MAXIMIZING PRINCIPLE AND ONE METHOD OF CONSISTENT DETERMINATION OF INPUT COEFFICIENTS OF LEONTIEF SYSTEM

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I. *Introduction*

Leontief system of input-output analysis deserves to be studied as one of the most interesting problems of economic theory and econometrics in the sense that it deals with the statistical application of the Walrasion economic equilibrium theory, although its theoretical scheme is rather simple. Quite recently in Japan the studies in this line have been much in vogue among younger scholars and the government.[1]

The present writer tries to find out a theoretical ground, both for the substitutional relationships between the factors of production the problems of which were discussed by N. Georgescu-Roegen,[2] P. A. Samuelson[3] and others, and for the consistent numerical determination of input coefficients with the production function itself. At the same time he introduces the maximization principle in this system,[4] which could attach some meaning to the idea that we make use of this system as that of planned economy or that of economic prediction.

II. *A Modified Leontief System*

The original form of Leontief system,

$$x_{1i}+x_{2i}+\cdots-X_i+\cdots+x_{ni}=0,\ (i=1, 2, \cdots, n), \tag{1}$$

where X_i is the output of the ith sector and x_{ji} the input of the jth sector distributed from the ith sector, has no term of x_{ii}, and takes into considera-

[1] Both the Economic Council Board and the Department of Commerce and Industry have been made Leontief tables, the former commencing the work at the end of 1952 and the latter half a year later than the former.

[2] N. Georgescu-Roegen, Leontief's System in the Light of Recent Results, *The Review of Economics and Statistics*, Vol. 32, 1950, pp. 214—222.

[3] P. A. Samuelson, Abstract of a System concerning Substitutability in Open Leontief Models, *Activity Analysis of Production and Allocation*, New York, 1951, pp. 142—146.

[4] So far, Samuelson and Klein attempted to introduce maximizing principle into the Leontief system. (P. A. Samuelson, *ibid.* Lawrence R. Klein, On the Interpretation of Professor Leontief's System, *The Review of Economic Studies*, Vol. 20, 1953, pp. 131—136).

tion the net output X_i which is the difference between the gross output Y_i and the diagonal element x_{ii}. Now we introduce x_{ii} into the system, and furthermore relax the condition that the output is always equal to the input in each row. Let S_i' be the balance of the ith sector carried from the previous period and S_i'' the balance of the ith sector carried to the next period. Then we get

$$-x_{1i}-x_{2i}-\cdots+(Y_i+S_i'-x_{ii})-\cdots-x_{ni}=S_i'', \ (i=1, 2, \cdots, n). \qquad (2)$$

Now we put $S_i''-S_i'\equiv S_i$, then we can write (2) as

$$-x_{1i}-x_{2i}-\cdots+(Y_i-x_{ii})-\cdots-x_{ni}=S_i, \ (i=1, 2, \cdots, n). \qquad (3)$$

Owing to the modification of (3), the original form of the Leontief system,

$$P_1x_{i1}+P_2x_{i2}+\cdots-\frac{P_iX_i}{B_i\beta}+\cdots+P_nx_{in}=0, \ (i=1, 2, \cdots, n), \qquad (4)$$

gets revision as follows:

$$-P_1x_{i1}-P_2x_{i2}-\cdots+P_i(Y_i-S_i-x_{ii})-\cdots-P_nx_{in}=\pi_i, \ (i=1,2,...,n), \qquad (5)$$

where P_i is the price of the output of the ith sector and π_i the economic surplus of the ith sector. The economic surplus of the business sector is, needless to say, equivalent to the profit of that sector, while that of the household sector, the nth sector, is the difference between income and expenditure. Entrepreneur doubtlessly tries to maximize his economic surplus, while consumer cannot be thought to do so, because the latter tries to maximize utility as the theory of consumption shows. Nevertheless, so far as we treat the household sector as an endogenous one,. we assume consumer tries to find out the maximum surplus as the first approximation. Moreover, it is rather reasonable to assume so in the statistical construction of the Leontief system because we cannot objectively calculate utility at the present stage of the consumer's theory.

Next we consider the production function. Instead of the limitational function, we introduce Cobb-Douglas type of production function,

$$Y_i=A_ix_{i1}^{b_{i1}}x_{i2}^{b_{i2}}\cdots x_{in}^{b_{in}}, \ (i=1, 2, \cdots, n), \qquad (6)$$

as the substitutional one. In (6), A_i and b_{ij} are parameters of the above production function respectively.

The input coefficient b_{ij} is given as follows:

$$b_{ij}=\frac{x_{ij}}{Y_i}\frac{\partial Y_i}{\partial x_{ij}}, \ (i=1, 2, \cdots, n; \ j=1, 2, \cdots, n). \qquad (7)$$

Now we maximize π_i in (5) in respect of x_{ij}. The result is,

$$\frac{\partial Y_i}{\partial x_{ij}}=\frac{P_j}{P_i}, \ (i=1, 2, \cdots, n; \ j=1, 2, \cdots, n). \qquad (8)$$

In this case we treat P_i and S_i as parameters. Combining (7) and (8), we get[5]

[5] The present writer attempted to introduce maximizing principle independently of Prof's Samuelson's and Klein's. The result is, however, just the same as Klein's in respect of (9). (cf. Isamu Yamada, Leontief System and Production Function, *The Economic Review*, Vol. 3, No. 3, July 1952, written in Japanese. L. R. Klein, *A Textbook of Econometrics*, Evanston, 1953, pp. 201—211.)

$$b_{ij}=\frac{P_j x_{ij}}{P_i Y_i},\ (i=1,2,\cdots,n;\ j=1,2,\cdots,n). \tag{9}$$

The above set-up of equations plays an important rôle in the numerical determination of the input coefficients, as we shall show later.

Some words must be added to (9) in this connection. The denominator $P_i Y_i$ and the numerator $P_j x_{ij}$ are respectively the actual numbers appeared in the Leontief table, and it is more worthwhile to mention that such input coefficients are compatible with those of the production function (6).

We have now completely constructed our modified Leontief system. The number of unknowns are n X_i's$(i=1,2,\cdots,n)$ and n^2 x_{ij}'s$(i,j=1,2,\cdots,n)$, while the number of equations are n in (3) and n^2 in (9). As we explained above, P_i, S_i and b_{ij} are all given as parameters.

III. *Aggregation of Sectors and Maximization of Surplus*

What relation exists between the maximization of surpluses of separate sectors and that of aggregated sector? This problem itself belongs in general to the "Problem of Aggregation," but in this section we are to discuss it in a specific way suitable for the Leontief system.

Let the sectors to be aggregated be those from the mth to the sth, and let this aggregated sector call the rth. Now we divide this aggregation problem into two.

A. The rth sector.

The production functions from the mth to the sth sector are as follows:

$$\begin{aligned} &Y_m=A_m x_{m1}^{b_{m1}}\cdots x_{mm}^{b_{mm}}\cdots x_{ms}^{b_{ms}}\cdots x_{mn}^{b_{mn}},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &Y_s=A_s x_{s1}^{b_{s1}}\cdots x_{sm}^{b_{sm}}\cdots x_{ss}^{b_{ss}}\cdots x_{sn}^{b_{sn}}. \end{aligned} \tag{10}$$

Let the production function of the rth sector, the aggregated one, be

$$Y_r=A_r x_{r1}^{b_{r1}}\cdots x_{rr}^{b_{rr}}\cdots x_{rn}^{b_{rn}}. \tag{11}$$

Now we assume the following relations:

$$Y_r=\left[Y_m^{P_mY_m}\cdots Y_s^{P_sY_s}\right]^{\alpha}, \tag{12}$$

$$x_{ri}^{b_{ri}}=\left[R_i\left(x_{mi}^{b_{mi}}\right)^{P_mY_m}\cdots\left(x_{si}^{b_{si}}\right)^{P_sY_s}\right]^{\alpha},\ (i=1,\cdots,m-1,s+1,\cdots,n), \tag{13}$$

$$x_{rr}^{b_{rr}}=\left[R_r\left(x_{mm}^{b_{mm}}\cdots x_{ms}^{b_{ms}}\right)^{P_mY_m}\cdots\left(x_{sm}^{b_{sm}}\cdots x_{ss}^{b_{ss}}\right)^{P_sY_s}\right]^{\alpha}, \tag{14}$$

where R_i is a parameter and $\alpha=1/(P_mY_m+\cdots+P_sY_s)$. We further assume

$$\begin{aligned} &x_{mi}=r_{m+1,i}x_{m+1,i}=r_{si}x_{si},\\ &x_{mm}=r_{m,m+1}x_{m,m+1}=\cdots=r_{ms}x_{ms}=r_{sm}x_{sm}=\cdots=r_{ss}x_{ss}, \end{aligned} \tag{15}$$

then we get

$$b_{ri}=\frac{b_{mi}P_mY_m+\cdots+b_{si}P_sY_s}{P_mY_m+\cdots+P_sY_s},\ (i=1,\cdots,m-1,s+1,\cdots,n),$$

$$b_{rr}=\frac{b'P_mY_m+\cdots+b''P_sY_s}{P_mY_m+\cdots+P_sY_s}, \tag{16}$$

where $b'=b_{mm}+\cdots+b_{ms}$; $b''=b_{sm}+\cdots+b_{ss}$.

The maximizing condition of the surpluses of the separate sectors is given by (9), then we put this condition into b in the right-hand side of (16),

$$b_{ri}=\frac{P_ix_{mi}+\cdots+P_ix_{si}}{P_mY_m+\cdots+P_sY_s},$$

$$b_{rr}=\frac{P_mx_{mm}+\cdots+P_sx_{ms}+\cdots+P_mx_{sm}+\cdots+P_sx_{ss}}{P_mY_m+\cdots+P_sY_s}, \tag{17}$$

are obtained. The above equations (17) are nothing but the maximizing conditions of the surplus of the aggregated sector. Accordingly, the maximizing condition of the separate sectors are compatible with those of the aggregated sector, so far as we admit the assumptions from (12) to (15).

B. The other sectors except the rth.

The production functions of the other sectors before aggregation are as follows:

$$Y_i=A_ix_{i1}^{b_{i1}}\cdots x_{im}^{b_{im}}\cdots x_{is}^{b_{is}}\cdots x_{in}^{b_{in}},\ (i=1,\cdots,m-1,s+1,\cdots,n). \tag{18}$$

After aggregation, however, they take the forms of

$$Y_i=A_ix_{i1}^{b_{i1}}\cdots x_{ir}^{b_{ir}}\cdots x_{in}^{b_{in}}. \tag{19}$$

In this case we also assume (13), and adding to it, we further assume

$$x_{im}=r_{i,m+1}x_{i,m+1}=\cdots=r_{is}x_{is}, \tag{20}$$

then we get

$$b_{ir}=\frac{(b_{im}+\cdots+b_{is})P_iY_i}{P_iY_i}=b_{im}+\cdots+b_{is}, \tag{21}$$

Making use of (9), we finally get

$$b_{ir}=\frac{P_mx_{im}+\cdots+P_sx_{is}}{P_iY_i}. \tag{22}$$

The above equation also shows the maximizing conditions of the surpluses of the other sectors after aggregation.

IV. *Fitting of Statistical Data*

—Leontief's Method—

We take, for example, three sectors, business, households and the others in the Leontief system. Symbolically, we get the following table. In this table, $\underset{)1(}{\Sigma}P_jx_{ij}$ means the sum over all items except $j=1$. Accordingly, $\underset{)1(}{\Sigma}P_jx_{1j}=P_2x_{12}+P_3x_{13}$. $\underset{)2(}{\Sigma}P_jx_{2j}$ and $\underset{)3(}{\Sigma}P_3x_{3j}$ are the sums over all items except $j=2$ and $j=3$ respectively. Using these symbols, Leontief's method of determining the numerical values of the input coefficients is as follows:

Table 1

i / j	Business	Households	Others	Total
Business		P_1x_{21}	P_1x_{31}	P_1X_1
Households	P_2x_{12}		P_2x_{32}	P_2X_2
Others	P_3x_{13}	P_3x_{23}		P_3X_3
Total	$\underset{(1)}{\Sigma}P_jx_{1j}$	$\underset{(2)}{\Sigma}P_jx_{2j}$	$\underset{(3)}{\Sigma}P_jx_{3j}$	

$$a_{ij}=\frac{P_jx_{ij}}{P_iX_i}B_i=\frac{P_jx_{ij}}{P_iX_i}\,\frac{P_iX_i}{\underset{(i)}{\Sigma}P_jx_{ij}}=\frac{P_jx_{ij}}{\underset{(i)}{\Sigma}P_jx_{ij}}, \tag{23}$$

The determinant of solving X_i in the original Leontief system is

$$\begin{vmatrix} -A_1B_1\beta & a_{21} & a_{31} \\ a_{12} & -A_2B_2\beta & a_{32} \\ a_{13} & a_{23} & -A_3B_3\beta \end{vmatrix}=D(\beta)=0. \tag{24}$$

Putting $A_i=1$ and $\beta=1$ in the above determinant,

$$\begin{vmatrix} -B_1 & a_{21} & a_{31} \\ a_{12} & -B_2 & a_{32} \\ a_{13} & a_{23} & -B_3 \end{vmatrix}=0, \tag{25}$$

is obtained. Now we substitute (23) into (25), we get

$$\begin{vmatrix} -\dfrac{P_1X_1}{\underset{(1)}{\Sigma}P_jx_{1j}} & \dfrac{P_1x_{21}}{\underset{(2)}{\Sigma}P_jx_{2j}} & \dfrac{P_1x_{31}}{\underset{(3)}{\Sigma}P_jx_{3j}} \\ \dfrac{P_2x_{12}}{\underset{(1)}{\Sigma}P_jx_{1j}} & -\dfrac{P_2X_2}{\underset{(2)}{\Sigma}P_jx_{2j}} & \dfrac{P_2x_{32}}{\underset{(3)}{\Sigma}P_jx_{3j}} \\ \dfrac{P_3x_{13}}{\underset{(1)}{\Sigma}P_jx_{1j}} & \dfrac{P_3x_{23}}{\underset{(2)}{\Sigma}P_jx_{2j}} & -\dfrac{P_3X_3}{\underset{(3)}{\Sigma}P_jx_{3j}} \end{vmatrix}=\frac{P_1P_2P_3}{\alpha\beta\gamma}\begin{vmatrix} -X_1 & x_{21} & x_{31} \\ x_{12} & -X_2 & x_{32} \\ x_{13} & x_{23} & -X_3 \end{vmatrix}=0, \tag{26}$$

where $\underset{(1)}{\Sigma}P_jx_{1j}=\alpha$, $\underset{(2)}{\Sigma}P_jx_{2j}=\beta$ and $\underset{(3)}{\Sigma}P_jx_{3j}=\gamma$. The original form of the production function of Leontief is, however, as follows:

$$a_{ij}=\frac{x_{ij}B_i}{X_i}, \qquad \text{assuming } A_i=1 \text{ and } \beta=1.$$

It will be clear that the above contradicts (23) *in principle.*[6]

V. *Determination of Input Coefficients by our Modified System*

We can symbolically arrange our modified system in section 2 in the following way.

Table 2

	Business	Households	Others	Total
Business	P_1x_{11}	P_1x_{21}	P_1x_{31}	$P_1(Y_1-S_1)$
Households	P_2x_{12}	P_2x_{22}	P_2x_{32}	$P_2(Y_2-S_2)$
Others	P_3x_{13}	P_3x_{23}	P_3x_{33}	$P_3(Y_3-S_3)$
Total	ΣP_ix_{1i}	ΣP_ix_{2i}	ΣP_ix_{3i}	

It is in general difficult in the Leontief table where all items are written in money value that we separate price or quantity from this money value. Needless to say, we can get price index or quantity index from the ordinary statistical data, but these indices are not suitable for the analyses, because the Leontief table deals with the absolute level of money value, instead of the relative one.

As we have pointed out, the equation

$$b_{ij}=\frac{P_jx_{ij}}{P_iY_i},\ (i=1, 2, \cdots, n;\ j=1, 2, \cdots, n), \tag{9}$$

shows that the input coefficients are expressed in money value, although they are to be the physical ratios by (7). This fact must be mentioned when we numerically calculate them from the Leontief table.

Adding to this, we supplement one annotation. Substituting (9) into (3), we derive

[6] Respecting the analysis of the 1939 American economy, Professor Leontief changed his systems as follows:

$$-x_{1i}-x_{2i}-\cdots+X_i-\cdots-x_{mi}=x_{ni},$$
$$x_{ij}=a_{ij}X_i,$$
$$(i=1, 2, \cdots, m;\ j=1, 2, \cdots, m),$$

while as to the analyses of 1919 and 1929 American economy he used such systems as:

$$x_{1i}+x_{2i}+\cdots-X_i+\cdots+x_{ni}=0,$$
$$x_{ij}=\frac{a_{ij}X_i}{A_iB_i\beta},\ (i=1, 2, \cdots, n;\ j=1, 2, \cdots, n),$$

(cf. Wassily W. Leontief, The Structure of American Economy 2nd ed., 1951). Although these two systems are slightly different, the methods of estimating the input coefficients are just the same.

$$-b_{1i}Y_1P_1-b_{2i}Y_2P_2-\cdots+(1-b_{ii})Y_iP_i-\cdots-b_{ni}Y_nP_n=S_iP_i,\quad (i=1, 2, \cdots, n). \tag{27}$$

From this set-up of equations, we can solve Y_i, so far as we treat P_i as a parameter. The latter cannot, however, get any meaning if we can determine its value previously from the other statistical data than the information of the Leontief table, as we showed earlier. Nevertheless, if we solve Y_iP_i instead of Y_i, we can get out of such a difficulty, and we do not meet any troubles when we construct a new Leontief table after equilibrium.

Some words must be further added to S_i. As we have defined in section 2, S_i consists of the balance carried from the previous period, S_i', and the balance carried to the next period, S_i''. Among these two balances, it is appropriate to regard S_i' as a parameter, while it seems to be reasonable to take S_i'' as an endogenous sector. Then we can rewrite (27) as

$$-b_{1i}Y_1P_1-b_{2i}Y_2P_2-\cdots+(1-b_{ii})Y_iP_i-\cdots-b_{ni}Y_nP_n=-S_i'P_i,\quad (i=1, 2, \cdots, n). \tag{28}$$

If we put $S_i'=0$ in the above equation, it becomes a simultaneous linear homogeneous equation. In this case, the following condition must be held.

$$\begin{vmatrix} 1-b_{11} & -b_{21} & \cdots & -b_{n1} \\ -b_{12} & 1-b_{22} & \cdots & -b_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ -b_{1n} & -b_{2n} & \cdots & 1-b_{nn} \end{vmatrix} = \Delta = 0. \tag{29}$$

It is very easy to prove (29). Putting b_{ij} in (9) into (29),

$$\begin{vmatrix} 1-\dfrac{P_1x_{11}}{P_1Y_1} & -\dfrac{P_1x_{21}}{P_2Y_2} & \cdots & -\dfrac{P_1x_{n1}}{P_nY_n} \\ -\dfrac{P_2x_{12}}{P_1Y_1} & 1-\dfrac{P_2x_{22}}{P_2Y_2} & \cdots & -\dfrac{P_2x_{n2}}{P_nY_n} \\ \cdots & \cdots & \cdots & \cdots \\ -\dfrac{P_nx_{1n}}{P_1Y_1} & -\dfrac{P_nx_{2n}}{P_2Y_2} & \cdots & 1-\dfrac{P_nx_{nn}}{P_nY_n} \end{vmatrix}$$

$$=\frac{1}{Y_1\cdots Y_n}\begin{vmatrix} Y_1-x_{11}-\cdots-x_{n1} & -x_{21} & \cdots & -x_{n1} \\ -x_{12}+Y_2-\cdots-x_{n2} & Y_2-x_{22} & \cdots & -x_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ -x_{1n}-\cdots+Y_n-x_{nn} & -x_{2n} & \cdots & Y_n-x_{nn} \end{vmatrix}=0.$$

Accordingly, (29) is always valid in our modified system. Putting $Y_1P_1=1$,

$$Y_iP_i=\frac{\Delta_{1i}}{\Delta_{11}},\quad (i=1, 2, \cdots, n), \tag{30}$$

where Δ_{11} and Δ_{1i} are co-factors of Δ.

Let us take the 1939 table of American economy which is compiled by Prof. Leontief. In this case we aggregate all sectors into three: business, households and the others.

Table 3

	Business	Households	Others	Total
Business	65,860	47,990	23,570	137,420
Households	46,187	31,591	14,852	92,630
Others	26,068	11,949	2,407	40,424
Total	138,115	91,530	40,829	270,474

(Units: million dollars)

First of all, we shall estimate the values of the input coefficients by (9). The result is tabulated as follows.

Table 4

j \ i	Business	Households	Others
Business	$b_{11}=\frac{65,860}{137,420}=0.4793$	$b_{21}=\frac{47,990}{92,630}=0.5181$	$b_{31}=\frac{23,570}{40,424}=0.5831$
Households	$b_{12}=\frac{46,187}{137,420}=0.3361$	$b_{22}=\frac{31,591}{92,630}=0.3410$	$b_{32}=\frac{14,852}{40,424}=0.3674$
Others	$b_{13}=\frac{26,068}{137,420}=0.1897$	$b_{23}=\frac{11,949}{92,630}=0.1290$	$b_{33}=\frac{2,407}{40,424}=0.0595$

In the calculation of b_{ij} in the table above, we assume $S_i'=0$. Then, we get the equilibrium values of Y_iP_i by solving (28) after the substitution of b_{ij} of table 4 in this equation.

$$Y_1{}^\circ P_1=1,\ Y_2{}^\circ P_2=0.6740,\ Y_3{}^\circ P_3=0.2941,$$

where $Y_i{}^\circ$ is the equilibrium value of Y_i.

Finally, we compile the results as shown in Table 5, after determining the inputs (in terms of money value) by (9). It must be remembered, in this connection, that all inputs and outputs are represented in the relative value when we take $Y_1{}^\circ P_1=1$, because in this case (28) becomes simultaneous linear homogeneous equations.

Table 5

	Business	Households	Others	Total
Business	0.4793	0.3492	0.1715	1
Households	0.3361	0.2298	0,1081	0.6740
Others	0.1897	0.0869	0.0175	0.2941
Total	1.0051	0.6659	0.2971	1.9681

VI. *Maximizing Principle And Economic Planning*

The rôles the Leontief analysis plays can be divided into three. It is clear, firstly, that the analysis is of use historically to describe the structure of economy more analytically than any other methods, so far as a statistical inquiry is concerned. Secondly, the analysis deserves to be mentioned as one and the most rigid scheme of economic planning. Thirdly, it may be used as a system of economic prediction. The first rôle need not be explained here. Let us now consider the other two.

Then, what relations maximizing principle holds in the Leontief system when we make use of it as an economic planning scheme? Among the parameters of our system, prices must be chosen as parameters of planning, while input coefficients, although they are parameters in our system, are not so important from the viewpoint of planning technique, because they only have technical implications. Now, entrepreneurs and households will try microeconomically to find out their maximum economic surpluses under the given prices which the planning authorities will control. What results will be expected, then, under these conditions? It seems reasonable in this case to introduce maximizing principle into the Leontief system. Our modified Leontief system will satisfy these conditions. At the same time, moreover, economic prediction will be equivalent to looking for the same conditions if prices are predetermined.

Let us consider the original Leontief system where maximizing principle is ruled out and take it for planning scheme. We must control, in this case, not only prices but both inputs and outputs which are obtained after solving this system.

Adding to this, the equation

$$-x_{1i}-x_{2i}-\cdots+(Y_i-x_{ii})-\cdots-x_{ni}=S_i,\quad (i=1, 2, \cdots, n) \qquad (3)$$

involves S_i which consists of S_i' and S_i'', as we have shown above. It is natural to regard S_i' as a parameter, as we have also made clear. With regard to S_i'', we may treat it as a planning parameter, or as an endogenous sector, which is, then, brought into the left-hand side of the equation. In the former case, we can decrease the degree of planning more than the original Leontief system also.