FUNDAMENTAL PROBLEMS OF COMPARATIVE STATICS

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I. *Introduction*

The object of comparative statics is to analyse and describe the shifts of economic equilibrium caused by changes of data. These are the main subjects contained in the theories of Pigou, Keynes, Hicks and Samuelson. But the method of analysis, especially the method to treat the change of the shape of a function, has not yet been fully examined. It is necessary to reexamine this method in order to obtain results in more details, and to clear up the problems exhaustively.

From the standpoint of pure mathematics there exist difficult problems, but it is not hard to obtain an exactitude sufficient for economic application. The question is to find a method of approximation which will express the variation of a function with a minimum number of parameters and yet does not miss the important economic problems. In the following I will describe my method and its tentative application.

II. *Comparative Statics and Equilibrium Theory*

In the system of equilibrium equations, economic data are given as parameters or functional relations. For example, population is a parameter and technique of production is expressed by a production function. The first step of the equilibrium theory is to find the essential variables and ascertain necessary and sufficient data which determine the values of the variables, and to express the relations between data and variables with the system of equations. It is difficult to solve these equations, but they are used to analyse most important economic problems as shown by Walras, Pareto and other authors.

The second step of the theory is to examine the stability of the equilibrium values. The equilibrium is said to be stable if the restoring movement occurs whenever the actual values diverge from equilibrium values, and unstable if the divergency becomes larger with time. The

equilibrium is said to be neutral if it is fixed to the new removed position. The earlier equilibrium theories such as those of Cournot and Walras were much concerned to analyse the stabilizing process, assuming the equilibrium to be stable. But recent theories such as those of Hicks and Samuelson have analysed the various cases of stability and instability, with the intention to clear up the conditions for stability.

The third step is to analyse the shifts of equilibrium values when the data change. This is the object of comparative statics. When we deduce theorems on comparative statics, we must necessarily assume stabilizing conditions. Therefore, there exists a logical connection between comparative statics and stabilizing conditions. This is the "correspondence principle" so named by Samuelson.[1]

The primitive theories of comparative statics were presented by Cournot, Walras, Marshall and Divisia. Cournot treated tax rate as a parameter, and analysed the effect of a change in tax rate on price and output under a monopoly and free competition. Marshall and Divisia used the graphical method in order to explain the effect of functional variation upon equilibrium values. Walras treated changes of parameters or functional relations with a literary method. Excepting Cournot, these methods were so simple that they could not add much to our knowledge.

The equilibrium theory, as far as statics is concerned, is most fruitful in the third step. According to Samuelson the object of economic analysis is to deduce a "meaningful theorem". Most of his achievements in *Foundations of Economic Analysis* are meaningful theorems deduced by means of comparative statics. The theories of Pigou and Keynes which contain many meaningful theorems both belong to comparative statics.

Changes of data, as mentioned above, are expressed in two different forms, one of which is the variation of parameters. For example, Hicks treated income or price as a parameter in his theory of consumers' demand. The other is the variation of a function. For example a change of taste is expressed by the shift of utility function, and a change of productive efficiency is expressed by the shift of a production function. These cases were analysed by Pigou and Samuelson.

III. *Comparative Statics and Stabilizing Process*

Comparative statics is based on the assumption that equilibrium is stable. But it requires more than stability. The process of adaptation to the variation of data is realized through a long causalistic movement towards equilibrium. This process can be analysed into two parts, the

[1] P. Samuelson, *Foundations of Economic Analysis* (Cambridge, 1949), pp. 5, 258.

normal process of adaptation without any oscillations, and the oscillating movement around the normal process. We assume that both processes are realized in an instant, that the velocity of adaptation and the velocity of convergency are both infinitely large.

But in reality these velocities are not infinite. We must be careful when we apply our conclusions to actual questions. Generally speaking, comparative statics can explain the actual process when the system converges more rapidly than the process of adaptation. For example when productive efficiency increases on account of technical improvement, price fluctuations converge more rapidly than the change of equilibrium values, because it takes a rather long time to replace machines and trained workers. On the contrary, when the capital market is in process of converging to its normal position, it is not realistic to describe the results of the change in liquidity preference by the method of comparative statics. Because the converging velocity of the capital market is considered to be smaller than the adapting velocity to the liquidity preference.

In this case we can explain the actual movement by transposing the converging process and adapting process. This is one of the applications of the "transposing principle" which is widely used in physics and astronomy. But process analysis is the subject of dynamics. Comparative statics describes the final result and not the process of change. We should not expect it to explain the actual process except in special cases.

IV. *Approximating Expression of Functional Variation*

As explained above, variation of data is expressed in two ways. When it is expressed as change of parameter, we can deduce the theorems on comparative statics by differentiating each variable with this parameter and by examining the sign of this derivative. When it is expressed as a functional variation, however, we have to consider first of all, how to express it analytically.

We have hitherto two methods, one of which was adopted by Pigou, and the other by Samuelson. Let the variables be $x_1, x_2, \ldots\ldots\ldots\ldots x_n$, the original function $f(x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$, and the changed function $g(x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$. Pigou's method in its general form is expressed in the following equation.[2]

$$g(x_1, x_2, \ldots\ldots\ldots, x_n) = cf(x_1, x_2, \ldots\ldots\ldots, x_n)$$

where c is a parameter. $c=1$ in the intial position, and is more or less than unity according to the direction of change. For example, let x_1 etc.

[2] A. C. Pigou, *Economics of Stationary States* (London, 1935), p. 269.
He applied this method to one variable function. Thus he puts $g(x) = cf(x)$.

be quantities of productive factors and f and g production function. $c > 1$ denotes the increase of productivity. So we can judge from the sign of $\frac{dx_1}{dc}$ etc. whether the factor used increases or not.

Samuelson's method is to use a finite number of parameters.[3] Let the parameters be $a_1, a_2, \ldots\ldots\ldots\ldots, a_m$ and he expressed the function z as follow:

$$z = f(x_1 \ldots\ldots\ldots x_n, a_1 \ldots\ldots\ldots a_m)$$

The function in the initial position is shown by fixing the values of parameters as $a_1^o, \ldots\ldots\ldots\ldots, a_m^o$. If, for example, f being production function, we express with a_i the productive efficiency of the factor x_i, the variation of x_i resulting from the increase of the productivity of those factors can be judged by the sign of

$$dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial a_j} da_j$$

Actually it is difficult to determine the sign when the parameters are numerous. In some cases Samuelson uses only one parameter, and adopts the function[4]

$$z = f(x_1 \ldots\ldots\ldots x_n, a)$$

or otherwise, he assumes that $\frac{\partial x_i}{\partial a_j}$ $(i \neq j)$ is so small that it can be neglected.[5]

Theoretically, Samuelson's method is more general than Pigou's method. The latter assumes that a function shifts upwards or downwards in proportion to its magnitude. This is approximately true in the small domain of variables and for the small variation of the parameter, but it is not exact when the domain is large, or for the large variation of c. The former method, however, can be made almost exact if we take a large number of variables. But if we take only one variable as Samuelson does in some cases, the degree of approximation is not necessarily so high as in the latter method.

Theoretically the number of behaviors of functional variations of $f(x)$ corresponds to transfinite number, i. e. non-countable infinity. It is therefore impossible to express these variations even with countable infinite numbers of parameters. We have to specify the form of the function if we wish to do so. We assume, therefore, that the function $f(x)$ is analytic and can be expanded into power series. If we take the coefficient of each term as a parameter, and express $f(x)$ as

[3] *Foundations*, p. 29.
[4] *Ibid.*, p. 34.
[5] *Ibid.*, p. 36.

$$f\,(x,\ a_0,\ a_1,\ \ldots\ldots\ldots)$$

we can approximate every kind of functional variation with the variation of these parameters.

For practical purposes even this function is too complicated. It is enough to express $f(x)$ as power series of finite degree. And when the domain of x is small, we need only a few terms of x for our purpose. Now if we adopt polynomials in x of m degree, we have $m+1$ parameters, and the function is expressed as

$$f\,(x,\ a_0,\ a_1,\ \ldots\ldots\ldots,\ a_m) \tag{1}$$

which Samuelson has adopted. The less the number of parameters, the lower the degree of approximation, but the more simple for treatment. If we intend to minimize the number of parameter, we have a linear equation for the small domain of x, and the function becomes

$$f\,(x,\ a_0,\ a_1) = a_0 + a_1 x \tag{2}$$

In general we cannot reduce the number of parameters further, but in special cases we might put

$$f\,(x,\ a_0) = x + a_0 \quad \text{or} \quad = a_0\ x \quad \text{or} \quad = x^{a_0} \quad \text{etc.}$$

If the function involves n variables, we need at least $n+1$ parameters including constant terms, i. e.

$$\begin{aligned} f\,(x_1\ \ldots\ldots\ldots\ x_n\,;\ a_0\ \ldots\ldots\ldots\ a_n) \\ = a_0 + a_1\ x_1 + \ldots\ldots\ldots + a_n\ x_n \end{aligned} \tag{3}$$

In general, to approximate a function with power series, we need at least one more parameters than variables. Therefore it is inadequate to use one or an arbitrary number of parameters as Samuelson has done.

Another method of approximation is to expand the difference between $g(x)$ and $f(x)$ into power series. Putting

$$g\,(x_1\ \ldots\ldots\ x_n) - f\,(x_1\ \ldots\ldots\ x_n) = \phi\,(x_1\ \ldots\ldots\ x_n) \tag{4}$$

and expanding $\phi\,(x_1\ \ldots\ldots\ldots\ x_n)$ into Maclaurin series, we have

$$\begin{aligned} \phi\,(x_1 \ldots\ldots\ldots x_n) &= \phi\,(o \ldots\ldots\ldots o) + \Sigma \frac{\partial \phi}{\partial x_i}\, x_i \\ &= a_0 + \Sigma\, a_i\, x_i \end{aligned}$$

$$\therefore \quad g\,(x_1 \ldots\ldots\ldots x_n) = f\,(x_1 \ldots\ldots\ldots x_n) + a_0 + \Sigma\, a_i\, x_i \tag{5}$$

In case of one variable, we have

$$g\,(x) = f\,(x) + a_0 + a_1\, x \tag{6}$$

This equation presents us a curve instead of a straight line, so we can have a higher degree of approximation than by the former method with the same number of parameters. We might even use only one parameter as

$$g(x) = f(x) + a_0 \tag{7}$$

or

$$g(x) = f(x) + a_1 x \tag{8}$$

as a general approximating formula.

In the case of n variables, we might use instead of (5)

$$g(x_1 \ldots\ldots\ldots x_n) = f(x_1 \ldots\ldots\ldots x_n) + a_0 \tag{9}$$

or

$$g(x_1 \ldots\ldots\ldots x_n) = f(x_1 \ldots\ldots\ldots x_n) + \Sigma\, a_i x_i \tag{10}$$

Sometime Samuelson uses these simplest forms. For example he adopts the formula (9) for consumption or investment function in Keynesian system [6]. He also uses formula (10) in another case.[7]

The third method is to express $g(x_1 \ldots\ldots\ldots x_n)$ as a function of $f(x_1 \ldots\ldots\ldots x_n)$ and to find an approximation of this function. In this case also, $g(x_1 \ldots\ldots\ldots x_n)$ can be expressed with the power series of $f(x_1 \ldots\ldots\ldots x_n)$, and among them the simplest formula is a linear equation;

$$g(x_1 \ldots\ldots\ldots x_n) = a_0 + a_1 f(x_1 \ldots\ldots\ldots x_n) \tag{11}$$

This method uses less parameters than the former method, for we need at least $n+1$ parameters in formula (5). But the above method is limited in its application, because $g(x)$ is fixed as a one-valued function of $f(x)$ according to the nature of power series. Therefore $g(x)$ has a maximum or minimum whenever $f(x_1 \ldots\ldots\ldots x_n)$ has a maximum or minimum. If for instance $g(x)$ increases monotonically while $f(x)$ increases at first and then decreases, we have two values for $g(x)$ against one value of $f(x)$, and $g(x)$ becomes two-valued function of $f(x)$. In this case g cannot be expressed as a power series of f.

If two functions rise and fall together or one rises while the other falls, we find correlation between them. When the correlation is linear, the formula (11) is a good approximation, and the larger is the coefficient of correlation absolutely, the larger is the degree of approximation. In this case we get a good formula with only two parameters. In a special case where g is proportional to f, we get

$$g(x_1 \ldots\ldots\ldots x_n) = a_1 f(x_1 \ldots\ldots\ldots x_n) \tag{12}$$

which is the general form of Pigou's approximation.

Now let us compare the three methods explained above. The first method is superior when we wish to specify the form of the function. The second and third methods do not give us the actual form of the function. But the second method provides us with a higher degree of approximation for the same number of parameters. The third method is also superior as

[6] *Foundations*, p. 276.
[7] *Ibid.*, p. 51.

for approximation, but it applies only to special cases where $f(x)$ correlates to $g(x)$ to a high degree. Moreover, when we have more than two variables, the third method cannot specify the variation attributable to each variable. In the first or second method, the coefficient of linear terms express the effects mainly attributable to each variable. Let the function be

$$f(x_1 \ldots\ldots\ldots x_n)+a_0+\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_i x_j + \ldots\ldots \tag{13}$$

Then a_0 shows the variation independent of all variables, and a_i the variation dependent on x_i, and a_{ij} the variation dependent on x_i and x_j, and so on.

From these considerations we arrive to the conclusion that the second method is the most applicable to general cases and easier for treatment. I think it is best to adopt this method for the general theory of comparaive statics.

V. *Comparative Statics applied to Keynesian System*

As an example let us apply our method to the Keynesian system. According to Samuelson the Keynesian system can be summarized into three equations as follows:

$$\left.\begin{array}{l} Y=C(i,\ Y)+\alpha+I \\ I=F(i,\ Y)+\beta \\ M=L(i,\ Y) \end{array}\right\} \tag{14}$$

where i, Y, I stand respectively for the interest rate, income and investment; C, F, L stand respectively for the consumption function, the investment function, and the schedule of liquidity preference. M stands for the existing amount of money; α represents an upward shift of consumption function; similarly β represents the increase of the marginal efficiency of capital.

We get a higher degree of approximation by rewriting these equations as follows:

$$\left.\begin{array}{l} Y=C(i,\ Y)+\alpha_0+\alpha_1\cdot Y+\alpha_2\cdot i+I \\ I=F(i,\ Y)+\beta_0+\beta_2\cdot i+\beta_1\cdot Y \\ M=L(i,\ Y) \end{array}\right\} \tag{15}$$

where α_0, α_1 and α_2 stand for the shift of consumption function, and β_0, β_1 and β_2 for the shift of investment function. If we confine ourselves to the shift of consumption function, the above equations are simplified:

$$\left.\begin{array}{ll} C(i,\ Y)-Y+\alpha_1 & Y+I=-\alpha_0 \\ F(i,\ Y) & -I=0 \\ L(i,\ Y) & =M \end{array}\right\} \tag{16}$$

where $a_2 \cdot i$ is neglected because the change of consumption is considered almost independent of interest rate. The increase of a_0 stands for income-independent upward shift, and of a_1 for income-proportional upward shift. Now for $\frac{\partial i}{\partial a_0}$, $\frac{\partial Y}{\partial a_0}$ and $\frac{\partial I}{\partial a_0}$, the result is similar to that already derived by Samuelson[8]. Differentiating totally with a_0 and solving the resulting linear equations, we have

$$\frac{\partial i}{\partial a_0} = \frac{-L_Y}{\triangle} \tag{17}$$

$$\frac{\partial Y}{\partial a_0} = \frac{L_i}{\triangle} \tag{18}$$

$$\frac{\partial I}{\partial a_0} = \frac{F_Y L_i - F_i L_Y}{\triangle} \tag{19}$$

where

$$\triangle = \begin{vmatrix} C_i & C_Y - 1 + a_1 & 1 \\ F_i & F_Y & -1 \\ L_i & L_Y & 0 \end{vmatrix}$$
$$= L_Y (F_i + C_i) + L_i (1 - C_Y - F_Y - a_1) \tag{20}$$

We know that

$$C_Y > 0,\ F_Y > 0,\ F_i < 0,\ L_Y > 0,\ L_i < 0 \tag{21}$$

and $C_i \geqq 0$, but C_i is so small in absolute value that it can be neglected.

In order to evaluate our derivatives $\frac{\partial i}{\partial a_0}$ etc., we must determine the sign of $\triangle$. As worked out by Samuelson, we must deduce stabilizing condition for that purpose. He used the following hypothesis; the rate of change of income is proportional to the difference between intended investment-savings and actual investment-savings. Then the above equations are replased by the dynamic ones:

$$\left.\begin{aligned} \dot{Y} &= I - [Y - C(i,\ Y) - a_0 - a_1 Y] \\ 0 &= F(i,\ Y) - I \\ 0 &= L(i,\ Y) - M \end{aligned}\right\} \tag{22}$$

By expanding these equations in the neighborhood of the equilibrium values (Y_0, i_0, I_0) where $\dot{Y}=0$, and neglecting the terms of higher orders, we get linear equations in the place of above, and the solutions of these are the form:

$$\left.\begin{aligned} Y &= Y_0 + a_1 e^{\lambda t} \\ i &= i_0 + a_2 e^{\lambda t} \\ I &= I_0 + a_3 e^{\lambda t} \end{aligned}\right\} \tag{23}$$

[8] *Foundations*, pp. 276-283.

where λ must satisfy the following conditions:

$$\begin{vmatrix} C_i & C_Y-1+\alpha_1-\lambda & 1 \\ F_i & F_Y & -1 \\ L_i & L_Y & 0 \end{vmatrix}$$

$$= \triangle + \lambda L_i = 0 \tag{24}$$

$$\therefore \quad \lambda = -\frac{\triangle}{L_i} \tag{25}$$

For the equilibrium to be stable, λ must be negative. While $L_i < 0$, $\triangle$ must be negative. This condition enables us to evaluate the derivatives and we have

$$\frac{\partial i}{\partial \alpha_0} > 0 \qquad \frac{\partial Y}{\partial \alpha_0} > 0 \tag{26}$$

Until this we followed the analysis presented by Samuelson. Next let us examine the results of proportional upward shift of consumption. Through a similar procedure we have;

$$\frac{\partial i}{\partial \alpha_1} = \frac{-Y \cdot L_Y}{\triangle} \tag{27}$$

$$\frac{\partial Y}{\partial \alpha_1} = \frac{Y \cdot L_i}{\triangle} \tag{28}$$

$$\frac{\partial I}{\partial \alpha_1} = \frac{Y(L_i F_Y - F_i L_Y)}{\triangle} \tag{29}$$

Comparing these equations with (17) (18) and (19), we can conclude that if the increase of consumption is proportional to income, the coefficient of variation of i, Y or I is Y times larger than the case of an equal increase. In other words, the movement of equilibrium value is larger when the functional variation is intensified according to magnitude.

The shift of consumption function also deffects the velocity of convergency. We have

$$\frac{\partial \triangle}{\partial \alpha_0} = 0 \qquad \therefore \frac{\partial \lambda}{\partial \alpha_0} = 0$$

$$\frac{\partial \triangle}{\partial \alpha_1} = -L_i > 0 \qquad \therefore \frac{\partial \lambda}{\partial \alpha_1} = 1$$

As λ is negative, the last equation means that the absolute value of λ decreases with the increase of α_1. Therefore we have the following conclusions:

(i) The equal upward shift of consumption has no effect on the velocity of convergency.

(ii) The income-proportional shift of consumption decreases the velocity of convergency.

Similar results are obtained when we consider the shift of investment

function. We see from these results that our method enables us to evaluate not only the sign and intensity of the effect of functional variation, but also the velocity of convergency towards equilibrium.