CHANGES OVER TIME OF PRODUCTION AND ADDED VALUE

- A Special Problem of Index Number Theory -

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1. Introduction

In order to estimate the rate of change over time of added value, we make very often use of production index instead of added value statistics, when we are engaged in the work of, say, the prediction of national income in the future period. We do so because the statistics of production index are more easily available than those of added value, and what is especially important, the former are classified into more detailed industries. This makes them absolutely indispensable. Is it right, then, that we should regard the changes of production indices as equivalent or parallel to those of added value indices if we can properly devise to construct the latter theoretically? We are rather difficult to find out studies about this problem at the present stage, which it is the purpose of the present paper to analyse.

Firstly we try to get the relationship between the production index of the theoretical Laspeyres type which is defined as the mean of individual production indices weighted with the value of production and that of the usual Laspeyres type weighted with values added.

We, then, adopt a new definition of added value index as a measure of changes over time of added value, and prove that such an index can almost be looked upon as that which shows a real level of added value.

Lastly, we verify that the usual method of measuring variations of added value by means of production indices is correct.

2. Meaning of Production Indices

It is quite clear that any production index measures changes over time of physical quantities of production whatever type it may take. Production index of Laspeyres type is, of course, written as follows:

$$Q_L = \frac{\sum p_0 q_1}{\sum p_0 q_0},\tag{1}$$

where Q_L represents a production index of Laspeyres type, and p_0 is the price of the base period 0, q_0 , q_1 are respectively the quantities of the base period 0 and

the current period 1. Rewriting equation (1),

$$Q_L = \frac{\Sigma\left(\frac{q_1}{q_0}\right) w_0}{\Sigma w_0}, \quad w_0 \equiv p_0 q_0, \tag{2}$$

where q_1/q_0 is the quantity index of an individual item.

The customary production index, on the other hand, is weighted with the values added in the base period, v_0 , instead of the values of production, w_0 . That is, in place of (2),

$$Q_{L'} = \frac{\Sigma\left(\frac{q_1}{q_0}\right) v_0}{\Sigma v_0}.$$
 (3)

If we then take the difference between (2) and (3),

$$Q_{L}-Q_{L}'=\frac{\Sigma\left(\frac{q_{1}}{q_{0}}\right)w_{0}}{\Sigma w_{0}}-\frac{\Sigma\left(\frac{q_{1}}{q_{0}}\right)v_{0}}{\Sigma v_{0}}$$
$$=\frac{\Sigma v_{0}\cdot\Sigma\left(\frac{q_{1}}{q_{0}}\right)w_{0}-\Sigma w_{0}\cdot\Sigma\left(\frac{q_{1}}{q_{0}}\right)v_{0}}{\Sigma w_{0}\cdot\Sigma v_{0}},$$

therefore, according to the following inequality,

$$\Sigma v_0 \cdot \Sigma \left(\frac{q_1}{q_0}\right) w_0 \gtrless \Sigma w_0 \cdot \Sigma \left(\frac{q_1}{q_0}\right) v_0, \qquad (4)$$

we can derive

Added value is, generally, defined as the difference between the value of production, w, and the inputs except wage, interest, etc. necessary for its production, c, and then $v=w=c.^1$ Taking this relation into our consideration,

$$\Sigma(w_0 - c_0) \cdot \Sigma\left(\frac{q_1}{q_0}\right) w_0 \ge \Sigma w_0 \cdot \Sigma\left[\frac{q_1}{q_0}(w_0 - c_0)\right],$$

 $Q_L \ge Q_L'$.

or, we can reduce from the above,

$$\frac{\Sigma\left(\frac{q_1}{q_0}\right)c_0}{\Sigma c_0} \gtrless \frac{\Sigma\left(\frac{q_1}{q_0}\right)w_0}{\Sigma w_0}.$$
(6)

The right-hand side of the equation (6) is, as a matter of course, the theoretical Laspeyres formula which is equivalent to the equation (2). Its left-hand side is, on the contrary, the mean of individual production indices weighted with the costs of inputs, which we conveniently denote $Q_{L'}$, i.e.,

$$Q_{L}^{\prime\prime} = \frac{\Sigma\left(\frac{q_{1}}{q_{0}}\right)c_{0}}{\Sigma c_{0}}.$$
(7)

Now we call Q_L'' a production index weighted with the costs of inputs, or shortly, a cost-of-input-weighted index. For the sake of contrast, let us call Q_L and

14

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(5)

¹ v=w—c is gross product. If it is necessary to take such a v as net product, we must add depreciation to c. But the results which come hereafter are the same whether we consider it or not.

 $Q_{L'}$ a theoretical production index and an added-value-weighted production index respectively.

We express (6) in ordinary language; according as the cost-of-input-weighted index, Q_L'' is greater than, equal to, or smaller than the theoretical production index, Q_L , the latter is greater than, equal to, or smaller than added-value-weighted production index, Q_L' .

Since Σw_0 and Σv_0 are both constants,

$$Q_{L} = \frac{\Sigma\left(\frac{q_{1}}{q_{0}}\right)w_{0}}{W_{0}} = \Sigma\left(\frac{q_{1}}{q_{0}}\right)\left(\frac{w_{0}}{W_{0}}\right),$$
$$Q_{L'} = \frac{\Sigma\left(\frac{q_{1}}{q_{0}}\right)v_{0}}{V_{0}} = \Sigma\left(\frac{q_{1}}{q_{0}}\right)\left(\frac{v_{0}}{V_{0}}\right),$$

where $W_0 = \sum w_0$, $V_0 = \sum v_0$. Subtracting the second equation from the first,

$$Q_{L}-Q_{L}'=\Sigma\left(\frac{q_{1}}{q_{0}}\right)\left(\frac{w_{0}}{W_{0}}-\frac{v_{0}}{V_{0}}\right).$$
(8)

If we can assume $w_0/W_0 \ge v_0/V_0$ as to every item, $Q_L \ge Q_L'$, but we cannot generally admit this inequality. It is impossible, then, to determine the unique relationship between Q_L and $Q_{L'}$.

3. Added Value Index

Now, we define our added value index as a measure of the changes over time of value added, but we must pay our attention to the fact that there are two kinds of added value indices, the one quantity index, and the other price index, just as in the ordinary indices. In the present paper, however, added value index always means quantity index of added value. We must, moreover, define the quantity of added value, i. e., real added value. For this purpose, we divide monetary added value by the price of the output in question. Then, added value index is defined as the mean of individual real added value indices weighted with the added value of each commodity in the base period.

The individual added value index of the *i*th commodity is given by

$$\frac{p_1^{(i)}q_1^{(i)}-c_1^{(i)}}{p_1^{(i)}}\Big/\frac{p_0^{(i)}q_0^{(i)}-c_0^{(i)}}{p_0^{(i)}}.$$

Then, added value index V is defined in the Laspeyres type as,

$$V = \frac{\sum \left(\frac{p_1 q_1 - c_1}{p_1} / \frac{p_0 q_0 - c_0}{p_0}\right) v_{\theta}}{\sum v_0}, \tag{9}$$

or, more concisely,

$$V = \frac{\Sigma\left(\frac{p_0}{p_1}\right)(p_1q_1 - c_1)}{\Sigma(p_0q_0 - c_0)}.$$
 (10)

If, in this case,

15

1957]

$$\frac{p_1'}{p_0'} = \frac{p_1''}{p_0''} = \cdots = \frac{p_1^{(n)}}{p_0^{(n)}} = P_{01},$$

we can derive the following relation,

$$V = \frac{\Sigma(p_{1}q_{1}-c_{1})}{P_{01}} / \frac{\Sigma(p_{0}q_{0}-c_{0})}{P_{00}},$$
(11)

where P_{01} is the price index of the Laspeyres type, i. e.,

$$P_{01} \equiv \frac{\Sigma\left(\frac{p_1}{p_0}\right)v_0}{\Sigma v_0}.$$

It is quite clear that $P_{00}=1$. Relation (11) is somewhat important in the customary sense, because we usually get added value index by dividing the real added value in the current period by that in the base period. As is clear, however, from the above mentioned, the ordinary method is correct so far as individual price indices are all equal to one another.

In order to see what the added value index (9) or (10) means, we now define a functional added value index. One of the true added value indices, I, is that which is the ratio of the added value in the current period, A_1 , to that in the base period, A_0 , which has the same price system as the former,² i. e.,

$$I \equiv A_1/A$$

(12)

Let c be the cost of materials and intermediate products and r be such a cost per unit output, then we get $r_0 = c_0/q_0$ in the base period and $r_1 = c_1/q_1$ in the current. The contents of such a cost are understood to be the following,

$$c_{0} = \pi_{0}' x_{0}' + \pi_{0}'' x_{0}'' + \dots + \pi_{0}^{(m)} x_{0}^{(m)} = \sum_{i=1}^{m} \pi_{0} x_{0},$$

$$c_{1} = \pi_{1}' x_{1}' + \pi_{1}'' x_{1}'' + \dots + \pi_{1}^{(m)} x_{1}^{(m)} = \sum_{i=1}^{m} \pi_{1} x_{i},$$

where $x', x'', ..., x^{(m)}$ are quantities of materials and intermediate products used and $\pi', \pi'', ..., \pi^{(m)}$ their respective prices. Further, the monetary value of materials and intermediate products in the current period which have the same prices as those in the base period is, of course,

$$c_{1}' = \pi_{0}' x_{1}' + \pi_{0}'' x_{1}'' + \dots + \pi_{0}^{(m)} x_{1}^{(m)} = \sum_{i=0}^{m} \pi_{0} x_{1}.$$

The r's in the base period and the current are respectively

$$r_{0} = c_{0}/q_{0} = (\tilde{\Sigma}\pi_{0}x_{0})/q_{0},$$

$$r_{1} = c_{1}/q_{1} = (\tilde{\Sigma}\pi_{1}x_{1})/q_{1},$$

and r'_1 is defined as

$$r_1' = c_1'/q_1 = (\Sigma \pi_0 x_1)/q_1$$

A true added value index I will be, according to its definition,

$$I = \frac{A_1}{A_0} = \frac{\tilde{\Sigma}(p_0 - r_1')q_1}{\tilde{\Sigma}(p_0 - r_0)q_0}.$$
 (13)

[October

² We have no intention to say that the definition of the true added value index (12) is a unique one, but we cannot help admitting it very hard to define it in the economic sense, for economic behavior has nothing to do with the maximization of added value as far as an individual firm is concerned.

Let us now take the difference between V of (10) and I of (13), then we get the following relation,

$$V-I = \frac{\sum_{i=1}^{n} \left[\frac{p_{0}}{p_{1}} (p_{1}-r_{1})q_{1} - (p_{0}-r_{1}')q_{1} \right]}{\sum_{i=1}^{n} (p_{0}q_{0}-c_{0})}$$
$$= \frac{\frac{\sum_{i=1}^{n} (r_{1}'-\frac{p_{0}r_{1}}{p_{1}})q_{1}}{\sum_{i=1}^{n} (p_{0}q_{0}-c_{0})}$$
$$= \frac{\sum_{i=1}^{n} (\sum_{i=1}^{n} x_{1}-\frac{p_{0}}{p_{1}} \sum_{i=1}^{n} x_{1}x_{1})}{\sum_{i=1}^{n} (p_{0}q_{0}-c_{0})}.$$
(14)

Since, generally $\sum_{n=0}^{n} (p_0 q_0 - c_0) > 0$,

$$D \equiv \sum_{n=0}^{m} \pi_{0} x_{1} - \frac{p^{0(j)}}{p_{1}^{(j)}} \sum_{n=1}^{m} \pi_{1} x_{1} \ge 0 \quad (j=1, 2, \cdots, n)$$
(15)

in order that $V \ge I$. (15) is a strong condition for this inequality. Rewriting (15),

$$D = \sum_{i=1}^{m} \left(\frac{\pi_{0}}{p_{0}^{(j)}} - \frac{\pi_{1}}{p_{1}^{(j)}} \right) p_{0}^{(j)} x_{1} \ge 0.$$
 (16)

We take, then, the profit function of the *j*th firm. In the base period, the profit function $g_0^{(j)}$ is,

$$g_0^{(j)} = p_0^{(j)} q_0^{(j)} - \sum_{n=0}^{m} \pi_0 x_0 - B_0^{(j)}$$
,
and in the current period, $g_1^{(j)}$ is

$$g_1^{(j)} = p_1^{(j)} q_1^{(j)} - \sum_{k=1}^{m} \pi_1 x_1 - B_1^{(j)}$$

where $B_0^{(j)}$ and $B_1^{(j)}$ are totals of the productive services in the base period and in the current respectively, which we assume to be constant with respect to x's from social point of view. The conditions of maximizing profit are, of course,

$$\begin{bmatrix} \frac{\partial g_{0}^{(j)}}{\partial x_{0}^{(j)}} = p_{0}^{(j)} \frac{\partial q_{0}^{(j)}}{\partial x_{0}^{(i)}} - \pi_{0}^{(i)} = 0, \\ \frac{\partial g_{1}^{(j)}}{\partial x_{1}^{(i)}} = p_{1}^{(j)} \frac{\partial q_{1}^{(j)}}{\partial x_{1}^{(i)}} - \pi_{1}^{(i)} = 0. \end{bmatrix}$$

From the above equations, we get

$$\begin{cases} \frac{\partial q_0^{(j)}}{\partial x_0^{(i)}} = \frac{\pi_0^{(i)}}{p_0^{(j)}}, \\ \frac{\partial q_1^{(j)}}{\partial x_1^{(i)}} = \frac{\pi_1^{(i)}}{p_1^{(j)}}, \end{cases}$$
(17)

 $\partial q_0{}^{(j)}/\partial x_0{}^{(i)}$ and $\partial q_1{}^{(j)}/\partial x_1{}^{(i)}$ are respectively marginal productivities of the *i*th material or intermediate product which are used by the *j*th firm. Some of such productivities in the current period are greater than those in the base period owing to the introduction of new techniques on the one hand, and some of them become nil, on the other, because of the disappearance of materials or intermediate products in the current period which were actually used in the base period.

1957]

Taking this fact into consideration, it is impossible to determine which are greater, marginal productivities in the base period or those in the current. Now we denote $\left(\frac{\pi_0}{p_0^{(j)}} - \frac{\pi_1}{p_1^{(j)}}\right) p_0^{(j)} x_1$ in (16), by *R*, and maximum in *R* by R_{max} , minimum by R_{min} . We, then, get the following relation,

$$mR_{max} \ge \sum_{j=1}^{m} \left(\frac{\pi_{0}}{p_{0}^{(j)}} - \frac{\pi_{1}}{p_{1}^{(j)}}\right) p_{0}^{(j)} x_{1} \ge mR_{min}$$
(18)

The more the development of techniques of production is achieved, the greater the difference between R_{max} and R_{min} becomes. R_{max} will be positive, and R_{min} will be liable to be negative. From inequality (18), we can conclude that D becomes zero or near to it. Even if otherwise, it will not be so great as R_{max} or R_{min} in absolute value. From this conclusion, we are able to say that the added value index V in (9) or (10) is equal to or near to the true added value index V defined by (12) or (13).

4. Changes Over Time of Production Index and Added Value Index

We classify this problem into two, one of which is to get the relation of variations over time of the theoretical production index (1) with those of added value index (10), and the other of which is to consider the relation of variations over time of added-value-weighted production index (3) with those of added value index (10).

A. Theoretical Index and Added Value Index

Let us examine what effect will happen to the index, when the quantity produced of the *i*th item in the current period changes. In order to look at it, we differentiate (1) partially with respect to $q_1^{(i)}$. That is,

$$\frac{\partial Q_L}{\partial q_1^{(i)}} = \frac{\partial}{\partial q_1^{(i)}} \frac{\Sigma\left(\frac{q_1}{q_0}\right)w_0}{\Sigma w_0} = \frac{p_0^{(i)}}{\Sigma w_0}.$$
(19)

In order to see the effect which is caused by a change in $q_1^{(i)}$, we, likewise, differentiate (10) partially with respect to $q_1^{(i)}$. and get

$$\frac{\partial V}{\partial q_1^{(i)}} = \frac{\left(\frac{p_0^{(i)}}{p_1^{(i)}}\right) (p_1^{(i)} - \partial c_1^{(i)} / \partial q_1^{(i)})}{\Sigma(p_0 q_0 - c_0)},$$

In this case, we assume that p and q are independent each other. Such an assumption will be allowed in the system which Prof. R. Frisch called atomistic approach.³ Further- more, marginal productivity $\partial c_1^{(i)}/\partial q_1^{(i)}$ is assumed to be zero from the viewpoint aforementioned. The above equation then becomes,

$$\frac{\partial V}{\partial q_1^{(i)}} = \frac{p_0^{(i)}}{\Sigma(p_0 q_0 - c_0)} = \frac{p_0^{(i)}}{\Sigma v_0}.$$
(20)

⁴ R. Frisch, Annual Survey of General Economic Theory: The Problem of Index Numbers, Econometrica, Jan. 1936, p.-3.

If we multiply $p_0^{(i)}/\Sigma v_0$, the right-hand side of (20) by $\Sigma v_0/\Sigma w_0$, the ratio of total added value to the total value of production, we get the same as the right-hand side of (19). This shows that the effect of any change in q on the theoretical production index is a multiple of its effect on the added value index, the multiplier of which is a constant and just equal to the aggregate income ratio to production, $\Sigma v_0/\Sigma w_0$. This conclusion is important when we make use of such a theoretical production index in case of estimating a dynamic change of added value. B. Changes of Added-value-weighted Production Index and Added Value Index

B. Changes of Added-value-weighted Production Index and Added Value Index The effect of any change in $q_1^{(i)}$ in the added-value-weighted index (3) is given by the partial differentiation of this index with respect to $q_1^{(i)}$. The result is,

$$\frac{\partial Q_L'}{\partial q_1^{(i)}} = \frac{v_0^{(i)}/q_0^{(i)}}{\Sigma v_0}.$$

Since this is an individual effect as to $q_1^{(i)}$, we can get the aggregate effects of all changes in $q_1^{(i)}$ by summing up the right-hand side of the above expression weighted with $q_1^{(i)}$:

$$\frac{\Sigma(v_0^{(i)}/q_0^{(i)} \cdot q_0^{(i)})}{\Sigma v_0} = \frac{\Sigma v_0}{\Sigma v_0} = 1.$$
 (21)

If, on the other hand, we sum up the right-hand side of (20) in the same way, we get the result as follows,

$$\frac{\Sigma p_0^{(i)} q_0^{(i)}}{\Sigma v_0} = \frac{\Sigma w_0}{\Sigma v_0}.$$
(22)

By comparing (21) with (22), we arrive at the conclusion that the aggregate effects of all changes in $q_1^{(i)}$ on the added-value-weighted production index are a multiple of such aggregate effects on the added value index, the multiplier of which is a constant and equal to the aggregate income ratio to production, $\Sigma v_0 / \Sigma w_0$. This result deserves our special attention because we can avail such an added-value-weighted production index for the purpose of estimating changes over time of added value as we usually do.

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