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LATTICE EQUATION AS LOGICAL EQUATION

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In the former part of this paper, the author will establish a solvability condition and a solution formula of the equation in distributive lattice whose sublattice has at least a maximal element and a minimal one. In the latter part, some theorems on the extension of Boolean lattice will be discussed, and their logical interpretation will be given.

In G. Birkhoff's Lattice Theory, he proposed a general programme to apply lattice theory to logic, and then explained briefly the various logics and corresponding lattice theories introduced by many writers up to this time.[1]

From the point of view of lattice equation, the author, in this paper, will discuss the elementary properties of a logical equation which has been fairly long treated,[2] but which is now almost neglected in spite of its great usefulness.

In section I, the author will state the essence of the theorems relating to the equation in Boolean lattice considered to correspond to classical logic. In section II, he wishes to consider the problem corresponding to section I with respect to the equation in distributive lattice which is situated between Boolean lattice and modular lattice, the latter being considered to correspond to the logic of quantum mechanics. Lastly in section III, one property of Boolean lattice will be considered which is interesting from the standpoint of classical logic.

I

The elementary theorems about the equation in Boolean lattice are as follows:

THEOREM I_1. *Simultaneous equations with one unknown in Boolean lattice can be transformed into a single equivalent equation, whose right-hand side is zero.*

This theorem means in the sense of classical logic that several propositions can be reduced to one proposition which is synonymous but of negative form.

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[1] G. Birkhoff, *Lattice Theory* (New York, 1940), pp. 122–131.

[2] J. Venn, *Symbolic Logic* (New York, 1894), pp. 289–330.
L. Couturat, *L'Algèbre de la Logique* (Paris, 1914), pp. 52–62.

THEOREM I_2 *The necessary and sufficient condition that the equation $f(x)=0$ in Boolean lattice is solvable (let an element of that Boolean lattice be its root) is as follows:*

$$f(1) \frown f(0) = 0$$

The solvability condition expressed in the above theorem is a generalization of many mediate inferences in classical logic.

THEOREM I_3 *When the equation $f(x)=0$ in Boolean lattice is solvable, the general form of its roots is*

$$f(0) \smile (f'(1) \frown u),$$

where u is an arbitrary element of that Boolean lattice.

This theorem has not been logically interpreted so far. But it is evident that this theorem shows the process to determine by calculation the concept satisfying various conditions.[3]

These theorems can be easily extended to the case of equation with many unknown and each of them can be also logically interpreted.[4]

These theorems regarding the equation in Boolean lattice include important and applicable properties both in theory and practice.

Therefore the device of mechanical calculation in Boolean lattice is practically significant and also possible.

II

In this section, the properties of the equation in distributive lattice will be discussed,[5] which correspond to section I.

If an equation in distributive lattice D has an element of D as root, this equation is said to be solvable.

In this section, we will especially deal with a distributive lattice D_m whose sublattice has a maximal element and a minimal one.

THEOREM II_1 *Simultaneous equations with one unknown in distributive lattice D_m satisfying the above conditions may be transformed into a single equivalent equation.*

Although this theorem will correspond to theorem I_1, its proof will be given later in this paper.

THEOREM II_2 *The necessary and sufficient condition that the equation in distributive lattice*

$$(1) \qquad (a \frown x) \smile b = (c \frown x) \smile d$$

is solvable, is expressed in the following way:

$$(2) \qquad (a \frown d) \smile b = (c \frown b) \smile d \text{ [6]}$$

[3] The author attempted general application of this root in "Elementary Applications of Complemented Distributive Lattice," *The Hitotsubashi Review*, Vol. **21** (1949), (written in Japanese).

[4] The author's "Equation in Boolean Algebra," *Otsuka Sūgaku-kai Shi (Journal of Ōtsuka Mathematical Society)*, Vol. **8** (1939), (written in Japanese).

[5] Assistant Seki considered some of these problems by representation theory of lattice. Now the author will directly deal with these problems not by means of representation theory.

[6] The condition that the equation $(a \frown x) \smile b=(c \frown x) \smile d$ in modular lattice is solvable, is

PROOF: Let the equation (1) be solvable and its root be r. Then we have,

$$(3) \qquad (a \frown r) \smile b = (c \frown r) \smile d.$$

By meeting b, we have

$$b = (b \frown c \frown r) \smile (b \frown d)$$

Again if c meets it, then

$$c \frown b = (b \frown c \frown r) \smile (b \frown c \frown d)$$

When we join d, we get

$$(c \frown b) \smile d = (b \frown c \frown r) \smile d$$

Then by the distributive law, we get

$$(4) \qquad (c \frown b) \smile d = ((c \frown r) \smile d) \frown (b \smile d)$$

Similarly if we substitute a for c and b for d respectively, we get

$$(5) \qquad (a \frown d) \smile b = ((a \frown r) \smile b) \frown (b \smile d)$$

By the equalities (3), (4) and (5), the equality (2) is the necessary condition.

Now assume that the equality (2) holds. When we substitute $s(=(a \frown d) \smile b)$ for x on the left-hand side of the equation (1), we get $(a \frown s) \smile b = (a \frown ((a \frown d) \smile b)) \smile b = (a \frown d) \smile b = s$. Also if $s(=(c \frown b) \smile d)$ is substituted for x on the right-hand side of the equation (1), we have $(c \frown s) \smile d = (c \frown ((c \frown b) \smile d)) \smile d = (c \frown b) \smile d = s$.

Therefore an element s of D satisfies the equation (1) and we see that the equality (2) is the sufficient condition. Q. E. D.

LEMMA 1 *The set of roots of the equation in distributive lattice D constitutes a sublattice. The root of the solvable equation in the above defined distributive lattice D_m is $l \smile (g \frown u)$ where g is the greatest root, l the least, and u is an arbitrary element of D_m.*

PROOF: Let the equation in D

$$(1) \qquad (a \frown x) \smile b = (c \frown x) \smile d$$

have r and s as roots. Then we have

$$(6) \qquad (a \frown r) \smile b = (c \frown r) \smile d, \quad (a \frown s) \smile b = (c \frown s) \smile d.$$

When we join these two equalities, then

$$(a \frown (r \smile s)) \smile b = (c \frown (r \smile s)) \smile d$$

When we meet these two equalities,

$$(a \frown (r \frown s)) \smile b = (c \frown (r \frown s)) \smile d$$

These two equalities tell that not only r and s but also $r \smile s$ and $r \frown s$ are the roots of the equation (1). Thus the set of roots of the equation (1) constitutes a sublattice W.

Especially when (1) is an equation in D_m, W will have a maximal element m. Let w be an arbitrary element of W. As mentioned above, $w \smile m$ is also a root of the equation (1) where $m \leqq w \smile m$. Hence $m = w \smile m$. And this shows

$(a \frown (c \smile d)) \smile b = (c \frown (a \smile b)) \smile d$. And this corresponds to the mediate inference in the logic of quantum mechanics.

that m is the greatest element of W. Let m be replaced by g. Similarly we see that the equation has the least root l.

As we have the relation, $l \leqq w \leqq g$, we get $w = l \smile (g \frown w)$ and w belongs to the set $l \smile (g \frown D_m)$. Conversely, any element of this set is $l \smile (g \frown u)$ while l and g satisfy the equation (1) and we get the following relation.

$$(7) \qquad (a \frown l) \smile b = (c \frown l) \smile d, \quad (a \frown g) \smile b = (c \frown g) \smile d$$

When we meet u to the second equality and join it to the first equality, we get

$$(a \frown (l \smile (g \frown u))) \smile b = (c \frown (l \smile (g \frown u))) \smile d$$

Hence $l \smile (g \frown u)$ is in general a root of the equation (1). Q. E. D.

LEMMA 2 *If the equation in* D_m

$$(8) \qquad (j \frown x) \smile k = (p \frown x) \smile q$$

is solvable, the least root of

$$(9) \qquad x \smile (k \frown q) = k \smile q$$

is the least root of (8)

PROOF: Let a root of the equation (9) be v, then

$$v \smile (k \frown q) = k \smile q.$$

By meeting j to it, we get

$$(j \frown v) \smile (j \frown k \frown q) = (j \frown k) \smile (j \frown q)$$

By joining k, $\quad (j \frown v) \smile k = (j \frown q) \smile k$

Similarly by substituting p for j and q for k respectively, we get

$$(p \frown v) \smile q = (p \frown k) \smile q$$

When the equation (8) is solvable, by theorem II_2, we have

$$(j \frown q) \smile k = (p \frown k) \smile q$$

Hence $\quad (j \frown v) \smile k = (p \frown v) \smile q$

Therefore all the roots of the equation (9) satisfy the equation (8).

Provided that the least root of the equation (9) be y and any root of the equation (8) be z, we obtain

$$(10) \qquad y \smile (k \frown q) = k \smile q, \quad (j \frown z) \smile k = (p \frown z) \smile q$$

When we substitute $y \frown z$ for x on the left-hand side of the equation (9),

$$(y \frown z) \smile (k \frown q) = (y \smile (k \frown q)) \frown (z \smile (k \frown q))$$

From the first equality of (10),

$$(y \frown z) \smile (k \frown q) = (k \smile q) \frown (z \smile k) \frown (z \smile q)$$

When z is joined to the second equality of (10),

$$z \smile k = z \smile q$$

Again when k is joined,

$$z \smile k = z \smile q = z \smile q \smile k$$

Hence $\quad (y \frown z) \smile (k \frown q) = (k \smile q) \frown (z \smile k \smile q)$

i.e. $\quad (y \frown z) \smile (k \frown q) = k \smile q$

This show that $y \frown z$ is also a root of the equation (9). As y is the least root of the equation (9), $y \leqq y \frown z \leqq z$. Hence $y \leqq z$. Q. E. D.

The least root of the equation (9) corresponds to the symmetric difference of k and q in Boolean lattice. Let it be denoted by $k \,!\, q$. Dually we have the following lemma.

LEMMA 3 *If the equation in* D_m, $(j \smile x) \frown k = (p \smile x) \frown q$ *is solvable, the greatest root of the equation* $x \frown (k \smile q) = k \frown q$ *is the greatest root of the above equation.*

The greatest root of the equation $x \frown (k \smile q) = k \frown q$ is the dual concept of $k \,!\, q$ defined above. Let it be denoted by $k \,¡\, q$.

THEOREM II_3 *If the equation in* D_m $(a \frown x) \smile b = (c \frown x) \smile d$ *is solvable, the general form of its roots is*

$$(b \,!\, d) \smile ((a \smile b) \,¡\, (c \smile d)) \frown u),$$

where u *is an arbitrary element of* D_m.

PROOF: The least root of the solvable equation D_m is $b!d$ by lemma 2. As the above equation can be transformed into $(b \smile x) \frown (a \smile b) = (d \smile x) \frown (c \smile d)$, the greatest root of this equation is $(a \smile b) \,¡\, (c \smile d)$ by lemma 3. Then by lemma 1, the general form of the root is $(b \,!\, d) \smile (((a \smile b) \,¡\, (c \smile d)) \frown u)$ where u is an arbitrary element of D_m. Q. E. D.

COROLLARY *The solvable equation in* D_m, $(a \frown x) \smile b = (c \smile x) \frown d$ *is equivalent to the equation* $x \smile (b \,!\, d) = x \frown ((a \smile b) \,¡\, (c \smile d))$.

PROOF: By theorem II_3, the set of the roots of a solvable equation in D_m is determined by its greatest and least roots. If $(a \frown x) \smile b = (c \frown x) \smile d$ is solvable, $b \,!\, d \leqq (a \smile b) \,¡\, (c \smile d)$. Then both of $b \,!\, d$ and $(a \smile b) \,¡\, (c \smile d)$ are also roots of $x \smile (b \,!\, d) = x \frown ((a \smile b) \,¡\, (c \smile d))$. Moreover as we have the following relation $(b \,!\, d) \leqq x \smile (b \,!\, d) = x \frown ((a \smile b) \,¡\, (c \smile d)) \leqq x$, $b \,!\, d$ is the least root and similarly $(a \smile b) \,¡\, (c \smile d)$ the greatest root. Hence the roots of the above two equations are identical. Q. E. D.

If the equation $(a \frown x) \smile b = (c \frown x) \smile d$ is solvable, $x \smile (b \,!\, d) = x \frown ((a \smile b) \,¡\, (c \smile d))$ will be called its *canonical form*.

PROOF OF THEOREM II_1: When simultaneous equations in D_m with one unknown $(a_i \frown x) \smile b_i = (c_i \frown x) \smile d_i$ $(i = 1, 2, \ldots, n)$ are reduced to their canonical forms, the above system is equivalent to

(11) $\quad x \smile (b_i \,!\, d_i) = x \frown ((a_i \smile b_i) \,¡\, (c_i \smile d_i))$ $(i = 1, 2, \ldots, n)$

Formally let us make the following equation

(12) $\quad x \smile (\check{i}\,(b_i \,!\, d_i)) = x \frown (\hat{i}\,((a_i \smile b_i) \,¡\, (c_i \smile d_i)))$[7]

If the simultaneous equations (11) are solvable, the following relation can be obtained for all values of i

(13) $\quad b_i \,!\, d_i \leqq x \leqq (a_i \smile b_i) \,¡\, (c_i \smile d_i)$ *Consequently*

(14) $\quad \check{i}\,(b_i \,!\, d_i) \leqq x \leqq \hat{i}\,((a_i \smile b_i) \,¡\, (c_i \smile d_i))$

[7] $\check{i}\, l_i$ denotes $l_1 \smile l_2 \smile \cdots \smile l_n$ and $\hat{i}\, l_i$ denotes $l_1 \frown l_2 \frown \cdots \frown l_n$.

It shows that all the roots of (11) are roots of (12). Conversely if (12) is solvable, we have the relation (14). And for all values of i, (13) will be established. Hence all the roots of the equation (12) are the roots of (11).

When we assume solvability, a system of the simultaneous equations with one unknown,

$$(a_i \frown x) \smile b_i = (c_i \frown x) \smile d_i \ (i=1, 2, \ldots, n)$$

is equivalent to the equation

$$x \smile (\breve{i}(b_i \mathbin{!} d_i)) = x \frown (\overset{\frown}{i}((a_i \smile b_i) \mathbin{i} (c_i \smile d_i))) \quad \text{Q. E. D.}$$

If the logic corresponding to lattice D_m (a logic situated between classical logic and the logic of quantum mechanics) exists, theorem II_1 will give a transformation rule of proposition in that logic. And theorem II_2 can be nothing but a mediate inference in it. Finally theorem III_3 offers the calculating process of constructing the assigned concept.

III

In this section, we will prove a theorem concerning the extension of Boolean lattice and explain its meaning in classical logic.

Def. 1 When the meaning of $'$ in Boolean lattice B is invariant in its sublattice S, i.e. without any modification I, O of B are I, O of S, this sublattice is called a *self-complemented sublattice* of B.

Def. 2 Let S be a self-complemented sublattice of a Boolean lattice B. An element of B, which satisfies an equation in S, is called an *algebraic element* of S, while other elements are called *transcendental.*

THEOREM III_1 *The set of all algebraic elements A of a self-complemented sublattice S of a Boolean lattice B is also a self-complemented sublattice of B.*

PROOF: Let two elements of A be a and b. As a satisfies an equation $(s \frown x) \smile (t \frown x')=O$ in S, $(s \frown a) \frown (t \smile a')=O$, where s and t are elements of S. As b also satisfies an equation $(u \frown x) \smile (v \frown x')=O$ in S, $(u \frown b) \smile (v \frown b')=O$ where u and v are elements of S. Now consider the equation $((s \smile u) \frown x) \smile ((t \frown v) \frown x')=O$. As $s \smile u$ and $t \frown v$ are also elements of S, that is an equation in S. When we substitute $a \frown b$ for x, we have $((s \smile u) \frown (a \frown b)) \smile ((t \frown v) \frown (a \frown b)')=(s \frown a \frown b) \smile (u \frown a \frown b) \smile (t \frown v \frown a') \smile (t \frown v \frown b')=O$. Hence $a \frown b$ also satisfies an equation in S and belongs to A. In a quite similar way, $a \smile b$ will also satisfy the equation $((s \frown u) \frown x) \smile ((t \smile v) \frown x')=O$ in S, and belong to A. Thus A constitutes a sublattice. Finally a' satisfies the equation $(t \frown x) \smile (s \frown x')=O$ in S, and belongs to A. Thus A is self-complemented. Q. E. D.

THEOREM III_2 *An algebraic element of A in the previous theorem is an algebraic element of S. In other words, A is algebraically closed.*

PROOF: Let $(a \frown x) \smile (b \frown x')=O$ be an equation in A, which is satisfied by its algebraic element c. Then we have

$$(15) \qquad (a \frown c) \smile (b \frown c') = O$$

where a and b are elements of A. As a and b are algebraic elements of S, we get (16) $(s \frown a) \smile (t \frown a') = 0, \quad (u \frown b) \smile (v \frown b') = 0$
where s, t, u and v are elements of S.

When the equalities (15) and (16) are joined, $(a \frown c) \smile (b \frown c') \smile (s \frown a) \smile (t \frown a') \smile (u \frown b) \smile (v \frown b') = 0$. It shows that the equation with unknown x and y, $(c \frown x) \smile (c' \smile y) \smile (s \frown x) \smile (t \frown x') \smile (u \frown y) \smile (v \frown y') = 0$ is solvable in B. The solvability condition shows that c will satisfy an equation in S. Thus an algebraic element of A is also an algebraic element of S. Q. E. D.

Def. 3 Let a be an element of a Boolean lattice B, which does not belong to a self-complemented sublattice S of B and s and t be two arbitrary elements of S. Then the set of elements $(s \frown a) \smile (t \frown a')$ will be denoted by $S(a)$.

THEOREM III_3 *$S(a)$ is the smallest self-complemented sublattice of B which contains S and a.*

PROOF: Provided that x and y are two elements of $S(a)$, we have $x = (s \frown a) \smile (t \frown a')$, $y = (u \frown a) \smile (v \frown a')$ where s, t, u and v are elements of S. Then $x \smile y = ((s \smile u) \frown a) \smile ((t \smile v) \frown a')$ where $s \smile u$ and $t \smile v$ belong to S. Hence $x \smile y$ is an element of $S(a)$. Similarly $x \frown y$ is also an element of $S(a)$. Consequently $S(a)$ is a sublattice of B. Again we $a' = (0 \frown a) \smile (1 \frown a')$. Moreover as 1 and 0 belong to S, a', together with a, belongs to $S(a)$. Thus $S(a)$ is self-complemented. Every element s of S, being expressed as $(s \smile a) \smile (s \smile a')$, belongs to $S(a)$. As $(1 \frown a) \smile (0 \frown a')$, a is an element of $S(a)$. Thus $S(a)$ is a self-complemented sublattice containing S and a.

Now let a self-complemented sublattice containing S and a be T. Then arbitrary elements s, t of S belong to T. And both a and a' belong to T. Consequently every element $(s \frown a) \smile (t \frown a')$ of $S(a)$ belong to T and we have $S(a) \leqq T$. Thus $S(a)$ is the smallest self-complemented sublattice containing S and a. Q. E. D.

By this theorem, $S(a)$ may be called an *extension of S by a*.

Def. 4 Let S be a self-complemented sublattice of a Boolean lattice B. When $S(a) = S(b)$ holds, a and b are said to be *equivalent with respect to S*.[8]

LEMMA Let a transcendental element of S be w. Then every element of $S(w)$ is in only and only one way expressed by $(s \frown w) \smile (t \frown w')$, s and t belonging to S.

PROOF: Put $(s \frown w) \smile (t \frown w') = (u \frown w) \smile (v \frown w')$ and assume that s, t, u and v are elements of S. Then we have

$$(((s \frown u') \smile (s' \frown u)) \frown w) \smile (((t \frown v') \smile (t' \frown v)) \frown w') = 0$$

As w is a transcendental element of S and the coefficients of the last equality belong S, we have $(s \frown u') \smile (s' \frown u) = 0$ and $(t \frown v') \smile (t' \frown v) = 0$. Hence we get $s = u$ and $t = v$. Q. E. D.

THEOREM III_4 *If a and b are transcendental elements of S, the neces-*

[8] As an isomorphic extension of lattice is considered to be logically meaningless, we regard extensions to the same lattice as equivalent.

sary and sufficient condition that a and b are equivalent with respect to S is that $(a \frown b') \smile (a' \frown b)$ *belongs to S.*

PROOF: Put $S(a)=S(b)$, then a belongs to $S(b)$ and $a=(s \frown b) \smile (t \frown b')$, s and t being elements of S. Again b belongs to $S(a)$ and $b=(u \frown a) \smile (v \frown a')$, u and v being elements of S. Then from these two equalities, we have

$$\begin{aligned} a &= (s \frown ((u \frown a) \smile (v \frown a'))) \smile (t \frown ((u' \frown a) \smile (v' \frown a'))) \\ &= (((s \frown u) \smile (t \frown u')) \frown a) \smile (((s \frown v) \smile (t \frown v')) \frown a') \end{aligned}$$

By lemma we get

$$(s \frown u) \smile (t \frown u') = I \text{ and } (s \frown v) \frown (t \frown v') = O$$

The first equality can be reduced to $(s' \frown u) \smile (t' \frown u') = O$. By joining it to the second, we have

$$(s' \frown u) \smile (t' \frown u') \smile (s \frown v) \smile (t \frown v') = O$$

It shows that the equation with unknown x and y

$$(s' \frown x) \smile (t' \frown x') \smile (s \frown y) \smile (t \frown y') = O$$

is solvable in S. The solvability condition is $(s' \frown t') \smile (s \frown t)=O$, i.e. $s=t'$. Hence $a=(b \frown t') \smile (b' \frown t)$ and $(a \frown b') \smile (a' \frown b)=t$ where t belongs to S.

Conversely, let $(a \frown b') \smile (a' \frown b)$ be an element r of S. From $(a \frown b') \smile (a' \frown b)=r$, we have $a=(b \frown r') \smile (b' \frown r)$. As r belongs to S, it shows that a is an element of $S(b)$. Then by theorem III_3, we get $S(a) \subseteqq S(b)$. Quite similarly $S(b)S(\subseteqq a)$. Thus $S(a)=S(b)$ and a and b are equivalent with respect to S. Q. E. D.

Def. 5 The equivalence relation defined by Def. 4 is evidently reflexive, symmetric and transitive. Therefore the elements of B may be classified by this relation. Every class of this classification is called an equivalent class.

As is well known, B constitutes a group with respect to symmetric difference. The self-complemented sublattice S of B is a subgroup of B.

COROLLARY *If a self-complemented sublattice A of a Boolean lattice B is algebraically closed, the equivalent classes of A coincide with the cosets of A as a group with respect to symmetric difference.*

PROOF: As A is algebraically closed, elements a and b of B are equivalent if and only if $(a \frown b') \smile (a' \frown b)$ belongs to A. This shows that a and b of B as a group belong to the same coset of A. Q. E. D.

Finally we will explain the logical meaning of various theorems proved in this section.

$S(a)$ means the smallest subuniverse which can be constructed of one subuniverse and an attribute "*a*" by "*and*", "*or*" and "*not*". That a is a transcendental element of S means that the attribute "*a*" cannot be defined as an attribute which satisfies a logical equation in subuniverse S.

Hence the principal theorem III_4 tells us how to decide whether two attributes can extend S to the same subuniverse respectively where each attribute cannot be defined as an attribute satisfying a logical equation in S.