A RECONSIDERATION ON THE CONCEPT OF UTILITY

By MASAO HISATAKE

Professor of Mathematical Economics and Business Mathematics

I. *Introduction*

The utility theory has been developed to explain the fundamental principles of price theory and welfare economics. Yet it cannot be said that the concept of utility itself has been clearly defined. Although the measurability of utility has been often questioned for more than 60 years, we have not succeeded in settling the problem.

Irving Fisher is the first to attempt in defining utility with logical exactitude. He defined utility as a quantity to express the order of choice of commodities. He proved that if the utility of a commodity is independent of some other commodity, the marginal utility of the former can be measured with the increment of the latter. Usually, however, utility being the function of many commodities, he gave a more general definition of utility that it expressed only the order of choice, not necessarily the measure of desirability. Pareto, Frisch, Marshall and Hicks followed Fisher and they enlarged and refined his idea. These authors, however, greatly differed in their effort to go beyond Fisher. Pareto and Hicks adopted his general definition and gave up the intention to measure utility. Hicks in particular tried to reform, in the light of ordinal utility, the traditional ideas such as related commodities or consumer's surplus. The adoption by Marshall and Frisch of his first definition led to the attempt to define cardinal utility. Frisch went so far as to measure utility using statistical data.

Recently, Neumann and Morgenstern developed a new method to measure utility. The aim of these writers was to apply it to the game theory. Their method has been highly appreciated by some mathematical economists and several papers have been published following their method.

The purpose of this paper is to reexamine the definitions of cardinal utility by Fisher, Marshall and Neumann-Morgenstern and to prove that the only successful definition hitherto was the one given by Marshall.

II. *Attachment to Cardinal Utility*

Hicks succeeded in reforming the price theory without using the cardinal utility. Yet there is a strong inclination to establish the concept of utility as a measurable quantity. We can enumerate several reasons for this.

First of all, the popular theorem of decreasing marginal utility can not exist without cardinal utility. As we will describe later, the concept of quantitative utility depends upon the comparison or choice between increments of commodities belonging to different positions of commodity space. The above theorem compares the successive increments which belong to the different points, so that we may say the theorem depends upon the measurement of utility. Hicks proposed to use the theorem of decreasing marginal rate of substitution in place of the former theorem. No one can deny the superiority of his theory from the logical standpoint, but his theory has such a big structure as to make us feel hard to connect his theory to our direct experience through the narrow channel of the above theorem.

Next we have the application of cardinal utility in the definitions of related goods, complimentary, alternative and independent. Of course we have Hicks' definitions without using cardinal utility. His definitions have their own merits which are shown in his successful applications in "Value and Capital". Yet they are not without weakness. For his definitions depend upon the effect of demand for the other commodity caused by the price change of one commodity, they can not be used as basic explanations of the relative price movement of related goods. He seems to try to hang a boot on its strap.

Another use of cardinal utility is the definition of consumers' surplus, which has been the fundamental tool in the welfare economics. Consumer's surplus is the integration of the differences between marginal utilities and marginal costs. Therefore, it cannot be defined without quantitative utility. Hicks seem to have succeeded in defining consumers' surplus without using cardinal utility. He used indifference curves between money and goods, and defined consumers' surplus as the difference between the amount of money on the indifference curve and that of money actually paid. Because the former amount means the maximum amount of money which a consumer will willingly pay for the acquisition of a commodity, we can see that there is little difference between his difinition and that of Marshall who defined utility in cardinal terms.

Another use of cardinal utility is in the definition of marginal utility of money (or of income) which should have important applications in the welfare economics. For instance we can compare the living standard using the flexibility of marginal utility of money. Frisch tried to fix the justifiable rate of income tax or to derive the supply function of labor by using the same tool.

These are the main reasons among many others, why many economists still cling to the quantitative utility.

III. *The Assumptions of Fisher, Frisch and Marshall*

Fisher describes that the sense in which utility is a quantity is determined by the following definitions.[1]

(1) For a given individual at a given time, the utility of A units of one commodity (a) is equal to the utility of B units of another (b), if the individual has no desire for the one to the exclusion of the other.

(2) For a given individual, at a given time, the utility of A units of (a) exceeds the utility of B units of (b) if the individual prefers A to the exclusion of B rather than for B to the exclusion of A. In the same case the utility of B is said to be less than that of A.

(3) The utility of any one commodity depends on the quantity of that commodity, but is independent of the quantities of other commodities.

From these definitions or assumptions he proves that the ratio of two infinitesimal utilities is measured by the ratio of two infinitesimal increments of the same commodity respectively equal in utility to the two utilities whose ratio is required, provided that these increments are on the margin of equal finite quantities.

As to his definitions (1) and (2) we can not find any objections. As to his definition (3), however, we might ask how we can confirm that commodity (a) is independent of (b) without appealing to measurable utility.

Suppose we have x of (a) and y of (b). Let the utility of a unit increment of x be denoted as follows:

$$u=u_x(x, y)$$

If we change y to y' and still we have the same utility of the above increment

$$u'=u_x(x, y')$$

then we can say (a) is independent of (b). But how can we know that u is equal to u' ? We could not know by act of choice, because (x, y) and (x, y') express the different points of the space of (a) and (b) and we are unable to make choice at the same time. Thus the definition (3) must be considered as an assumption because it can not generally confirmed by direct experience.

Frisch has enlarged the definition of Fisher[2]. He clarified the necessary axioms for Fisher's definition (3), but he expressed no doubt for their validity. His assumptions were grouped into two categories of axioms. The first group of axioms was developed as follows:

I. Axioms relative to a given position.

(a) Axiom of choice.

When an individual finds himself on a point $\mathbf{x}$ and is asked to choose between two displacements $\mathbf{p}$ and $\mathbf{q}$, we assume that his choice is always determined and

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[1] I. Fisher: Mathematical Investigations into the Theory of Value and Prices. 1892. p. 12

[2] R. Frisch: Sur un problème d'économie pure. 1926. pp. 3–5.

belongs to one of the three following cases.

(1) He prefers **p** to **q**.

(2) He prefers **q** to **p**.

(3) The choice between **p** and **q** is not related.

For the sake of simplicity we use the following notations to express these three cases.

$$(\mathbf{xp}) \gtreqqless (\mathbf{xq})$$

This axiom I is just equivalent to Fisher's definitions (1) and (2).

Frisch pointed out other axioms which were not explicitly expressed but implicitly used in Fisher's argument.

(b) Axiom of coordination

If $(\mathbf{xp}) > (\mathbf{xq})$

and $(\mathbf{xq}) > (\mathbf{xr})$

The choice will be

$$(\mathbf{xp}) > (\mathbf{xr})$$

And the same is true for the sign$=$, $<$ and non $>$.

(c) Axiom of addition

If $(\mathbf{xp}) > (\mathbf{xq})$

and $(\mathbf{xr}) > (\mathbf{xs})$

Then we have

$$(\mathbf{x}, \mathbf{p}+\mathbf{r}) > (\mathbf{x}, \mathbf{q}+\mathbf{s})$$

Now we have axiom II.

II. Axioms relative to different positions.

(a) Axiom of choice

When an individual finds himself at two different occasions in the positions **x** and **y** respectively, and has to choose between a displacement **p** in the position **x** and a displacement **q** in the position **y**, we assume that his choice is always well determined. That is, we have always one of the following three cases.

$$(\mathbf{xp}) \gtreqqless (\mathbf{yq})$$

(b) Axiom of coordination

If $(\mathbf{xp}) > (\mathbf{yq})$

and $(\mathbf{yq}) > (\mathbf{zr})$

We have $(\mathbf{xp}) > (\mathbf{zr})$

and the same for the other signs.

(c) Axiom of addition

If $(\mathbf{xp}) > (\mathbf{yq})$

and $(\mathbf{xr}) > (\mathbf{ys})$

we have

$$(\mathbf{x}, \mathbf{p}+\mathbf{r}) > (\mathbf{y}, \mathbf{q}+\mathbf{s})$$

and the same for the other signs.

We have shown that Fisher's definition of independent utility depends upon the choice between the increments belonging to the defferent points of commodity space. Therefore we can say that the axiom II of Frisch is an enlargement or a refinement of Fisher's definition (3).

Frisch has developed a method of measuring marginal utility in the same paper. He used sugar as an independent commodity which meant that he applied his axiom II in a special form.[3]

We have another method of measuring marginal utility, i.e. the one proposed by Marshall.[4] His method is to measure marginal utility in terms of the price which an individual is just willing to pay for any one unit of a commodity rather than go without that unit altogether. In other words, the utility of increment of a commodity is measured by increment of money just equivalent to it.

This definition is very well adapted to our common sense understanding of utility and has been widely accepted by many economists except for few specialists who are interested in basic theory. Those criticisms were directed towards the logical vagueness or inconsistency of his definition.

Marshall himself admitted that in his definition the basic assumption is that the marginal utility of money itself must be constant. In this respect his definition was criticized by some authors such as Pareto and Hicks. I am of the opinion that his assumption may be considered as another expression of independency of money utility. Compared with that of Fisher and Frisch, however, his definition has somewhat different aspect.

I wish to examine Marshall's definition further in detail in order to find that whether we can accept those criticisms or not.

IV. *An Examination of the Marshallian Definition*

If the marginal utility of money is independent of the quantity of the commodity the utility of which is to be measured, we can accept Marshall's definition as a special case of Fisher's or Frisch's definition. The question, therefore, is to be raised as to the independency of money utility. The criticism by Pareto or Hicks did not concern itself with this point directly, but with the question of constancy of marginal utility of money. Certainly this is another problem if we succeed to establish the independency of money utility as we shall discuss later. But if this is not the case, the two problems are interrelated to each other and sometimes confused in discussion.

Now according to Pareto[5], let the quantities of commodities to be purchased by an individual be

$$x, y, z, \cdots\cdots\cdots\cdots$$

x being the quantity of hoarded money. Let the prices of the commodities be

$$1, p_y, p_z, \cdots\cdots\cdots\cdots$$

By the condition of maximum index of utility, we have

[3] R. Frisch, ibid. p. 24, p. 30

[4] A. Marshall, Principles of Economics. 8th ed. 1938. pp. 93–94

[5] V. Pareto, Manuel d'économie politique. trad. par A. Bonnet. 2me ed. 1927. pp. 579–589

$$\varphi_x=\frac{1}{p_y}\varphi_y=\frac{1}{p_z}\varphi_z=\cdots\cdots\cdots\cdots \tag{1}$$

in which φ is an index of utility which is an arbitrary function subject to a certain condition. φ_x is a derivative of φ in relation to x, φ_y, φ_z being defined similarly. Let the marginal utility index of money be denoted with m. Then we have

$$\varphi_x=m, \qquad \varphi_y=p_y m, \qquad \varphi_z=p_z m, \cdots\cdots \tag{2}$$

When we express the quantities of goods possessed by him before exchange with x_0, y_0, $z_0, \cdots\cdots$, and the quantities after exchange with x, y, $z, \cdots\cdots$, we always have

$$x-x_0+p_y(y-y_0)+p_z(z-z_0)+\cdots\cdots=0 \tag{3}$$

From (2) and (3) we can deduce

$$\frac{\partial m}{\partial p_y}=-\frac{(y-y_0)\mathrm{R}+m\mathrm{M}_{2,1}}{\mathrm{M}} \tag{4}$$

where $$\mathrm{M}=-\begin{vmatrix} 0 & 1 & p_y & p_z\cdots\cdots \\ 1 & \varphi_{xx} & \varphi_{xy} & \varphi_{xz}\cdots\cdots \\ p_y & \varphi_{yx} & \varphi_{yy} & \varphi_{yz}\cdots\cdots \\ p_z & \varphi_{zx} & \varphi_{zy} & \varphi_{zz}\cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{vmatrix}$$

and $\mathrm{M}_{2,1}$ is a confactor of M.

In the simplest case where every commodity is independent of each other, we have

$$\varphi_{xy}=0, \qquad \varphi_{xz}=0, \cdots\cdots$$

In this case the equation (4) can be reduced as follows:

$$\frac{\partial m}{\partial p_y}=-\frac{y-y_0+\dfrac{\varphi_y}{\varphi_{yy}}}{\mathrm{T}} \tag{5}$$

where $$\mathrm{T}=\frac{1}{\varphi_{xx}}+\frac{p_y^2}{\varphi_{yy}}+\frac{p_z^2}{\varphi_{zz}}+\cdots\cdots$$

Even in this case we cannot say that the marginal utility index of money is constant.

Pareto pointed out that according to Marshall $\dfrac{\partial m}{\partial p_y}$ should be zero. This was the proof for his assertion that Marshall's definition could not logically be held.

In my opinion, the presumption of Pareto's criticism is that marginal utility should be regarded as a marginal rate of substitution of a commodity with money, holding money income constant, or to express it more correctly, along the income and expenditure equation (3). In equations (2), the right hands denote the marginal rate of substitution of commodity with money.

This is of course one way of defining marginal utility along the Marshallian line. But we can also define marginal utility as a marginal rate of substitution of a commodity for money, holding the total utility constant. This definition is essentially the same as that given by Hicks as his interpretation of the Marshallian concept of utility. He argues, however, that this definition will not hold if the marginal utility of money varies with the quantity purchased of the commodity.

According to Hicks,[6] Marshall's assupmtion on the constancy of marginal utility of money is a simplification in which he neglects the income effect caused by the change of price. This simplification is valid when income effect is small, for instance, when the commodity concerned is a relatively unimportant part of the consumer's budget. In this case we may only consider the substitution effect, i.e. the variation along the same indifference curve. This is the reason for the statement above that Hick's interprelation of the Marshallian utility depends upon the constancy of the total utility. Hicks, however, affirms that income effect can not be neglected and so Marshallian utility should not be used in the general case.

Pareto and Hicks have enough reasons to blame Marshall if their interpretation of the Marshallian utility should be right. I believe, however, there is another way of defining marginal utility for measurement purpose. When we compare an increment of a commodity or a group of commodities with an increment of money, we need not necessarily regard them as substitutes. We need not buy a commodity to evaluate it. We need only imagine a situation where we have an increase in our consumption or stock of commodities, and a situation where we have a greater amount of monetary income. Then we compare the two situations and make a choice between them. This is a natural behavior, and the result of such a choice can be expressed objectively. This, I think, is the method which Marshall had in mind when he defined marginal utility in terms of money.

It may be said that this method is nothing but a kind of substitution. But there is a fundamental difference between this kind of substitution and the kind in the cases of Pareto and Hicks described above. In those two cases the two increments compared can be summed (ignoring sign), and, if they are summed, the total money income or total utility, originally considered as being constant, will be some what changed. In this third case, however, the sum of the utilities of the commodity and money increments is meaningless because they belong to different positions of commodity space which cannot be realized simultaneously. When we consider the increase of money income we hold the prices of commodities constant, while when we consider the commodity increment we evaluate it in terms of a price which may vary. Suppose one is in an equilibrium position and consumes n commodities whose quantities are expressed as $(x_1, x_2, \cdots\cdots, x_n)$. If M represents his money income, then the utility of dx_1, an increment of x_1, can be measured by dM so that the following relation is satisfied:

$$F(x_1+dx_1, x_2, \cdots\cdots, x_n)=U(M+dM) \qquad (6)$$

Here F and U denote the indices of utility of commodities and money respectively. F and U are assumed to be continuous and differentiable. Now $U(M)$ can be determined through the following three equations.

$$M=p_1x_1+p_2x_2+\cdots\cdots+p_nx_n \qquad (7)$$

[6] J Hicks, Value and Capital. 1939. pp. 20-32

$$U=F(x_1, x_2, \cdots\cdots, x_n) \tag{8}$$

$$\frac{F_1}{p_1}=\frac{F_2}{p_2}=\cdots\cdots=\frac{F_n}{p_n} \tag{9}$$

These are well-known equilibrium equations and determine U as a function of M, equations (9) being essentially the same as the equations (2). This is generally recognized as a definition of money utility.

Now when we consider $U(M+dM)$ we regard p_1 as constant, while when we evaluate dx_1, we evaluate it as $p_1'=\frac{dM}{dx_1}$ which might be different from p_1. So we can say the utility index of dx_1 is independent of the utility index of dM. From this point of view, I wish to affirm that the Marshallian method of utility measurement belongs to the case, which Fisher described, in which utility could be measured with the increments of independent commodities.

It seems that we have succeeded in evaluating marginal utility in terms of money. But still we have to consider the question whether a unit of money expresses a unit of utility. We assumed the continuity and differentiability of the index function. This, I think, is justifiable when we consider relatively small increments of commodities. From this assumption it follows that the marginal utility index of money can be considered almost constant when dM is small. But it does not hold in those cases involving so large a degree of variation that the total utility can not be deduced by integrating marginal utility. Or from this proposition we can not say anything about the second derivative of total utility, so that the law of decreasing marginal utility should not be deduced from our definition.

We know from our experience that, in general, marginal utility in terms of money decreases when the quantity of a commodity purchased increases. Now we can see that this fact places some restrictions on the form of utility index function, Let one of the utility index function of commodities $(x_1, x_2, \cdots\cdots, x_n)$ for an individual be denoted with $\varphi(x_1, x_2, \cdots\cdots, x_n)$, then any index function of the same commodities can be expressed as

$$I=F\{\varphi(x_1, x_2, \cdots\cdots, x_n)\}.$$

F is an arbitrary continuous function subject to the following condition

$$F'>0.$$

The marginal utility index of commodity x_i is

$$I_i=F'\varphi_i(x_1, x_2, \cdots\cdots, x_n)$$

and the marginal utility index of money being

$$\mu=\frac{1}{p_i}F'\varphi_i(x_1, x_2, \cdots\cdots, x_n)$$

If we measure I_i with μ as a unit, we can easily see that

$$I_i=\mu p_i$$

which shows that marginal utility in terms of money is p_i as Marshall described.

Now when we evaluate one more unit of the commodity in Marshallian way, we measure $I_1'=F'\varphi_1(\bar{x}_1, \cdots\cdots, x_n)$ not by $\mu=\frac{1}{p_1}F'\varphi_1(x_1, x_2, \cdots\cdots, x_n)$ but by $\mu'=\frac{1}{p_1}F'\varphi_1(x_1', x_2', \cdots\cdots, x_n')$ where $\bar{x}_1$ denotes $x_1+\Delta x_1$, Δx_1 being a unit of x_1, and $(x_1', x_2', \cdots\cdots, x_n')$ denote the quantities purchased by $M+\Delta M(\Delta M=p_1\Delta x_1)$. The decreasing marginal utility in terms of money shows that

$$I_1<\mu' p_1.$$

This is quite natural because when we evaluate money utility we assume the most favorable distribution of expense among various commodities, while when we evaluate one special commodity, we suppose the increase of only one commodity, which is not necessarily the best combination for the expense. From the above consideration we can conclude that the function $F'\varphi_i(x_1\cdots\cdots x_i\cdots\cdots x_n)$ must be a decreasing function of x_i.

Now F is an arbitrary function of x_1 under the restriction $F'>0$, we can put $F'=\text{constant}>0$ as a possible case. In this case φ_i must be decreasing along the x_i ax. Any utility index φ should be subject to this condition. Though F is an arbitrary function, the arbitariness is restricted because it has to conserve the proposition that

$$\frac{\partial\{F'\varphi_i\}}{\partial x_i}<0.$$

The constancy of the marginal utility of money can not be maintained in our interpretation of the Marshallian utility, but we can easily see that the variation of μ is smaller than I_i, so that his definition can be applied to a small increment of commodity, but we should say that it must not be used in a case involving so large a degree of variation that the total utility can not be deduced by integrating margimal utility.

V. *An Examination of Neumann-Morgenstern's definition*

Neumann and Morgenstern proposed to measure utility on a hypothesis in which they assumed that we could compare and make choice between a group of events of which their respective probabilities were known and another event or a similar group of events.

Let us denote the utility index of an event with u. We can use another index by appliying monotonic transformation to u, which we are going to express as

$$\rho=V(u).$$

If we denote another utility which is different and alternative to u with v, the transformation V must satisfy the following relations of correspondence.

(i) If $u>v$, then $V(u)>V(v)$

(ii) When we combine the two utilities u, v and apply transformation V, the result is the same whether we apply tranformation before they are combined or after. In algebraic expression it is

$$V\{\alpha u+(1-\alpha)v\}=\alpha V(u)+(1-\alpha)V(v)$$

wehre α is the probability for the utility u.

From these two hypotheses they deduce the measurability of utility. Suppose we have two tranformation formulae V and V' which satisfy the above two conditions. We shall express them as

$$\rho=V(u) \qquad \rho'=V'(u) \tag{10}$$

Let the relation of ρ and ρ' be denoted as

$$\rho'=\phi(\rho) \tag{11}$$

and then examine the characteristics of the function ρ.

From the equations (10) (11) we can put

$$\phi(\rho)=V'(u)$$

We also have the similar equation for v.

$$\phi(\sigma)=V'(v)$$

in which σ denotes $V(v)$.

Now we can easily see that ϕ is also a monotonic transformation. Therefore the assumed conditions (i) (ii) must also hold for ϕ. We have then

(i)′ If $\rho>\sigma$, then $\phi(\rho)>\phi(\sigma)$

(ii)′ $\phi\{\alpha\rho+(1-\alpha)\sigma\}=\alpha\phi(\rho)+(1-\alpha)\phi(\sigma)$

The last equation shows that ϕ is a linear function, so that we can put

$$\phi(\rho)=w_0\rho+w_1$$

where w_0 and w_1 are constant. From this we can deduce that any utility index function is subject to linear transformation with each other. This means that utility can be measured when we fix the unit of utility and the zero point.

The above reasoning following Neumann and Morgenstern appears perfect. But if we examine it in detail, we find that another basic assumption is made tacitly. That is the proposition that a utility of combined events is equal to the mathematical expectation of the separate utilities of the events. This we can find in the hypothesis (ii). For if we should consider the mathematical expectation $\alpha u+(1-\alpha)v$ as the utility for the combined events, we might be able to apply the transformation V to it and put it equal to the mathematical expectation of the separate transformed utilities. But if the mathematical expectation does not express the utility of the events, we have no reason to prove the validity of the transformation. In this case we need another assumption which implies the linearity of the transformation V. Hence, their method of utility measurement fails to sustain its own perfection. Of course, in some cases, their method might be used as an approximation which is also unavoidable in Marshall's method.