A MEASUREMENT OF MONEY UTILITY AND FUNCTIONAL VALUES OF THE COST OF LIVING INDEX

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I. *Measurement of Marginal Utility of Money*

The measurement of individual subjective utility has not yet been attained up to the present. The measurement of money utility, however, was attempted by Ragnar Frisch and F. V. Waugh.[1] In this paper we shall adopt the former method.

The fundamental idea of Frisch's method is as follows. We take, say, sugar as a well defined and physically measurable commodity, which is called a *commodity of comparison.* There are two ways of defining the marginal utility of a commodity of comparison. One-is the utility measured per one pound, and the other that measured per one dollar. Firstly we consider the utility per one pound. u represents the marginal utility of one pound of any commodity, which is dependent only upon its consumption x, i.e.,

$$(1) \qquad u = u(x).$$

Let p represent the price of this commodity of comparison, then the expenditure ξ in terms of dollar per unit time is, of course,

$$(2) \qquad \xi = xp.$$

Let μ, moreover, represent the marginal utility measured per one dollar of the commodity of comparison, then

$$(3) \qquad u = \mu p.$$

The marginal utility μ of one dollar depends upon the expenditure ξ and the price p, i.e.,

$$(4) \qquad \mu = \mu(\xi, p).$$

Taking (2), (3) and (4) into consideration, we get

$$(5) \qquad \mu(\xi, p) = \frac{1}{p} u\left(\frac{\xi}{p}\right).$$

We have considered, so far, the marginal utility of a commodity of

[1] Ragnar Frisch, *Sur un Problème d'Économie pure, Norsk Matematisk Forenings Skrifter* (Oslo, 1926).
Ditto, *New Methods of Measuring Marginal Utility* (Tübingen, 1932).
F. V. Waugh, "The Marginal Utility of Money in the United States from 1917 to 1921 and from 1922 to 1932," *Econometrica*, Vol. 3, 1935, pp. 376–399.

comparison. The same formulae will be obtained, if we consider the marginal utility of money. Corresponding to the expenditure ξ of a commodity of comparison per one year, the nominal income in terms of money may be formulated. Let P represent the price of living, then we get the real income $r=\rho/P$, corresponding to the physical volume of consumption x of the commodity of comparison per one year. The only difference between ξ, p and ρ, r is that the former has the connection with any special commodity of comparison, while the latter with the common commodities. We distinguish the marginal utility of money measured per one dollar from that measured per one unit of real purchasing power, corresponding to the distinction between the marginal utility of commodity per one dollar and that of one pound. The former is represented by ω which is called *nominal money utility*, while the latter by w, which is called *real* or *deflated money utility*.[2]

Corresponding to (3), (4) and (5), we get the following relations:

$$(6) \qquad w = \omega P,$$

$$(7) \qquad \omega = \omega(\rho,\ P),$$

$$(8) \qquad \omega(\rho,\ P) = \frac{1}{P} w\left(\frac{\rho}{P}\right).$$

The last equation has the very important relations with the following extension.

According to the static equilibrium theory, individuals distribute their expenditures to each item which appears in their budgets, in such a way that the product of nominal money utility of a special commodity, say, food and its price equals the marginal utility of the commodity of comparison measured in terms of the unit of physical quantity. That is:[3]

$$(9) \qquad \omega(\rho, P)\cdot p = u(x).$$

Substituting (8) into $\omega(\rho,\ P)$ in the above equation,

$$(10) \qquad w\left(\frac{\rho}{P}\right) = \frac{P}{p} u(x).$$

Moreover, putting

$$(11) \qquad a = \frac{P}{p},$$

we get

$$(12) \qquad w(r) = a\,u(x).$$

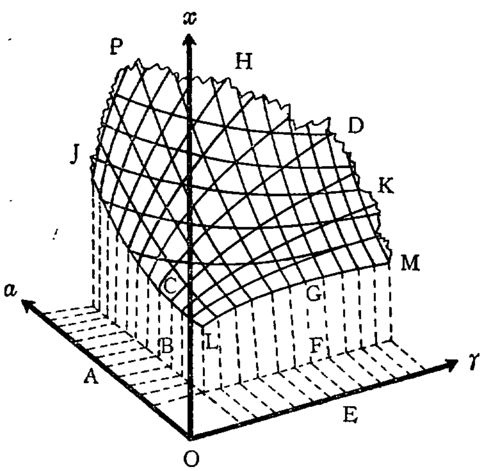
This is called the *equilibrium equation*. The above equation shows that the marginal utility of money $w(r)$ is proportional to the marginal utility of commodity, and at the same time it involves three variables, a, w, and r, which define one surface. We call it the *surface of consumption*, and by means of it we can evaluate the utility. Involving three variables, the

[2] Money utility in Frisch's terms always means marginal utility of money, as well as in the case of commodity.

[3] It is quite easy to derive equation (9), from the general law of equi-marginal utility.

surface of consumption is plotted in the space prescribed by three axes. Fig. 1 shows it.[4] The surface LMKDHPJL is the surface of consumption. In order to depict it as curves in a plane, we may pursue the relations between two variables, putting any one of them constant. We may distinguish three cases, according as what variable is held constant.

Fig. 1.



In this respect, it is the most interesting to consider the case where the inverted price a and the real income r relate each other, the volume of consumption x being held constant. In this case, from the equilibrium equation (12) we obtain

$$(13) \qquad a = \frac{1}{u(x)} w(r).$$

x and therefore $u(x)$ being constant, a is the product of $w(r)$ and a constant. This means money utility equation $w=w(r)$ if we leave the constant part out of our consideration. In order to plot this, the section parallel to the plane aOr, passing through any point on the x axis, is to be paid attention. Such a section is, e.g., a curve JK. A set of such curves is called *isoquant curves*.

We shall now consider the method which makes possible the comparison between the money utilities of different individuals. More correctly speaking, this method is to compare the money flexibility $w(r)$ of one individual with $w(r)$ of another individual. The *flexibility of money utility* in this case is defined as follows:

$$(14) \qquad \lim_{\substack{\Delta r \to 0 \\ \Delta w \to 0}} \frac{\Delta w(r)}{w(r)} \bigg/ \frac{\Delta r}{r} = \frac{r}{w(r)} \cdot \frac{d\,w(r)}{d\,r}.$$

To determine the consumption surface statistically, we follow Frisch's method, which makes use of the functional relationships between social average of consumption x and social income r.[5]

The most simple method of determining the money utility is to get the isoquant curve on the consumption surface.[6]

[4] Fig. 1 is drawn by altering Frisch's, in order to understand it easily.
[5] Frisch proves such an assumption theoretically. Cf. Frisch, *New Methods*, pp. 22–27.
[6] R. Frisch, *Sur un Problème*.

II. *Extension of Frisch's Index Number Formula*

A theory of functional index number of distant comparisons is studied by Ragnar Frisch.[7] The result of his own is described as follows:[8]

$$(15) \qquad 1+\breve{\omega}_0 = \frac{1+\breve{\omega}_t}{1-\breve{P}_{0t}} + \frac{\dfrac{d\ln\breve{P}_{0t}}{d\ln\rho_t}}{(1-\breve{P}_{0t})^2}.$$

In the above equation ln shows natural logarithm, and $\breve{P}_{0t}$ is defined as:

$$(16) \qquad \breve{P}_{0t} = \frac{d\ln P_{0t}}{d\ln\rho_t},$$

where P_{0t} is the functional index number of time t, time 0 taken as base. ρ_0 and ρ_t are respectively money expenditures at time 0 and t. In addition $\breve{\omega}_0$ and $\breve{\omega}_t$ are respectively nominal money flexibilities at time 0 and t. That is:

$$(17) \qquad \begin{cases} \breve{\omega}_0 = \dfrac{d\ln\omega_0}{d\ln\rho_0}, \\ \breve{\omega}_t = \dfrac{d\ln\omega_t}{d\ln\rho_t}, \end{cases}$$

where ω_0 and ω_t are respectively nominal money utilities at time 0 ant t. Let I be the choice indicator, then they are represented as follows:

$$(18) \qquad \begin{cases} \omega_0 = \dfrac{dI}{d\rho_0}, \\ \omega_t = \dfrac{dI}{d\rho_t}. \end{cases}$$

Since it is inconvenient directly to compute index number so far as (15) is concerned, we consider the method to derive P_{0t} from (15).

Rewriting (15),

$$\breve{\omega}_0 = \frac{\breve{\omega}_t+\breve{P}_{0t}}{1-\breve{P}_{0t}} - \frac{\dfrac{d(1-\breve{P}_{0t})}{d\ln\rho_t}}{(1-\breve{P}_{0t})^2}$$

$$= \frac{\breve{\omega}_t+\breve{P}_{0t}}{1-\breve{P}_{0t}} - \frac{\dfrac{d\ln(1-\breve{P}_{0t})}{d\ln\rho_t}}{1-\breve{P}_{0t}}.$$

Further we get

$$(\breve{\omega}_0-\breve{\omega}_t)\,d\ln\rho_t + d\ln(1-\breve{P}_{0t}) = (1+\breve{\omega}_0)\,d\ln P_{0t}.$$

Solving this differential equation,[9]

$$\rho_t^{\breve{\omega}_0-\breve{\omega}_t}(1-\breve{P}_{0t}) = c_1\,P_{0t}^{1+\breve{\omega}_0},$$

[7] Ragnar Frisch, "Annual Survey of General Economic Theory: The Problem of Index Numbers," *Econometrica*, Jan. 1936, pp. 31–38.

[8] *Ibid.*, p. 37.

[9] In integrating this differential equation, $\breve{\omega}_0$ and $\breve{\omega}_t$ are assumed to be constants, because in the neighborhood of time 0 and t both may be regarded as constants respectively.

where c_1 is a constant of integration. If we put $t=0$, $P_{0t}=1$ and $\check{P}_{0t}=0$, and accordingly $c_1=1$.[10] Then the above equation becomes

$$\rho^{\check{\omega}_0-\check{\omega}_t}(1-\check{P}_{0t}) = P_{0t}^{1+\check{\omega}_0}.$$

We can derive further

$$\rho^{\check{\omega}_0-\check{\omega}_t}(d\ln\rho_t-d\ln P_{0t}) = P_{0t}^{1+\check{\omega}_t}d\ln\rho_t.$$

Solving this differential equation,

$$\frac{1}{(1+\check{\omega}_0)e^{(1+\check{\omega}_0)(\ln\rho_t+\ln P_{0t})}}=\frac{e^{(1+\check{\omega}_t)\ln\rho_t}}{1+\check{\omega}_t}+c_2,$$

where c_2 is another constant of integration. Putting $t=0$ and $\ln\rho_0=-\frac{1}{2}$, $c_2=0$.[10] Then, we get from the above equation,

$$(19)\qquad P_{0t}=\sqrt[1+\check{\omega}_0]{\frac{1+\check{\omega}_t}{1+\check{\omega}_0}}\exp\left\{\frac{\check{\omega}_0-\check{\omega}_t}{1+\check{\omega}_0}\ln\rho_t\right\}.$$

As a general case, moreover, we may define

$$\check{P}_{0t} = \frac{d\ln P_{0t}}{d\ln(\rho_t/\rho_0)},$$

where ρ_0=constant. Then, we finally get

$$(20)\qquad P_{0t} = \sqrt[1+\check{\omega}_0]{\frac{1+\check{\omega}_t}{1+\check{\omega}_0}}\exp\left\{\frac{\check{\omega}_0-\check{\omega}_t}{1+\check{\omega}_0}\ln\left(\frac{\rho_t}{\rho_0}\right)\right\}.$$

Equation (20) is the desired one by means of which we calculate the numerical values of the functional index numbers.

III. *Application of the Theory to Japanese Economic Data*

Before applying the above theory to the actual data of Japan, we shall still consider the Frisch's actual procedure. Frisch introduces the following method.[11]

The relation between the consumption, the inverted price and the real money income is derived in the following way:

$$(21)\qquad x_1 = \alpha x_2 + \beta x_3,$$

where $x_1=x-\bar{x}$, $x_2=a-\bar{a}$, and $x_3=a\log r-\overline{a\log r}$, log being common logarithm. Here, $\bar{x}$, $\bar{a}$ and $\overline{a\log r}$ are their means of x, a and $a\log r$ respectively. From (21),

$$(22)\qquad a = \frac{\dfrac{x+(\alpha\bar{a}+\beta\overline{a\log r}-\bar{x})}{\beta}}{\log r+\dfrac{\alpha}{\beta}}.$$

[10] c_1 and c_2 are in general not constants. All process throughout this paper, however, is that of comparative statics. Accordingly we have no means to make c_1 and c_2 dependent variables theoretically.

[11] R. Frisch, *Sur un Problème*, pp. 36–39.

Let the marginal utility be represented as:

$$(23)\qquad w(r)=\frac{l}{\log r-\log b},$$

and let the marginal utility of a commodity as:

$$(24)\qquad u(x)=\frac{c}{x+m},$$

where b, c, l and m are all constants. From the equilibrium equation (13),

$$(25)\qquad a=\frac{\dfrac{x+m}{\dfrac{c}{l}}}{\log r-\log b}.$$

Comparing this equation with (22), we get $c/l=\beta$. Then, according to (23), the marginal money utility is represented as follows:

$$(26)\qquad w(r)=\frac{\dfrac{c}{\beta}}{\log r+\dfrac{a}{\beta}}.$$

At the same time, the equation of the marginal utility of the commodity becomes

$$(27)\qquad u(x)=\frac{\beta l}{x+(a\bar{a}+\beta\overline{a\log r}-\bar{x})}$$

from (24). Referring to (14), the flexibility of marginal money utility is expressed as:

$$(28)\qquad \breve{w}(r)=\frac{\dfrac{dw(r)}{dr}}{\dfrac{w(r)}{r}}=\frac{-0.4343}{\log r+\dfrac{a}{\beta}}.$$

So far we have traced the Frisch's method to compute the flexibility of the marginal money utility. We shall now discuss how to derive the numerical values of cost of living index of Japan, making use of the method above-mentioned.

Rewriting (9), we get

$$(29)\qquad \omega(\rho,P)=\frac{1}{p}u(x).$$

In the above equation we assume P is constant, i. e., we consider a short period when P is assumed not to change, or negligibly to change. Then, substituting $r=\rho/P$ in (23),

$$(30)\qquad \frac{l}{\log\dfrac{\rho}{P}-\log b}=\frac{l}{\log\rho-\log b'},$$

where $b'=Pb$ which is a constant. Therefore, (30) becomes a function of ρ, which we make equal to $\omega(\rho)$. Moreover, using (24), (29) becomes

$$(31) \qquad \omega(\rho) = \frac{l}{\log \rho - \log b'} = \gamma \frac{c}{x+m},$$

where $\gamma = \frac{1}{p}$. From the above equation,

$$(32) \qquad \beta = \frac{\dfrac{x+m}{\dfrac{c}{l}}}{\log \rho - \log b'}.$$

The flexibility of marginal nominal money utility $\breve{\omega}(\rho)$ is derived from (30):

$$(33) \qquad \breve{\omega}(\rho) = \frac{-0.4343}{\log \rho - \log b'}.$$

From this equation we can get $\breve{\omega}_0$ and $\breve{\omega}_t$, and then P_{0t}. In this case it is to be mentioned not to take the price of living index P into consideration throughout the computation of P_{0t}.

We shall show, next, the statistical data of Japan for the numerical computation of $\breve{\omega}$ and P_{0t}. We take rice as a commodity of comparison. Rice is a staple good of consumption not only in Japan but in the East Orient. In order to make use of the volume of consumption and price of rice, the data of family budget inquiry in Japan is to be available. We now look at the" Kakei Chōsa Hōkoku," or "Report of Family Budget Inquiry," compiled by Statistics Bureau of Prime Minister's Office, which is shown in Table 1. In this table, the figures of September, 1931 to August, 1932, the beginning year of this inquiry, and of September, 1939 to August, 1940, the final year, are given.

Table 1 (1) (1931—1932)

Income Class	Consumption Unit	Consumption of Rice		Real Expenditure		Expenditure for Rice	
		per Unit Family	per Cons. Unit (x)	per Unit Family	per Cons. Unit (ρ)	per Unit Family	Price of Rice per *To* (p)
Yen		*To*	*To*	*Yen*	*Yen*	*Yen*	*Yen*
Under 50	2.90	3.15	1.09	46.04	15.88	7.14	2.27
60	3.05	3.06	1.00	51.73	16.96	7.00	2.29
70	2.90	3.05	1.05	59.31	20.45	7.02	2.30
80	2.99	3.04	1.02	66.74	22.32	7.01	2.31
90	3.02	3.08	1.02	75.65	25.05	7.14	2.32
100	3.11	3.15	1.01	82.13	25.41	7.28	2.31
Mean			1.032		25.41		

(N. B. *To* is a unit of volume, approximately equal to 18.04 litres.)

The reason why we adopt consumption unit instead of unit family is that we consider consumption to be variable as to sex and years. Real expenditure involves costs of food, housing, etc., not savings, insurance, etc. Price of rice per *to* is obtained by dividing expenditure for rice per

Table 1 (2) (1939—1940)

Income Class	Consumption Unit	Consumption of Rice		Real Expenditure		Expenditure for Rice	
		per Unit Family	per Cons. Unit (x)	per Unit Family	per Cons. Unit (ρ)	per Unit Family	Price of Rice per *To* (p)
Yen Under 60	2.74	*To* 3.44	*To* 1.26	*Yen* 63.88	*Yen* 23.31	*Yen* 14.39	*Yen* 4.18
70	2.02	2.87	1.10	59.09	22.55	12.46	4.34
80	2.78	2.91	1.05	69.08	24.85	12.81	4.40
90	2.94	3.00	1.02	75.50	25.68	13.18	4.39
100	2.96	3.06	1.03	83.54	28.22	13.47	4.40
Mean			1.092		24.92		

Table 2 (1) (1931—1932)

$\gamma=\frac{1}{p}$	$\gamma \log \rho$	x'	γ'	$(\gamma \log \rho)'$
0.4405	0.5290	105.6	101.3	92.3
0.4367	0.5367	96.9	100.4	93.7
0.4348	0.5700	101.7	100.0	99.5
0.4329	0.5840	98.8	99.6	101.9
0.4310	0.6030	98.8	99.1	105.2
0.4329	0.6156	97.9	99.6	107.4
0.4348	0.57305	100.0	100.0	100.0

Table 2 (2) (1939—1940)

$\gamma=\frac{1}{p}$	$\gamma \log \rho$	x'	γ'	$(\gamma \log \rho)'$
0.2392	0.3272	115.4	103.8	101.8
0.2304	0.3117	100.7	100.0	97.0
0.2273	0.3171	96.2	98.7	98.7
0.2278	0.3212	93.4	98.9	99.9
0.2273	0.3298	94.3	98.7	102.6
0.2304	0.3214	100.0	100.0	100.0

unit family by consumption of rice per unit family.

Table 2 shows the figures necessary to calculate $\breve{\omega}$ and P_{0t}. In this table x', γ' and $(\gamma \log \rho)'$ are respectively 100-fold as large as the values divided by the means of x, γ and $\gamma \log \rho$ repectively.

Let the deviations from the means of x', γ' and $(\gamma \log \rho)'$ be x_1, x_2 and x_3 respectively, then we get

$$(34) \qquad x_1 = 4.84\, x_2 + 0.29\, x_3,$$

by applying the usual method of least squares to (21), in respect to the figures of 1931-1932. In the above equation $x_1=x'-100$, $x_2=\gamma'-100$ and

$x_3=(\gamma \log \rho)'-100$. Then we get

$$(35) \qquad \gamma' = \frac{\dfrac{x+413}{0.29}}{(\log \rho)'+16.69}.$$

Computing $\breve{\omega}_0(\rho)$ by means of (33),

$$(36) \qquad \breve{\omega}_0(\rho) = \frac{-0.4343}{(\log \rho)'+16.69} = -\ 0.0232.$$

By the same way we can get the results from the figures of 1939–1940:

$$(37) \qquad x_1 = 4.18\, x_2 - 0.27\, x_3,$$

$$\gamma' = \frac{\dfrac{x+291}{-0.27}}{(\log \rho)'-15.48},$$

$$(38) \qquad \breve{\omega}_t(\rho) = \frac{-0.4343}{(\log \rho)'-15.48} = 0.0322.$$

Putting these values of $\breve{\omega}_0$ and $\breve{\omega}_t$ in (20), we finally get

$$(39) \qquad P_{0t} = \sqrt[1-0.0232]{\frac{1+0.0322}{1-0.0232}}\left(\frac{24.92}{25.41}\right)^{\frac{-0.0232-0.0322}{1-0.0232}} = 1.059.$$

For the sake of comparison, we look at 150.6, 1940 index (base=1932), the atomistic value of cost of living index compiled by *The Asahi*.