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SUPPORT PRICES FOR OPTIMAL PROGRAMS OF CAPITAL ACCUMULATION IN A GENERAL REDUCED MODEL UNDER UNCERTAINTY

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I. *Introduction*

The purpose of this paper is to prove the existence of support prices for optimal programs of capital accumulation under uncertainty. The theorem established here is a generalization of those of Radner (1973) and Zilcha (1976). Also it is a natural extension of the deterministic case proved by McKenzie (1986).

In the economy considered in this paper, there is uncertainty in production technologies and utility functions. At each period in time, there is no uncertainty in production technology and utility function. However future technologies and utilities are uncertain at each period in time. Our economic model is presented in a general reduced form so that it includes many cases considered so far.

In proving the existence of support prices for optimal programs, there are two key arguments. One is the induction argument of existence of prices, which has been developed by McKenzie (1986) in deterministic cases and was first applied by Zilcha (1976) to uncertain cases. The other one is the decomposition argument of finitely additive measures by Yoshida & Hewitt (1952), which is used to show that prices are integrable functions. The argument was first used by Bewley (1972) in models of general equilibrium, and also applied by Radner (1973) and Zilcha (1976) to models of economic growth under uncertainty. In their model there is uncertainty in production technologies, and the uncertainty is stationary. In Radner (1973), support prices for stationary optimal programs were considered. In this paper we will consider a general non-stationary model with uncertainty in production technology and utility, and will establish by those two arguments a general support price theorem for optimal programs of capital accumulation under uncertainty.

II. *Model*

The model considered here is a generalized reduced model of capital accumulation where future utilities and production technologies are uncertain. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Each element in Ω denotes one possible state (a set of environments in past, present, and future). Family $\mathscr{F}$ is the set of all possible events and P denotes the probability distribution of states. Let $T=\{0, 1, 2, \ldots\}$ be the space of time. The uncertainty about states is described by a filtration $\{\mathscr{F}_t \mid t \in T\}$, i.e., $\mathscr{F}_t$ is a family of subsets of Ω and $\mathscr{F}_t \subset \mathscr{F}_{t+1}$ for each time $t \in T$. Family $\mathscr{F}_t$ is interpreted as the informations about states that become available until time t.

The production technology available at time $t>0$ is described by a correspondence Y_t, $\omega \in \Omega \rightarrow Y_t(\omega) \subset R^n \times R^n$, where R^n is an n-dimensional Euclidean space. The graph of Y_t defined by $G(Y_t)=\{(x, y, \omega) \mid (x, y) \in Y_t(\omega)\}$ is assumed to be $\mathscr{B}(R^n) \times \mathscr{B}(R^n) \times \mathscr{F}_t$-measurable, where $\mathscr{B}(R^n)$ is the family of all Borel subsets of R^n. By set $Y_t(\omega)$ we represent the possibility of transformation of capital stocks, i.e., $(x, y) \in Y_t(\omega)$ means that it is possible to transform capital stock x at time t-1 into capital stock y at time t under state ω.

The satisfaction in the economy at time $t>0$ is described by a utility function, $u_t: G(Y_t) \rightarrow R$, which is assumed to be a $\mathscr{B}(R^n) \times \mathscr{B}(R^n) \times \mathscr{F}_t$-measurable function on $G(Y_t)$ to the real line R. Value $u_t(x, y, \omega)$ is interpreted as the utility obtained at time t if capital stocks at time t-1 and t are x and y respectively.

In order to describe a program of capital accumulation, we will use a stochastic process, i.e., a function $f: T \times \Omega \rightarrow R^n$ such that $f(t, \cdot)$ is $\mathscr{F}_t$-measurable for each $t \in T$. For any stochastic process $f: T \times \Omega \rightarrow R^n$, we denote by $f_t: \Omega \rightarrow R^n$ a function defined by $f_t(\omega)=f(t, \omega)$.

Definition 2.1: If a function $k: T \times \Omega \rightarrow R^n$ is called a *program*, it is a stochastic process such that $k_t \in L_\infty(\mathscr{F}_t)$ for each $t \in T$, where $L_\infty(\mathscr{F}_t)$ denotes the set of all essentially bounded $\mathscr{F}_t$-measurable functions on Ω to R^n.

We should note that by this definition we restrict ourselves to essentially bounded programs.

Definition 2.2: A program k is said to be *feasible* if $(k_{t-1}(\omega), k_t(\omega)) \in Y_t(\omega)$ a.s. for each $t>0$.

To evaluate feasible programs, we use the so-called overtaking criterion. For a feasible program k, we denote by $U_t(k)$ the sum of expected utilities which will be obtained until time t in program k, i.e.,

$$U_t(k)=\int_\Omega \Big[\sum_{s=1}^{t} u_s(k_{s-1}(\omega), k_s(\omega), \omega)\Big] dP(\omega)$$

Definition 2.3: A feasible program k is said to be *optimal* if there is no other feasible program k' with $k_0=k'_0$ such that

$$\limsup_{t \rightarrow +\infty} [U_t(k')-U_t(k)]>0 .$$

III. *Support Prices for Optimal Programs*

First we assume the convexity of the model.

Assumption 1: For each t, $Y_t(\omega)$ is convex and $u_t(x, y, \omega)$ is concave in (x, y) a.s.

Moreover, we assume the monotonicity of utility functions with respect to initial capital stock at each period in time.

Assumption 2: If $(x, y) \in Y_t(\omega)$ and $x \leqq x'$, then $(x', y) \in Y_t(\omega)$ and $u_t(x, y, \omega) \leqq u_t(x', y, \omega)$.

By the next assumption we assume the boundedness of the model.

Assumption 3: For each t, there exists a number b such that if $(x, y) \in Y_t(\omega)$, then $||y|| \leqq b||x||$ and $u(x, y, \omega) \leqq b||x||$.

The set of all possible transformations of capital stocks between time $t\text{-}1$ and time t is defined by

$$\mathscr{Y}_t = \{(f, g) \in L_\infty(\mathscr{F}_{t-1}) \times L_\infty(\mathscr{F}_t) | (f(\omega), g(\omega)) \in Y_t(\omega) \text{ a.s.}\} .$$

Also, for each t, we define

$$\mathscr{X}_t = \{g | (f, g) \in \mathscr{Y}_t \text{ for some } f\} .$$

Remark 3.1: Assumption 3 implies that given $f \in L_\infty(\mathscr{F}_{t-1})$, there exists a number B such that $||g||_\infty \leqq B$ and $\int u_t(f, g, \cdot)dP \leqq B$ for all $(f, g) \in \mathscr{Y}_t$. Namely, if capital stock at time $t\text{-}1$ is bounded, then capital stock and expected utility at time t must be bounded.

Let $k: T \times \Omega \to R^n$ be an optimal program. In order to define a value function we assume:

Assumption 4: $\int u_t(k_{t-1}, k_t, \cdot)dP > -\infty$ for each $t > 0$.

Remark 3.2: Together with Assumption 3, the above assumption implies that in optimal program k, $\int u_t(k_{t-1}, k_t, \cdot)dP$ is finite for each $t > 0$.

For each $t \in T$, we define the normalized utility function $v_t: G(Y_t) \to R$ by

$$v_t(x, y, \omega) = u_t(x, y, \omega) - u_t(k_{t-1}(\omega), k_t(\omega), \omega) .$$

Moreover, for each $t \in T$ and $f \in L_\infty(\mathscr{F}_t)$, by $\mathscr{K}_t(f)$ we denote the set of all feasible programs after time t from capital stock f, i.e.,

$$\mathscr{K}_t(f) = \{h | h \text{ is a program such that } h_t = f \text{ and } (h_s(\omega), h_{s+1}(\omega)) \in Y_{s+1}(\omega) \text{ a.s. for each } s \geqq t\} .$$

Now, by virtue of Remark 3.2, we can define the so-called normalized value functions. For each $t \in T$ and $f \in L_\infty(\mathscr{F}_t)$, define

$$V_t(f) = \sup_{h \in \mathscr{K}_t(f)} \{\liminf_{r \to +\infty} \int [\sum_{s=t+1}^{r} v_s(h_{s-1}, h_s, \cdot)]dP\} .$$

Also, for each $t \in T$, we define a set by

$$\mathscr{D}_t = \{f \in L_\infty(\mathscr{F}_t) | V_t(f) > -\infty\} .$$

Remark 3.3: By Assumption 1, we can easily show that V_t is concave and $\mathscr{D}_t$ is convex for each $t \in T$. Also, by Assumption 2, V_t is an increasing function. Moreover, a map defined by

$$(f, g) \in \mathscr{Y}_t \to \int u_t(f, g, \cdot) dP \in R$$

can be shown to be concave for each $t > 0$.

Assumption 5: $k_0 \in \text{int} \mathscr{D}_0$ and int $\mathscr{X}_t \cap \mathscr{D}_t \neq \phi$ for each $t > 0$, where $\mathscr{X}_t$ and $\mathscr{D}_t$ are subsets of space $L_\infty(\mathscr{F}_t)$ and symbol 'int' means the interior in the topology of the space.

We should notice that Assumptions 4 and 5 are made for a particular optimal program k and they depend on the program.

Support prices for optimal programs are also described by stochastic processes. However we are interested in particular prices in the sense of the following definition.

Definition 3.4: If a function $p: T \times \Omega \to R^n$ is called a *price system*, it is a stochastic process such that $p_t \in L_1(\mathscr{F}_t)$ for each $t \in T$, where $L_1(\mathscr{F}_t)$ denotes the set of all integrable $\mathscr{F}_t$-measurable functions on Ω to R^n.

The following is the main result of this paper.

Main Theorem: Let k be an optimal program satisfying Assumptions 4 and 5. Then, under Assumptions 1, 2, and 3, there exists a price system p such that for each $t \in T$

1) $V_t(k_t) - \int p_t k_t dP \geqq V_t(f) - \int p_t f dP$ for all $f \in \mathscr{D}_t$, and

2) $\int u_{t+1}(k_t, k_{t+1}, \cdot) dP - \int p_t k_t dP + \int p_{t+1} k_{t+1} dP$
$\geqq \int u_{t+1}(f, g, \cdot) dP - \int p_t f dP + \int p_{t+1} g dP$ for all $(f, g) \in \mathscr{Y}_{t+1}$,

where V_t is the normalized value function for program k.

IV. *Proof of the Main Theorem*

To prove the Main Theorem, let k be an optimal program and V_t be the normalized value function for program k.

For each $t \in T$, let $L_\infty^*(\mathscr{F}_t)$ denote the set of all continuous linear function on $L_\infty(\mathscr{F}_t)$ to R. For simplicity we use the following notations. For each $t \in T$, define

$$\partial V_t(k_t) = \{\pi_t \in L_\infty^*(\mathscr{F}_t) | V_t(k_t) - \pi_t(k_t) \geqq V_t(f) - \pi_t(f) \text{ for all } f \in L_\infty(\mathscr{F}_t)\} .$$

Moreover, for each $t > 0$, define

$$\partial u_t(k_{t-1}, k_t) = \{(\pi_t, -\pi_{t+1}) \in L_\infty^*(\mathscr{F}_t) \times L_\infty^*(\mathscr{F}_{t+1}) | \int u_{t+1}(k_t, k_{t+1}, \cdot) dP - \pi_t(k_t) + \pi_{t+1}(k_{t+1}) \geqq \int u_{t+1}(f, g, \cdot) dP - \pi_t(f) + \pi_{t+1}(g) \quad \text{for all } (f, g) \in \mathscr{Y}_{t+1}\} .$$

Lemma 1: If $\pi_t \in \partial V_t(k_t)$, then there exists $\pi_{t+1} \in L_\infty^*(\mathscr{F}_{t+1})$ such that

1) $(\pi_t, -\pi_{t+1}) \in \partial u_t(k_t, k_{t+1})$ and

2) $\pi_{t+1} \in \partial V_{t+1}(k_{t+1})$.

Proof: Define a number w_{t+1} by

(4.1) $w_{t+1}=\int u_{t+1}(k_t, k_{t+1}, \cdot)dP+V_{t+1}(k_{t+1})-\pi_t(k_t)=V_t(k_t)-\pi_t(k_t)$.

Also, define two sets,

$$A=\{(w, g)\in R\times L_\infty(\mathscr{F}_{t+1})|\, w>w_{t+1}-\int u_{t+1}(f, g, \cdot)dP+\pi_t(f) \text{ for some } f \text{ with } (f, g)\in \mathscr{Y}_{t+1}\} ,$$

and

$$B=\{(w, g)\in R\times L_\infty(\mathscr{F}_{t+1})|\, V_{t+1}(g)\geqq w\} .$$

Clearly, point $(V_{t+1}(k_{t+1}), k_{t+1})\in B$, and $(V_{t+1}(k_{t+1}), k_{t+1})\in bd\ A$. By Assumption 1, these sets are convex. In addition, by Assumption 5, set B has non-empty interior.

Suppose $A\cap B\neq\phi$. Then there exists $(f, g)\in \mathscr{Y}_{t+1}$ such that

$$V_{t+1}(g)>w_{t+1}-\int u_{t+1}(f, g, \cdot)dP+\pi_t(f) .$$

By (4.1) and the definition of value function V_t, we have

$$V_t(f)-\pi_t(f)>V_t(k_t)-\pi_t(k_t) ,$$

which implies that $\pi_t\notin\partial V_t(k_t)$, a contradition. Hence $A\cap B=\phi$.

By a separation theorem [Dunford & Schwartz (1964), Thm.V.2.8, p. 417], there exists a non-zero continuous linear function (c, $-\pi_{t+1}$), i.e., a number c and a function $\pi_{t+1}\in L_\infty{}^*(\mathscr{F}_{t+1})$ such that

$$cw-\pi_{t+1}(g)\geqq cw'-\pi_{t+1}(g') \text{ for all } (w, g)\in A \text{ and } (w', g')\in B.$$

This implies that

(4.2) $c[w_{t+1}-\int u_{t+1}(f, g, \cdot)dP+\pi_t(f)]-\pi_{t+1}(g)$
$\geqq cV_{t+1}(g')-\pi_{t+1}(g')$ for all $(f, g)\in \mathscr{Y}_{t+1}$ and $g'\in\mathscr{D}_{t+1}$.

Suppose $c=0$. (4.2) implies that $\pi_{t+1}(g'-g)\geqq 0$ for all $(f, g)\in \mathscr{Y}_{t+1}$ and $g'\in\mathscr{D}_{t+1}$. By Assumption 5, $\pi_{t+1}=0$, which is a contradition to $(c, \pi_{t+1})\neq 0$. Hence we can assume $c=1$.

Put $g'=k_{t+1}$ in (4.2). Then, by (4.1), we have

$$\int u_{t+1}(k_t, k_{t+1}, \cdot)dP-\pi_t(k_t)+\pi_{t+1}(k_{t+1}) \geqq \int u_{t+1}(f, g, \cdot)dP-\pi_t(f)+\pi_{t+1}(g) \quad \text{for all } (f, g)\in \mathscr{Y}_{t+1} .$$

This implies 1). Moreover, put $f=k_t$, and $g=k_{t+1}$ in (4.2). Then, we have

$$V_{t+1}(k_{t+1})-\pi_{t+1}(k_{t+1})\geqq V_{t+1}(g')-\pi_{t+1}(g') \quad \text{for all } g'\in\mathscr{D}_{t+1} .$$

This implies 2). ■

For time t and function $p_t\in L_1(\mathscr{F}_t)$, a map defined by

$$f\in L_\infty(\mathscr{F}_t) \to \int p_t f dP\in R$$

is an element of $L_\infty{}^*(\mathscr{F}_t)$. Therefore function $p_t\in L_1(\mathscr{F}_t)$ can be considered as an element of $L_\infty{}^*(\mathscr{F}_t)$.

Lemma 2: If $(\pi_t, -\pi_{t+1})\in\partial u_{t+1}(k_t, k_{t+1})$ and $\pi_{t+1}\in\partial V_{t+1}(k_{t+1})$, then there exists $p_t\in L_1(\mathscr{F}_t)$ such that

1) $(p_t, -\pi_{t+1})\in\partial u_{t+1}(k_t, k_{t+1})$ and

2) $p_t \in \partial V_t(k_t)$.

In addition, if $(\pi_{t-1}, -\pi_t) \in \partial u_t(k_{t-1}, k_t)$, then

3) $(\pi_{t-1}, -p_t) \in \partial u_t(k_{t-1}, k_t)$.

Proof: First we will show that there exists $p_t \in L_1(\mathscr{F}_t)$ such that 1) and 2) hold. Since $\pi_t \in L_\infty^*(\mathscr{F}_t)$, by a theorem [Dunford & Schwartz (1964), Thm.IV.8.16, p. 296] there is a bounded finitely additive vector-values measure ν such that

$$\pi_t(f) = \int f d\nu \qquad \text{for all } f \in L_\infty(\mathscr{F}_t)\,.$$

Since $(\pi_t, -\pi_{t+1}) \in \partial u_{t+1}(k_t, k_{t+1})$, $\pi_t \geqq 0$ by Assumption 2. Therefore it follows that $\nu \geqq 0$. Hence, measure ν can be decomposed into two measures by a theorem [Yoshida & Hewitt (1952), Thm.1.23, p. 52], that is

$$\nu = \nu_c + \nu_p\ ,$$

where ν_c is a non-negative countably additive measure on $(\Omega, \mathscr{F}_t)$ which is absolutely continuous with respect to P and ν_p is a non-negative purely finitely additive measure on $(\Omega, \mathscr{F}_t)$. Therefore, by the Radon-Nikodym theorem [Dunford & Schwarts (1964), Thm. III.10.7, p. 181] there is a unique $p_t \in L_1(\mathscr{F}_t)$ such that

$$\int f d\nu_c = \int p_t f dP \qquad \text{for all } f \in L_\infty(\mathscr{F}_t)$$

Moreover, by a theorem [Yoshida & Hewitt (1952), Thm.1.22, p. 52] there is a sequence $\{A_n\}_{n=1}^{\infty}$ such that $A_n \subset A_{n+1}$ and $\nu_p(A_n) = 0$ for all n and $\lim_{n\to\infty} P(A_n) = 1$.

To prove 1), let $(f, g) \in \mathscr{Y}_{t+1}$. For each n, define f_n by

$$f_n(\omega) = \begin{cases} f(\omega) & \text{for } \omega \in A_n \\ k_t(\omega) + b & \text{otherwise}\,. \end{cases}$$

where $b \in R^n$ is taken so that $(f_n, g) \in \mathscr{Y}_{t+1}$. Then, since $(\pi_t, -\pi_{t+1}) \in \partial u_{t+1}(k_t, k_{t+1})$, we have

$$\begin{aligned} &\int u_{t+1}(k_t, k_{t+1}, \cdot\,) dP - \int p_t k_t dP - \int_{\Omega/A_n} k_t d\nu_p + \pi_{t+1}(k_{t+1}) \\ &\quad \geqq \int u_{t+1}(f_n, g, \cdot\,) dP - \int p_t f_n dP - \int_{\Omega/A_n} f_n d\nu_p + \pi_{t+1}(g) \end{aligned}$$

for all n. Since b is small if (f, g) is close to (k_t, k_{t+1}), we have in the limit

$$\begin{aligned} &\int u_{t+1}(k_t, k_{t+1}, \cdot\,) dP - \int p_t k_t dP + \pi_{t+1}(k_{t+1}) \\ &\quad \geqq \int u_{t+1}(f, g, \cdot\,) dP - \int p_t f dP + \pi_{t+1}(g)\,. \end{aligned}$$

This proves 1).

Next we will show that 2) holds. In fact, since $\pi_{t+1} \in \partial V_{t+1}(k_{t+1})$,

$$V_{t+1}(k_{t+1}) - \pi_{t+1}(k_{t+1}) \geqq V_{t+1}(g) - \pi_{t+1}(g) \qquad \text{for all } g \in L_\infty(\mathscr{F}_{t+1})\,.$$

Also, 1) implies that

$$\begin{aligned} &\int u_{t+1}(k_t, k_{t+1}, \cdot\,) dP - \int p_t k_t dP + \pi_{t+1}(k_{t+1}) \\ &\quad \geqq \int u_{t+1}(f, g, \cdot\,) dP - \int p_t f dP + \pi_{t+1}(g) \qquad \text{for all } (f, g) \in \mathscr{Y}_{t+1}\,. \end{aligned}$$

Hence, by the above two inequalities, we have

$$\begin{aligned} &\int u_{t+1}(k_t, k_{t+1}, \cdot\,) dP + V_{t+1}(k_{t+1}) - \int p_t k_t dP \\ &\quad \geqq \int u_{t+1}(f, g, \cdot\,) dP + V_{t+1}(g) - \int p_t f dP \qquad \text{for all } (f, g) \in \mathscr{Y}_{t+1}\,. \end{aligned}$$

Thus, by definition of the value function V_t, we can conclude that

$$V_t(k_t)-\int p_t k_t dP \geqq V_t(f)-\int p_t f dP \qquad \text{for all } f\in L_\infty(\mathscr{F}_t) .$$

That is, 2) holds.

Moreover, to prove 3), let $(f, g)\in \mathscr{Y}_t$. For each n, define f_n and g_n by

$$(f_n(\omega), g_n(\omega))=\begin{cases}(f(\omega), g(\omega)) & \text{for } \omega\in A_n \\ (f(\omega)+b, k_t(\omega)) & \text{otherwise,}\end{cases}$$

where $b\in R^n$ is taken so that $(f_n, g_n)\in \mathscr{Y}_t$ for all n. Then, since $(\pi_{t-1}, -\pi_t)\in\partial u_t(k_{t-1}, k_t)$, we have

$$\begin{aligned}&\int u_t(k_{t-1}, k_t, \cdot)dP-\pi_{t-1}(k_{t-1})+\int p_t k_t dP+\int_{\Omega/A_n} k_t d\nu_p \\ &\geqq \int u_t(f_n, g_n, \cdot)dP-\pi_{t-1}(f_n)+\int p_t g_n dP+\int_{\Omega/A_n} g_n d\nu_p\end{aligned}$$

for all n. Since b is small if (f, g) is close to (k_{t-1}, k_t), we have in the limit

$$\begin{aligned}&\int u_t(k_{t-1}, k_t, \cdot)dP-\pi_{t-1}(k_{t-1})+\int p_t k_t dP \\ &\geqq \int u_t(f, g, \cdot)dP-\pi_{t-1}(f)+\int p_t g dP .\end{aligned}$$

This proves 3). ■

Now we are ready to complete the proof of the Main Theorem. Since $k_0\in\text{int}\,\mathscr{D}_0$ by Assumption 5, there exists $\pi_0\in L_\infty^*(\mathscr{F}_t)$ such that $\pi_0\in\partial V_0(k_0)$. Therefore, by virtue of Lemma 1, for each $t\in T$ there exists $\pi_t\in L_\infty^*(\mathscr{F}_t)$ such that

$$\pi_t\in V_t(k_t) \quad\text{and}\quad (\pi_t, -\pi_{t+1})\in\partial u_{t+1}(k_t, k_{t+1}).$$

Hence, when $t=0$, by Lemma 2 there exists $p_0\in L_1(\mathscr{F}_0)$ such that $p_0\in\partial V_0(k_0)$ and $(p_0, -\pi_1)\in\partial u_1(k_0, k_1)$. Therefore, when $t=1$, again by Lemma 2, there exists $p_1\in L_1(\mathscr{F}_1)$ such that $(p_0, -p_1)\in\partial u_1(k_0, k_1)$, $p_1\in\partial V_1(k_1)$ and $(p_1, -\pi_2)\in\partial u_2(k_1, k_2)$. Similarly, by virtue of Lemma 2, for each $t\in T$ there exists $p_t\in L_1(\mathscr{F}_t)$ such that

$$p_t\in\partial V_t(k_t) \quad\text{and}\quad (p_t, -p_{t+1})\in\partial u_{t+1}(k_t, k_{t+1}).$$

This implies the Main Theorem.

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