SOME NOTES ON UTILITY FUNCTION*

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§. 1

In this paper, it is assumed that there are two kinds of consumption goods, labeled by x_1 and x_2 respectively, and that the utility function $u=u(x_1, x_2)$ is "well behaved" in the sense that $\frac{\partial u}{\partial x_i}\equiv u_i>0$ and $\frac{\partial^2 u}{\partial x_i^2}\equiv u_{ii}<0$ $(i=1, 2)$. Let M and p_i be the amount of money income and the price of x_i respectively which are all given autonomously. Under the assumption of rational behavior of consumers, we know that the marginal rate of substitution between x_1 and x_2 must be equal to their price ratio, namely $\frac{u_1}{u_2}=\frac{p_1}{p_2}$.

The object of this paper is to analyse some properties of the utility function in connection with the rational behavior of consumers. In §. 2, it will be examined the property of the utility function in which the income elasticity of x_2 is always zero in the sense that

$$\frac{\partial x_2}{\partial M}\Big/\frac{x_2}{M}=0.$$

In §. 3 on the contrary, we examine the property of the utility function in which the income elasticity of x_2 is always unit in the sense of

$$\frac{\partial x_2}{\partial M}\Big/\frac{x_2}{M}=1.$$

And in §. 4 lastly, it will be examined the property of the utility function in which the cross elasticity of x_2 to p_1 is always zero in the sense that

$$\frac{\partial x_2}{\partial p_1}\Big/\frac{x_2}{p_1}=0.$$

§. 2

In the utility function

$$u=u(x_1, x_2) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2.1)$$

let x_2 be the consumption goods whose income elasticity is always zero. In terms of Fig. 1, this means that, as far as the price ratio $\frac{p_1}{p_2}\left(=\frac{u_1}{u_2}\right)$ is given, x_2 is kept at x_2^0 in spite of

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the increase of money income from M' to M''. Therefore we have

$$x_2=H\left(\frac{u_2}{u_1}\right). \quad \cdots\cdots\cdots\cdots (2.2)$$

Remembering that the indifference curves are all "well behaved" as shown by the figure, it is easy to show that the function H has its inverse, so that we obtain

$$\frac{u_2}{u_1}=\phi(x_2). \quad \cdots\cdots\cdots\cdots (2.3)$$

Now, we can prove the following

THEOREM 1: *For the income elasticity of x_2 in the utility function $u=u(x_1, x_2)$ to be always zero, it is necessary and sufficient that it is demoted by*

$$u=u(x_1+f(x_2)),$$

where $f'(x_2)=\phi(x_2)$.

Proof of Sufficiency.

Let $u(x_1, x_2)$ be $u(x_1+f(x_2))$. Apparently $u_1=u'$ and $u_2=u'\cdot f'(x_2)$. Therefore it follows

$$\frac{u_2}{u_1}=f'(x_2)=\phi(x_2).$$

Proof of Necessity.

For a moment, let $u=A_0$ (a given constant). Thus we obtain

$$u_1dx_1+u_2dx_2=0.$$

Dividing both sides by u_1 and using (2. 3), we have

$$dx_1+\phi(x_2)dx_2=0.$$

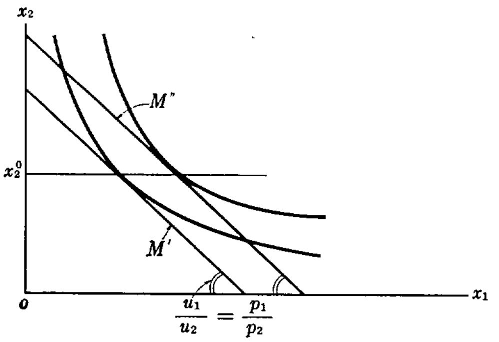
The integration of this last equation yields

$$x_1+f(x_2)=C_0,$$

where C_0 is the integral constant associated with A_0. Thus, the general solution of (2. 3) is given by

$$u=u(x_1+f(x_2)). \qquad \text{Q.E.D.}$$

FIG. 1



It will be easy to know that

$$\phi'(x_2)=f''(x_2)<0.$$

Thus, for example, $u=u(x_1+x_2^{0.5})$ is a case which satisfies the requirements in connection with THEOREM 1.

§. 3

Our next case to be considered is the one in which the income elasticity of x_2 is always unit. As is easily seen from Fig. 2, this case means that the ratio of x_1 to x_2 remains constant for a given price ratio $\frac{p_1}{p_2}\left(=\frac{u_1}{u_2}\right)$ in spite of the increase of money income from M' to M''. In the same way as the case in §. 2, we may express this case by the equation

$$\frac{u_2}{u_1}=\phi\left(\frac{x_2}{x_1}\right). \quad\cdots\cdots\cdots\cdots\cdots\cdots (3.1)$$

The utility function which satisfies (3. 1) is often called *homothetic*. Our problem in the second case is thus to analyse the property of an homothetic utility function.

Then we can prove the following

THEOREM 2: *For the utility function $u(x_1, x_2)$ to satisfies (3. 1), it is necessary and sufficient that it is denoted by*

$$u=u\left(\frac{1}{x_1}f\left(\frac{x_2}{x_1}\right)\right)$$

where $$-\frac{f'}{f}=\phi\left(\frac{x_2}{x_1}\right)\Big/1+\phi\left(\frac{x_2}{x_1}\right)\cdot\frac{x_2}{x_1}.$$

Proof of Sufficiency.

Let $u(x_1, x_2)=u\left(\frac{1}{x_1}f\left(\frac{x_2}{x_1}\right)\right)$. It is apparent that $u_1=u'\cdot\frac{1}{x_1^2}\left\{-f-f'\cdot\frac{x_2}{x_1}\right\}$ and $u_2=u'\cdot\frac{f'}{x_1^2}$. Then it follows

$$\frac{u_2}{u_1}=-\frac{f'}{f+f'\cdot\frac{x_2}{x_1}}\equiv\phi\left(\frac{x_2}{x_1}\right).$$

Proof of Necessity.

Let us suppose that $u=A_0$ (a given constant). Then we obtain

$$u_1dx_1+u_2dx_2=0.$$

Dividing both sides by u_1 and using (3. 1), we have

$$dx_1+\phi\left(\frac{x_2}{x_1}\right)dx_2=0.$$

Let us put $z=\frac{x_2}{x_1}$ or $x_2=z\cdot x_1$. Then we obtain

$$dx_2=z\cdot dx_1+x_1\cdot dz.$$

Putting this into the above equation and making rearrangement, we have

$$\frac{dx_1}{x_1}=-\frac{\phi(z)}{1+\phi(z)\cdot z}dz.$$

The integration of both sides gives us

$$\log f(z) = \log x_1 + \log C_0,$$

where C_0 is the integral constant associated with A_0 and

$$f(z) = e^{-\int \frac{\phi(z)}{1+\phi(z)\cdot z}dz}.$$

Thus finally we get the result desired, namely

$$u = u\left(\frac{1}{x_1}f\left(\frac{x_2}{x_1}\right)\right). \qquad \text{Q.E.D.}$$

As an example of the homothetic utility function, we may present the well-known Cobb-Douglasian function $u = x_1^{\alpha} \cdot x_2^{\beta}$. It is easy to show that

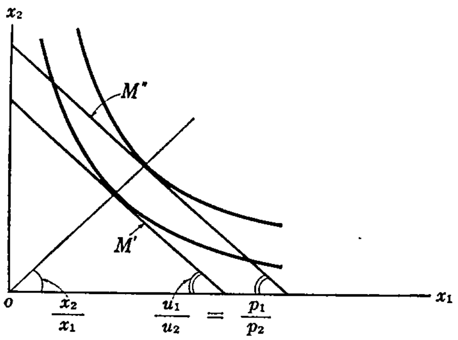
$$u = x_1^{\alpha} \cdot x_2^{\beta} = \left\{\frac{1}{x_1}\left(\frac{x_2}{x_1}\right)^{-\frac{\beta}{\alpha+\beta}}\right\}^{-(\alpha+\beta)} \equiv \left\{\frac{1}{x_1}f\left(\frac{x_2}{x_1}\right)\right\}^{-(\alpha+\beta)} \equiv u\left(\frac{1}{x_1}f\left(\frac{x_2}{x_1}\right)\right).$$

It should be remembered that the homogeneity of utility function of some degree is only a sufficient condition, not a necessary condition, for the function to be homothetic. For example,

$$u = x_1^{\alpha} \cdot x_2^{\beta} + \text{a given constant}$$

is homothetic, but not homogeneous.

FIG. 2



§. 4

Our last case to examine is the one in which the cross elasticity of x_2 to p_1 is always zero. By the help of Fig. 3, we can describe this case by showing that, as far as the amount of money income M $(=p_1x_1+p_2x_2)$ is given, x_2 is kept at x_2^0 in spite of the change of p_1 from p_1' to p_1''. Thus, the amount of x_2 demanded solely depends on

$$\frac{M}{p_2} = \frac{p_1}{p_2}x_1 + x_2 = \frac{u_1}{u_2}x_1 + x_2.$$

Therefore we have

$$\frac{u_1}{u_2}=\frac{1}{x_1}\phi(x_2). \qquad (4.1)$$

Now we can prove the following

TEHOREM 3: *For x_2 to be always independent of x_1 (namely, for the cross elasticity of x_2 to p_1 to be always zero) in the utility function $u=u(x_1, x_2)$, it is necessary and sufficient that the utility function is described by*

$$u=u\left(\frac{1}{x_1}f(x_2)\right),$$

where $-\frac{f'}{f}=\phi(x_2)$.

Proof of Sufficiency.

Let $u(x_1, x_2)$ be $u\left(\frac{1}{x_1}f(x_2)\right)$. Apparently $u_1=\frac{u'}{x_1^2}\{-f(x_2)\}$ and $u_2=\frac{u'}{x_1^2}\{f'(x_2)\cdot x_1\}$. Then it follows

$$\frac{u_1}{u_2}=\left\{\frac{-f(x_2)}{f'(x_2)}\right\}\frac{1}{x_1}\equiv\phi(x_2)\frac{1}{x_1}.$$

Proof of Necessity.

Let us suppose that $u=A_0$ (a given constant). Thus we obtain

$$u_1dx_1+u_2dx_2=0.$$

Dividing both sides by u_2 and using (4. 1), we get

$$\frac{dx_1}{x_1}+\frac{1}{\phi(x_2)}dx_2=0.$$

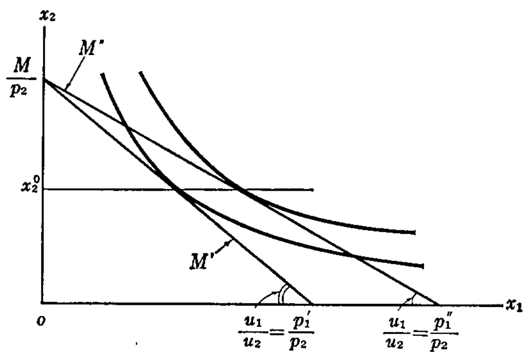
The integration of this last equation yields

$$\log f(x_2)=\log x_1+\log C_0,$$

where C_0 is the integral constant associated with A_0 and

$$f(x_2)=e^{-\int\frac{1}{\phi(x_2)}dx_2}.$$

FIG. 3



As the general solution, thus, we reach the final result desired, namely

$$u=u\left(\frac{1}{x_1}f(x_2)\right). \qquad \text{Q.E.D.}$$

§. 5

So far, we have made the simplified assumption that there exist only two kinds of commodity. Although we have made some effort to extend our analysis to three commodities case and have got some results about that case, we shall retain to publish such results until the more general case is analysed.