in s^*} L^i(N) > \sum_{i \in s^*} L_i (N)$.

Proof. Let L^{*N} be an autarkic labour allocation supporting $v^N (a^N)$. We have from Assumption 4 $\sum_{k \in N} L^{*ik}_{ic} > 0$ for each $i \in N$ and for each $c \in M$. In the following, we write $L^i_j = \sum_{k \in N} \sum_{C \in M} L^{ik}_{jc}$ We can easily obtain from L^{*N} a Pareto-optimal labour allocation L^N which satisfies

$$\sum_{j \in s^*} L^i_j = \varepsilon \qquad (i \in \overline{S^*})$$

$$\sum_{j \in \overline{s^*}} L^j_i = (|\overline{S^*}| \,/\, |S^*|)\, \varepsilon \qquad (i \in S^*)$$

$$L^i_j = 0 \qquad (i \neq j,\ i, j \in S^* \text{ or } i, j \in \overline{S^*}),$$

where ε is a sufficiently small positive number. We have $u^s (L^N) \notin V^s$ for all $S \not\supset S^*$, since $v^s (a^N) \notin V^s$ for all $S \not\supset S^*$. We also have $u^s (L^N) \notin V^s$ for all $S \supset S^* (S \neq N)$, since

$$\sum_{i \in s} \sum_{j \in s} (L^i_j - L^j_i) = -[\,|\overline{S^*}| - (|S| - |S^*|)]\,\varepsilon = -(n - |S|)\,\varepsilon < 0,$$

for all $S \supset S^* (S \neq N)$. Therefore $u^N (L^N) \in C^N$ and (3) holds for all $i \in S^*$.

The latter half of Proposition 3 follows from the fact that $v^s (a^N) \in V^S$. q. e. d.

The condition $|\tilde{S}| = 1$ means that there exists a coalition S^* which monopolizes the efficient processes a^N. Proposition 3 says that the coalition S^* as a whole never lose, and are able to gain, labour. To put it in another way, $v^N(a^N)$ is the most unprofitable in the core C^N for the coalition S^*, whereas it is the most profitable in the complementary coalition $\overline{S^*}$. Thus, the agents naturally divide into two groups, *i. e.*, S^* and $\overline{S^*}$. It is worth noticing that for an appropriate positive number δ the price system $p=(p_c)=((1+\delta)/a_c^N)$, together with the money-wage rate which is taken as numeraire, supports a point of the core C^N such that (3) holds for all $i \in S^*$. (Of course, this price system does not necessarily provide production prices.)

In order to complete the story, we have to examine the case $|\tilde{S}| \geqq 2$.

Proposition 4. Suppose that Assumptions 1, 2 and 3 hold. Suppose further that there exist at least two disjoint coalitions belonging to $\tilde{S}$. Then $v^N(a^N)=C^N$.

Proof. Let $(S^*, \overline{S^*})$ be a partition of N such that $a^{S^*}=a^{\overline{S^*}}$. By virtue of Proposition 1, it suffices to see that a point of the core C^N cannot be sustained by a labour allocation which doesnot satisfy condition(1). Suppose to the contrary that L^N sustains a point of the core C^N, while (1) does not hold for L^N. Write $e(S)=\sum_{i \in S}(L^i(N)-L_i(N))$. There exists anonempty coalition $T=\{i \in N | L^i(N)>L_i(N)\}$. Since $e(N)=0$ and $e(S \cup S') = e(S)+e(S')-e(S \cap S')$, we have $e(T \cup S^*)+e(T \cup \overline{S^*})=2e(T) -e(T \cap S^*)-e(T \cap \overline{S^*})=e(T)>0$. Therefore we may suppose $e(T \cup S^*)>0$ without loss of generality. In view of the fact that $a^{T \cup S^*}=a^N$, we have $v^{T \cup S^*}(a^{T \cup S^*}) \geq u^{T \cup S^*}(L^N)$. Noticing that $v^{T \cup S^*}(a^{T \cup S^*}) \in V^{T \cup S^*}$, we then obtain $u^{T \cup S^*}(L^N) \in \text{int } V^{T \cup S^*}$, a contradiction to the fact $u^N(L^N) \in C^N$. q.e.d.

Proposition 4 says that none of the agents gain labour because of competition between the coalitions belonging to $\widetilde{S}$. When there is an agent who possesses none of the efficient processes a^N, it seems very likely that the members of the coalitions belonging to $\widetilde{S}$ cooperate in order to increase their bargaining power. Once they form a union (*i. e.*, a coalition which never splits into subcoalitions during the bargaining with the outside agents), the situation reduces itself to the monopoly case described by Proposition 3. It can readily be seen that the counter union is of no effect. In order to state these facts formally, we define

$$S^* = \cup \{S \mid S \in \widetilde{S}\},$$

$$\widetilde{T} = \{S \subset N \mid S \cap S^* = S^* \text{ or } S \cap S^* = \phi\}.$$

Proposition 5. Suppose that Assumptions 1, 2, 3 and 4 hold and that $|\widetilde{S}| \neq 0$. Suppose further that $S^* \neq N$. Define

$$C' = \{u \in R^N \mid u^S \notin \text{int } V^S \text{ for all } S \in \widetilde{T}\},$$

$$C'' = \{u \in R^N \mid u^S \notin \text{int } V^S \text{ for } S = S^*, \ \overline{S^*}, \text{ and } N\}.$$

Then the conclusions of Proposition 3 hold for C' and $C^{(5), (6)}$.

Proof. Note that $v^N(a^N) \in C^N \subset C' \subset C''$; therefore we can obtain the conclusions in the same way as the proof of Proposition 3. q. e. d.

There remains the borderline case, that is, the case where $|\widetilde{S}| \geqq 2$ and there exist no disjoint coalitions belonging to $\widetilde{S}$. If $v^N(a^N) \neq C^N$ the situation is almost the same as that described by Proposition 3. (The former differs from the latter in that there is no point of the core C^N such that $u^S \notin V^S$ for all $S \neq N$.) Needless to say, if $v^N(a^N) = C^N$, the situation is the same as that of Proposition 4. At any rate, Proposition 5 applies in this borderline case as well

4. Concluding Remarks

We have shown the possibility that a group which monopolizes the efficient processes obtains more goods than can be produced with the amount of labour provided by the group. It must be noted that our argument does not rely on such factors as the existence of a reserve army of unemployed labourers and direct force outside economic conditions.

Footnotes

(1) This definition of the appropriation of surplus labour is due to Weizsäcker〔11〕.

(2) With respect to the meaning and interpretation of simple commodity production, there seems to be no agreement even among the Marxian theorists. For instance, see 〔4〕, 〔5〕 and 〔7〕.

(3) S. Marglin 〔1〕 denies A. Smith's thesis to the effect that the division of labour in a capitalist economy derives from the pursuit of technical efficiency.

(4) As for the case where labour lacks mobility, we can construct a game in characteristic form in the same way. By virtue of Scarf's theorems (〔8〕,〔9〕), we can easily prove that if, in addition, the utility functions are assumed to be quasi-concave, the core C^s is not empty for all $S \subset N$.

(5) To investigate what labour allocations result from the bargaining within the union is beyond the scope of this paper.

(6) It is interesting to see that A. Smith regarded the conflict between masters and workmen as a game situation. See 〔10〕 pp. 83–85.

References

〔1〕 Marglin, S. 'What do Bosses do?', *Harvard Institute of Economic Research Discussion Paper*, No. 222 (1971).

〔2〕 Marx, K. *Capital*, Moscow (1971).

〔3〕 May, K. 'The Structure of Classical Value Theories', *Review of Economic Studies* 17 (1949–50), 60–61.

〔4〕 Meek, R. *Studies in the Labour Theory of Value* : Introduction to the Second Edition, London (1973).

〔5〕 ________ 'Is there an "Historical Transformation Problem"? A Comment', *Economic Journal* 86 (1976), 342-347.

〔6〕 Morishima, M. *Marx's Economics : A Dual Theory of Value and Growth,* Cambridge (1973).

〔7〕 ma, M. and Catephores, C. 'Is there an "Historical Transformation Problem"?', *Economic Journal* 85 (1975), 309-328.

〔8〕 Scarf, H. 'The Core of an n Person Game', *Econometrica* 37 (1967), 50-69.

〔9〕 ________ 'On the Existence of a Cooperative Solution for a General Class of n Person Games, *Jonrnal of Economic Theory* 3 (1971), 169-181.

〔10〕 Smith, A. *An Inquiry into the Nature and Causes of the Wealth of Nations,* (eds.) R.H. Campbell, A.S. Skinner, and W.B. Wood, Oxford (1976).

〔11〕 von Weizsäcker, C.C. 'Modern Captial Theory and the Concept of of Exploitation', *Kyklos* 26 (1973), 245-281.

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