Econometric Theory, **18**, 2002, 1197–1220. Printed in the United States of America.
DOI: 10.1017.S0266466602185082

THE LIMITING PROPERTIES OF THE CANOVA AND HANSEN TEST UNDER LOCAL ALTERNATIVES

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This paper investigates the limiting properties of the Canova and Hansen test, testing for the null hypothesis of no unit root against seasonal unit roots, under a sequence of local alternatives with the model extended to have seasonal dummies and trends or no deterministic term and also only seasonal dummies. We derive the limiting distribution of the test statistic and its characteristic function under local alternatives. We find that the local limiting power is an inverse function of the spectral density at frequency π ($\pi/2$) when we test against a negative unit root (annual unit roots). We also theoretically show that the local limiting power of the Canova and Hansen test against a negative unit root (annual unit roots) does not increase when the true process has annual unit roots (a negative unit root) but not a negative unit root (annual unit roots), which has been observed in Monte Carlo simulations in such research as Caner (1998, *Journal of Business and Economic Statistics* 16, 349–356), Canova and Hansen (1995, *Journal of Business and Economic Statistics* 13, 237–252), and Hylleberg (1995, *Journal of Econometrics* 69, 5–25).

1. INTRODUCTION

This paper deals with the seasonal model with unit roots. Dickey, Hasza, and Fuller (1984) consider the model such as $(1 - B^d)y_t = e_t$ for $d = 1,2,4$, and 12 where B denotes the backshift operator. They develop the asymptotic theory of the least squares estimator of the coefficient associated with y_{t-d}. For the quarterly seasonal model, $(1 - B^4)$ can be decomposed into $(1 + B)(1 + B) \times (1 + B^2)$, and we can consider four roots, ± 1 and $\pm i$ where $i = \sqrt{-1}$. The roots -1 and $\pm i$ are called seasonal unit roots, and we call the root -1 a negative unit root and $\pm i$ annual unit roots.

Tests for seasonal unit roots are considered in the literature. Ahtola and Tiao (1987) and Chan and Wei (1988) investigate the limiting distributions of the least squares estimators of the autoregressive model with complex roots. Using their results, Hylleberg, Engle, Granger, and Yoo (1990) investigate the testing procedure for seasonal unit roots. This test may be seen as an extension of the

We are grateful to anonymous referees, a co-editor, K. Tanaka, and T. Yamamoto for useful comments on the earlier version of the paper. Address correspondence to: Eiji Kurozumi, Department of Economics, Hitotsubashi University, 2-1, Naka, Kunitachi-shi, Tokyo 186-8601, Japan; e-mail: kurozumi@stat.hit-u.ac.jp.

augmented Dickey–Fuller test, whereas Breitung and Franses (1998) consider seasonal unit root tests by extending the test of Phillips and Perron (1988).

As in the preceding literature, many unit root tests consider the test for the null hypothesis of nonstationarity against the alternative of stationarity, and then if the null hypothesis is not rejected, we cannot have much confidence in the existence of a unit root. On the other hand, Kwiatkowski, Phillips, Schmidt, and Shin (1992) consider the test for stationarity against a unit root. Their test is derived as the Lagrange multiplier (LM) test, which is equivalent to the locally best invariant (LBI) test under some conditions. Leybourne and McCabe (1994) also investigate the LBI test for stationarity. The difference between these two tests is that Kwiatkowski et al. (1992) correct autocorrelation nonparametrically as do Phillips and Perron (1988), whereas Leybourne and McCabe (1994) correct it parametrically. It is shown in Leybourne and McCabe (1994) that the former test is consistent in the order T/l where l indicates the lag truncation number used in estimating the long-run variance in a nonparametric way, whereas the latter test is of order T under the alternative hypothesis.

The preceding two tests were generalized to the seasonal model. Canova and Hansen (1995) investigate the testing procedure for the null of stationarity with seasonal dummies against the alternative of seasonal unit roots by extending the Kwiatkowski et al. test. Hylleberg (1995) compares the Canova and Hansen test with the Hylleberg et al. test and concludes that, in a practical analysis, "the best advice is to apply both tests, as they complement each other in several respects." In a similar way as Canova and Hansen (1995), Caner (1998) generalizes the test of Leybourne and McCabe (1994) assuming the parametric structure in serial correlation.

In this paper, we investigate the limiting power properties of the Canova and Hansen test with the model extended to have seasonal dummies and trends or no deterministic term and only seasonal dummies. Because many economic time series seem to be trending variables, inclusion of seasonal trends may be seen as an important model specification. We derive the limiting distribution of the Canova and Hansen test against a negative unit root (a seasonal unit root at frequency π) or against annual unit roots (seasonal unit roots at frequency $\pi/2$) under a sequence of local alternatives using the Fredholm approach, which is extensively developed in Nabeya and Tanaka (1988) and Tanaka (1990a, 1990b, 1996). By deriving the local limiting distribution, we will see that the power of the test depends not only on the local parameter, c, but also on the reciprocal of the spectral density of the stationary component of the time series at frequency π or $\pi/2$. We also derive the characteristic function of the limiting distribution. In addition to its theoretical interest, it can be used to calculate the asymptotic power by the inversion formula as in Tanaka (1996) with high accuracy saving computational time compared with the simulation method. By deriving explicitly the asymptotic power curve, we can see the effects of the departure from the null hypothesis on the power and also the effects of the parameters in the model.

This paper proceeds as follows. In Section 2 we present the model and notation and briefly review the Fredholm approach. In Section 3 we investigate the

Canova and Hansen test for the null of stationarity against the alternative of seasonal unit roots. The limiting distribution and its characteristic function will be derived under a sequence of local alternatives. Section 4 studies the finite sample performance of the test statistic for the model with seasonal dummies and trends. Section 5 concludes the paper. All proofs are given in the Appendix.

2. THE MODEL AND NOTATION

Consider the following seasonal model:

$$y_t = x_t'\beta + r_t + u_t \qquad (t = 1, \ldots, T),$$

$$A_m(B) r_t = \epsilon_t, \quad u_t = \sum_{i=0}^{\infty} a_i v_{t-i}, \quad \text{with } a_0 = 1, \qquad \sum_{i=0}^{\infty} i|a_i| < \infty, \qquad \sum_{i=0}^{\infty} a_i \neq 0 \tag{1}$$

for $m = \pi$ and $\pi/2$, where $A_\pi(B) = 1 + B$ and $A_{\pi/2}(B) = 1 + B^2$, x_t is a deterministic component, and $\{v_t, \epsilon_t\}'$ is jointly independently and identically normally distributed with mean zero and $E[v_t^2] = \sigma^2 > 0$, $E[\epsilon_t^2] = \sigma_\epsilon^2 \geq 0$, and $E[v_t \epsilon_t] = 0$. We set $r_0 = r_{-1} = 0$ and assume that $N = T/4$ is an integer. Note that $\{r_t\}$ has a negative unit root (a seasonal unit root at frequency π) when $m = \pi$ whereas it has annual unit roots (seasonal unit roots at frequency $\pi/2$) when $m = \pi/2$.

Stacking each variable from $t = 1$ to T, we have

$$y = X\beta + r + u, \qquad r = L_m \epsilon,$$

where, e.g., $y' = [y_1, \ldots, y_T]$ and

$$L_m = \begin{bmatrix} Q_{m,0} & & & \mathbf{0} \\ Q_{m,1} & Q_{m,0} & & \\ \vdots & \ddots & \ddots & \\ Q_{m,1} & \cdots & Q_{m,1} & Q_{m,0} \end{bmatrix} \tag{2}$$

for $m = \pi$ and $\pi/2$ with

$$Q_{\pi,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \qquad Q_{\pi,1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},$$

$$Q_{\pi/2,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \qquad Q_{\pi/2,1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Note that L_m^{-1} corresponds to $A_m(B)$. In the following discussion we abbreviate L_m, $Q_{m,0}$, and $Q_{m,1}$ as L, Q_0, and Q_1, respectively, unless there is confusion.

We specify the deterministic term X as follows.

Case A. No deterministic term.

Case B. Seasonal dummies.

$$X = [I_4, \dots, I_4]',$$

where I_j denotes the $j \times j$ identity matrix.

Case C. Seasonal dummies and trends.

$$X = \begin{bmatrix} I_4 & I_4 & \dots & I_4 \\ 1I_4 & 2I_4 & \dots & NI_4 \end{bmatrix}'.$$

It follows that the dimension of β varies according to the definition of X.

Our model (1) is slightly different from that of Canova and Hansen (1995) and Caner (1998). In their model, e.g., the nonstochastic component at frequency π, $\{r_t^*\}$, is defined as

$$r_t^* = \cos(\pi t)\eta_t, \qquad \eta_t = \eta_{t-1} + e_t.$$

Then, assuming $e_t = 0$ for $t \le 0$, r_t^* can be expressed as $r_t^* = (-1)^t \sum_{j=1}^t e_j$, whereas the corresponding component in our model, $(1 + B)r_t = \epsilon_t$, is expressed as $r_t = \sum_{j=1}^t (-1)^{t-j}\epsilon_j$. If $\{e_t\}$ is independently, identically, and symmetrically distributed, such as an independently and identically distributed (i.i.d.) normal distribution, $\{(-1)^t e_t\}$ has the same distribution as $\{e_t\}$, and then the distribution of $\{r_t^*\}$ is the same as $(-1)^t \sum_{j=1}^t (-1)^j e_j = \sum_{j=1}^t (-1)^{t-j} e_j$, which is the same expression as r_t in our model. In this sense, our model may be seen to be equivalent to that in Canova and Hansen (1995) and Caner (1998). However, because the nonstochastic component at frequency π is often defined as $(1 - B)r_t = \epsilon_t$ in the literature, as in Hylleberg et al. (1990) and Breitung and Franses (1998), our definition of the seasonal unit root process directly corresponds to theirs. Similarly, the equivalence between the models with annual unit roots can also be shown.

Here we briefly review the Fredholm approach, which will be used to investigate the limiting properties of the Canova and Hansen test in the next section. Let us consider the quadratic form, $S_T = T^{-1} \sum_{j,k=1}^T K(j/T, k/T) u_j u_k$, where $\{u_t\} \sim i.i.d.(0,1)$ and $K(s,t)$ $(\not\equiv 0)$ is a symmetric, continuous, and nearly definite function. We also consider the integral equation of the second kind,

$$f(t) = \lambda \int_0^1 K(s,t) f(s)\, ds, \tag{3}$$

and denote a sequence of eigenvalues associated with (3) as $\{\lambda_n\}$ and an orthonormal sequence of eigenfunctions as $\{f_n(t)\}$. Using Mercer's theorem, it can be shown that

$$S_T = \frac{1}{T}\sum_{j,k=1}^{T}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n} f_n\left(\frac{j}{T}\right) f_n\left(\frac{k}{T}\right)\right) u_j u_k$$
$$= \sum_{n=1}^{T'}\frac{1}{\lambda_n}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_n\left(\frac{j}{T}\right) u_j\right)^2 + o_p(1)$$

for $T' \to \infty$ as $T \to \infty$. Because $T^{-1/2}\sum_{j=1}^{T} f_n(j/T)u_j$ converges to a standard normal distribution by Lemma 1 of Nabeya and Tanaka (1988), we have $S_T \xrightarrow{d} \sum_{n=1}^{\infty}\zeta_n^2/\lambda_n$ where $\{\zeta_n\} \sim NID(0,1)$ and $\xrightarrow{d}$ signifies convergence in distribution. Then, because the limiting distribution is the weighted sum of independent $\chi^2(1)$ distributions, its characteristic function is given by

$$\lim_{T\to\infty} E[e^{i\theta S_T}] = \left\{\prod_{n=1}^{\infty}\left(1 - \frac{2i\theta}{\lambda_n}\right)\right\}^{-1/2}. \tag{4}$$

Now we introduce the Fredholm determinant of $K(s,t)$. Let us consider $f_T = (\lambda/T)K_T f_T$ as an approximation of the integral equation (3), where $f_T = [f(1/T), f(2/T), \ldots, f(T/T)]'$ and K_T is a $T \times T$ matrix with the (j,k) element $K(j/T, k/T)$. The Fredholm determinant is defined as

$$D(\lambda) = \lim_{T\to\infty}\left| I_T - \frac{\lambda}{T} K_T \right|.$$

Moreover, according to Hochstadt (1973, p. 251), the Fredholm determinant can be expressed as

$$D(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right),$$

and applying this result to the characteristic function (4), we have

$$\lim_{T\to\infty} E[e^{i\theta S_T}] = [D(2i\theta)]^{-1/2}.$$

Then, the characteristic function of the limiting distribution of S_T can be expressed using the Fredholm determinant, and it may be used to calculate the percentage point of the limiting distribution and also the limiting power by Lévy's inversion formula. See, e.g., Hochstadt (1973) for the integral equation and Nabeya and Tanaka (1988) and Tanaka (1996) for its application to the statistical problems.

3. THE LIMITING PROPERTIES OF THE CANOVA AND HANSEN TEST UNDER LOCAL ALTERNATIVES

Let us consider the testing problem

$$H_0 : \rho = 0 \text{ v.s. } H_1^m : \rho = \frac{c^2}{T^2}, \tag{5}$$

where $\rho = \sigma_\epsilon^2/\sigma^2$, c is a constant, and H_1^m ($m = \pi$ and $\pi/2$) denotes the particular alternative with $A_m(B)$, i.e., H_1^π denotes the alternative of a negative unit root, whereas $H_1^{\pi/2}$ denotes that of annual unit roots. Then, (5) signifies the testing problem, the null hypothesis of no unit roots against a sequence of local alternatives of the particular seasonal unit roots.

It can be shown that, for a stylized model in which $u_t = v_t$ is assumed, the LM test for (5) is given by

$$\frac{1}{\tilde{\sigma}^2}(y - X\tilde{\beta})' L_m L_m' (y - X\tilde{\beta}) \tag{6}$$

as rejecting H_0 when (6) takes large values, where $\tilde{\beta}$ and $\tilde{\sigma}^2$ are the maximum likelihood estimators of β and σ^2 under H_0 and given by $\tilde{\beta} = (X'X)^{-1}X'y$ and $\tilde{\sigma}_u^2 = T^{-1}y'My$ with $M = I_T$ for Case A and $M = I_T - X(X'X)^{-1}X'$ for Cases B and C. Note that the LM test (6) or (7) (which follows) for Cases B and C is equivalent to the Canova and Hansen test as shown in the Appendix,[1] and we call (7) the Canova and Hansen test statistic, although the calculation of (7) for Case A is different from Canova and Hansen (1995).

For the general model (1) we define the test statistic as

$$S_T^m = \frac{1}{\hat{q}_m T^2} y' M L_m L_m' M y \tag{7}$$

for $m = \pi$ and $\pi/2$, where

$$\hat{q}_\pi = \sum_{j=-\ell}^{\ell} w(j,\ell)\cos(j\pi)\hat{\gamma}(j), \qquad \hat{q}_{\pi/2} = \frac{1}{4}\sum_{j=-\ell}^{\ell} w(j,\ell)\cos(j\pi/2)\hat{\gamma}(j) \tag{8}$$

are the estimators of the constant multiple of the spectral density of $\{u_t\}$ at frequencies π and $\pi/2$, respectively, where $\hat{\gamma}(j) = T^{-1}\sum_t \tilde{y}_{t+j}\tilde{y}_t$ with $\tilde{y}_t$ regression residuals of y_t on x_t and $w(\cdot,\ell)$ is a kernel function such as Bartlett, Parzen, or quadratic spectral with $\ell = O(T^{1/5})$ as used in Canova and Hansen (1995). Here $\hat{q}_m$ is essentially the same as the diagonal element of $\hat{\Omega}^f = \sum_j w(j,\ell)T^{-1}\sum_t f_{t+j}\tilde{y}_{t+j} f_t'\tilde{y}_t$ in Canova and Hansen (1995), where $f_t = [\cos(t\pi/2), \sin(t\pi/2), \cos(t\pi)]'$. Noting that $\cos((t+j)\pi)\cos(t\pi) = \{\cos((2t+j)\pi) + \cos(j\pi)\}/2 = \cos(j\pi)$, the third diagonal element of $\hat{\Omega}^f$ can be expressed as

$$\begin{aligned}
&\sum_{j=-\ell}^{\ell} w(j,\ell)\frac{1}{T}\sum_t \cos((t+j)\pi)\tilde{y}_{t+j}\cos(t\pi)\tilde{y}_t \\
&\quad = \sum_{j=-\ell}^{\ell} w(j,\ell)\cos(j\pi)\frac{1}{T}\sum_t \tilde{y}_{t+j}\tilde{y}_t,
\end{aligned}$$

which is the same as $\hat{q}_\pi$. Similarly, using the relation of $\cos((t+j)\pi/2)\times \cos(t\pi/2) = \cos(j\pi/2)(1 + (-1)^t)/2$ and $\sin((t+j)\pi/2)\sin(t\pi/2) =$

$\cos(j\pi/2)(1-(-1)^t)/2$, the sum of the first two diagonal elements of $\hat{\Omega}^f$ becomes

$$
\begin{aligned}
&\sum_{j=-\ell}^{\ell} w(j,\ell)\,\frac{1}{T}\sum_t (\cos((t+j)\pi/2)\tilde{y}_{t+j}\cos(t\pi/2)\tilde{y}_t \\
&\qquad\qquad + \sin((t+j)\pi/2)\tilde{y}_{t+j}\sin(t\pi/2)\tilde{y}_t) \\
&\quad = \sum_{j=-\ell}^{\ell} w(j,\ell)\,\frac{1}{T} \\
&\qquad \times \sum_t \left(\frac{1+(-1)^t}{2}\cos(j\pi/2) + \frac{1-(-1)^t}{2}\cos(j\pi/2)\right)\tilde{y}_{t+j}\tilde{y}_t \\
&\quad = \sum_{j=-\ell}^{\ell} w(j,\ell)\cos\frac{j\pi}{2}\,\frac{1}{T}\sum_t \tilde{y}_{t+j}\tilde{y}_t,
\end{aligned}
$$

which is the same as $\hat{q}_{\pi/2}$.

Remark 1. Spectral densities at frequencies π and $\pi/2$ are related to the long-run variance matrix of the annualized process of $\{u_t\}$, which is defined as $\{U_j\} = [u_{4j-3}, u_{4j-2}, u_{4j-1}, u_{4j}]'$ for $j = 1, \ldots, N$. From the definition of u_t in (1), U_j can be expressed as

$$
U_j = \sum_{l=0}^{\infty} A_l V_{j-l}, \quad A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_1 & 1 & 0 \\ a_3 & a_2 & a_1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_4 & a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 & a_2 \\ a_6 & a_5 & a_4 & a_3 \\ a_7 & a_6 & a_5 & a_4 \end{bmatrix}, \ldots,
$$

where V_j is defined as U_j. Then the long-run variance matrix of $\{U_j\}$, Σ, is expressed as

$$
\Sigma = \sigma^2 AA', \quad \text{where } A = \sum_{l=0}^{\infty} A_l = \begin{bmatrix} s_0 & s_3 & s_2 & s_1 \\ s_1 & s_0 & s_3 & s_2 \\ s_2 & s_1 & s_0 & s_3 \\ s_3 & s_2 & s_1 & s_0 \end{bmatrix},
$$

with $s_i = \sum_{j=0}^{\infty} a_{4j+i}$ for $i = 0$, 1, 2, and 3. Direct calculations show that $P'\Sigma P = \sigma^2 \operatorname{diag}\{(s_0+s_1+s_2+s_3)^2, (s_0-s_1+s_2-s_3)^2, (s_0-s_2)^2 + (s_1-s_3)^2, (s_0-s_2)^2 + (s_1-s_3)^2\}$, where

$$
P = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & -\sqrt{2} \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & 1 & \sqrt{2} & 0 \end{bmatrix}, \tag{9}
$$

and, because $f(\omega) = (2\pi)^{-1}\sigma^2|\sum_{j=0}^{\infty} a_j e^{ij\omega}|^2$, we have

$$\sigma_0^2 \equiv 2\pi f(0) = \sigma^2(s_0 + s_1 + s_2 + s_3)^2,$$

$$\sigma_\pi^2 \equiv 2\pi f(\pi) = \sigma^2(s_0 - s_1 + s_2 - s_3)^2,$$

$$\sigma_{\pi/2}^2 \equiv 2\pi f(\pi/2) = \sigma^2\{(s_0 - s_2)^2 + (s_1 - s_3)^2\}.$$

Thus, the diagonal elements of $P'\Sigma P$ consist of 2π times spectral densities at frequencies 0, π, and $\pi/2$. Notice that Σ appears in the Canova and Hansen test because we can easily show that Ω^f in Canova and Hansen (1995) is equal to $R_1'\Sigma R_1/4 = \text{diag}\{\sigma_{\pi/2}^2/2, \sigma_{\pi/2}^2/2, \sigma_\pi^2\}$ where $R_1 = [\,f_1, f_2, f_3, f_4]'$.

Because S_T^m is equivalent to the Canova and Hansen test, our purpose is to investigate the limiting behavior of the Canova and Hansen test under a sequence of local alternatives.

The following theorem gives the limiting distribution of S_T^m under H_1^m and its characteristic function for $m = \pi$ and $\pi/2$. For the expression of the characteristic function we use the following Fredholm determinants:

$$D_A(\lambda) = \cos\sqrt{\lambda},$$

$$D_B(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}},$$

$$D_C(\lambda) = \frac{12}{\lambda^2}\,(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}),$$

associated with the kernels

$$K_A(s,t) = 1 - \max(s,t),$$

$$K_B(s,t) = \min(s,t) - st,$$

$$K_C(s,t) = \min(s,t) - 4st + 3st(s+t) - 3s^2t^2.$$

See, e.g., Theorem 6 of Nabeya and Tanaka (1988) and equations (5.34) and (9.94) of Tanaka (1996). We denote a sequence of eigenvalues associated with $K_j(s,t)$ as $\{\lambda_{jn}\}$ for $j = A$, B, and C. Note that every zero of $D_j(\lambda)$ is an eigenvalue of K_j for each j. For example, the eigenvalues of K_A are $((n - \frac{1}{2})\pi)^2$ and those of K_B are $n^2\pi^2$ for $n = 1,2,\ldots$, whereas we cannot explicitly express the eigenvalues of K_C.

THEOREM 1.

(i) Under H_1^π,

$$S_T^\pi \xrightarrow{d} \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{jn}} + \frac{c^2\sigma^2/\sigma_\pi^2}{\lambda_{jn}^2}\right)\zeta_n^2,$$

and its characteristic function, $\phi_\pi(\theta;c)$, is given by

$$\phi_\pi(\theta;c) = \left[D_j(i\theta + \sqrt{-\theta^2 + 2ic^2\sigma^2\theta/\sigma_\pi^2}) \times D_j(i\theta - \sqrt{-\theta^2 + 2ic^2\sigma^2\theta/\sigma_\pi^2})\right]^{-1/2}$$

for $j = A, B$, and C according to Cases A, B, and C, respectively, where $\{\zeta_n\} \sim NID(0,1)$ and $\{\lambda_{jn}\}$ and $D_j(\cdot)$ are a sequence of eigenvalues and the Fredholm determinant, respectively.

(ii) Under $H_1^{\pi/2}$,

$$S_T^{\pi/2} \xrightarrow{d} \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{jn}} + \frac{c^2\sigma^2}{4\sigma_{\pi/2}^2\lambda_{jn}^2}\right)\zeta_{1,n}^2 + \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{jn}} + \frac{c^2\sigma^2}{4\sigma_{\pi/2}^2\lambda_{jn}^2}\right)\zeta_{2,n}^2,$$

and its characteristic function, $\phi_{\pi/2}(\theta;c)$, is given by

$$\phi_{\pi/2}(\theta;c) = \left[D_j(i\theta + \sqrt{-\theta^2 + ic^2\sigma^2\theta/(2\sigma_{\pi/2}^2)}) \times D_j(i\theta - \sqrt{-\theta^2 + ic^2\sigma^2\theta/(2\sigma_{\pi/2}^2)})\right]^{-1}$$

for $j = A, B$, and C according to Cases A, B, and C, respectively, where $\{\zeta_{1,n}\}$ and $\{\zeta_{2,n}\}$ are independent and $NID(0,1)$ and $\{\lambda_{jn}\}$ and $D_j(\cdot)$ are a sequence of eigenvalues and the Fredholm determinant, respectively.

Remark 2. Under H_0, $c = 0$ so that

$$S_T^{\pi} \xrightarrow{d} \sum_{n=1}^{\infty}\frac{1}{\lambda_{jn}}\zeta_n^2, \qquad S_T^{\pi/2} \xrightarrow{d} \sum_{n=1}^{\infty}\frac{1}{\lambda_{jn}}\zeta_{1,n}^2 + \sum_{n=1}^{\infty}\frac{1}{\lambda_{jn}}\zeta_{2,n}^2, \tag{10}$$

and their characteristic functions are given by

$$\phi_\pi(\theta;0) = D_j(2i\theta)^{-1/2}, \qquad \phi_{\pi/2}(\theta;0) = D_j(2i\theta)^{-1} \tag{11}$$

for $j = A$, B, and C, respectively.

From the preceding theorem, we can obtain the null distribution function by inverting the characteristic function (11). In general when the nonnegative random variable Y has the characteristic function $\phi(\theta)$, we have, using Lévy's inversion formula,

$$P(Y \le x) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{1 - e^{-i\theta x}}{i\theta}\phi(\theta)\right]d\theta.$$

Table 1 shows percentage points of the limiting null distributions of S_T^π and $S_T^{\pi/2}$, which are calculated by numerical integration. From each table, we can see that the limiting null distribution of S_T^π is located to the left compared with that of $S_T^{\pi/2}$. On the other hand, the more complicated the deterministic term is, the further the limiting null distribution of S_T^m is shifted to the left for a fixed m. For example, the 95% point of S_T^π is 1.66, 0.46, and 0.15 for Cases A, B, and C, respectively.

TABLE 1. Percentage points of limiting null distributions

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
				S_T^{π}			
Case A	0.034460	0.056460	0.076536	0.290476	1.195820	1.655739	2.787459
Case B	0.024798	0.036562	0.046015	0.118880	0.347305	0.461361	0.743459
Case C	0.017269	0.023409	0.027886	0.055548	0.119220	0.147890	0.217747
				$S_T^{\pi/2}$			
Case A	0.126913	0.199049	0.260318	0.757496	2.062210	2.624054	3.928615
Case B	0.078830	0.109425	0.132220	0.277571	0.607037	0.747520	1.073664
Case C	0.049118	0.062648	0.071843	0.120873	0.210670	0.246535	0.328622

We also calculate the limiting power using the upper 5% points in Table 1. Note that from Theorem 1 the asymptotic local power only depends on the model parameters through the ratio $c^2\sigma^2/\sigma_m^2$. To illustrate how these parameters affect the power of the test, we consider the case when $\{u_t\}$ obeys an AR(1) process, $u_t = au_{t-1} + v_t$ with $\{v_t\} \sim NID(0,1)$. In this case, $c^2\sigma^2/\sigma_m^2 = c^2/\sigma_m^2$, and we can easily check that $\sigma_\pi^2 = (1+a)^{-2}$ and $\sigma_{\pi/2}^2 = (1+a^2)^{-1}$. Using these relations, we can calculate the limiting power as a function of c for a fixed a.

Figures 1a–c are the limiting powers of S_T^{π} for Cases A, B, and C for $a = -0.8$, -0.4, 0.0, 0.4, and 0.8, which correspond to $\sigma_\pi^2 = 25$, 2.78, 1, 0.51, and 0.31, respectively. In each case, the closer to -1 the value of a becomes, the less powerful is the test statistic. Intuitively this may be explained as follows: when $a = -0.8$, $u_t = -0.8u_{t-1} + v_t$ and this process is difficult to distinguish from the negative unit root process, $u_t = -u_{t-1} + v_t$, so that the power increases slowly as a function of c compared with the other cases such as $a = 0.8$. We also note that, for a fixed value of a, the power of Case A (Figure 1a) dominates that of Case B (Figure 1b) and the latter dominates that of Case C (Figure 1c). That is, the more complicated the deterministic term becomes, the less powerful is the test.

Figures 2a–c are the limiting powers of $S_T^{\pi/2}$. In this case, $\sigma_{\pi/2}^2 = 1$, 0.86, and 0.61 for $a = 0$, ± 0.4, and ± 0.8, respectively. From the figures, the power function corresponding to the larger absolute value of a dominates that corresponding to the smaller absolute value of a. On the other hand, the relation between the power and the deterministic term is the same as in Figures 1a–c.

Next, we investigate the limiting properties of S_T^m under H_1^i for $i, m = \pi, \pi/2$ $(i \neq m)$; i.e., we examine the asymptotic behavior of S_T^m when the true process has seasonal unit roots different from those we assumed under the alternative.

COROLLARY 1. *S_T^{π} and $S_T^{\pi/2}$ converge in distribution to the null distributions (10) under $H_1^{\pi/2}$ and H_1^{π}, respectively.*

The preceding corollary indicates that the local limiting power of S_T^{π} $(S_T^{\pi/2})$ under $H_1^{\pi/2}$ (H_1^{π}) does not increase from the significance level, i.e., the test

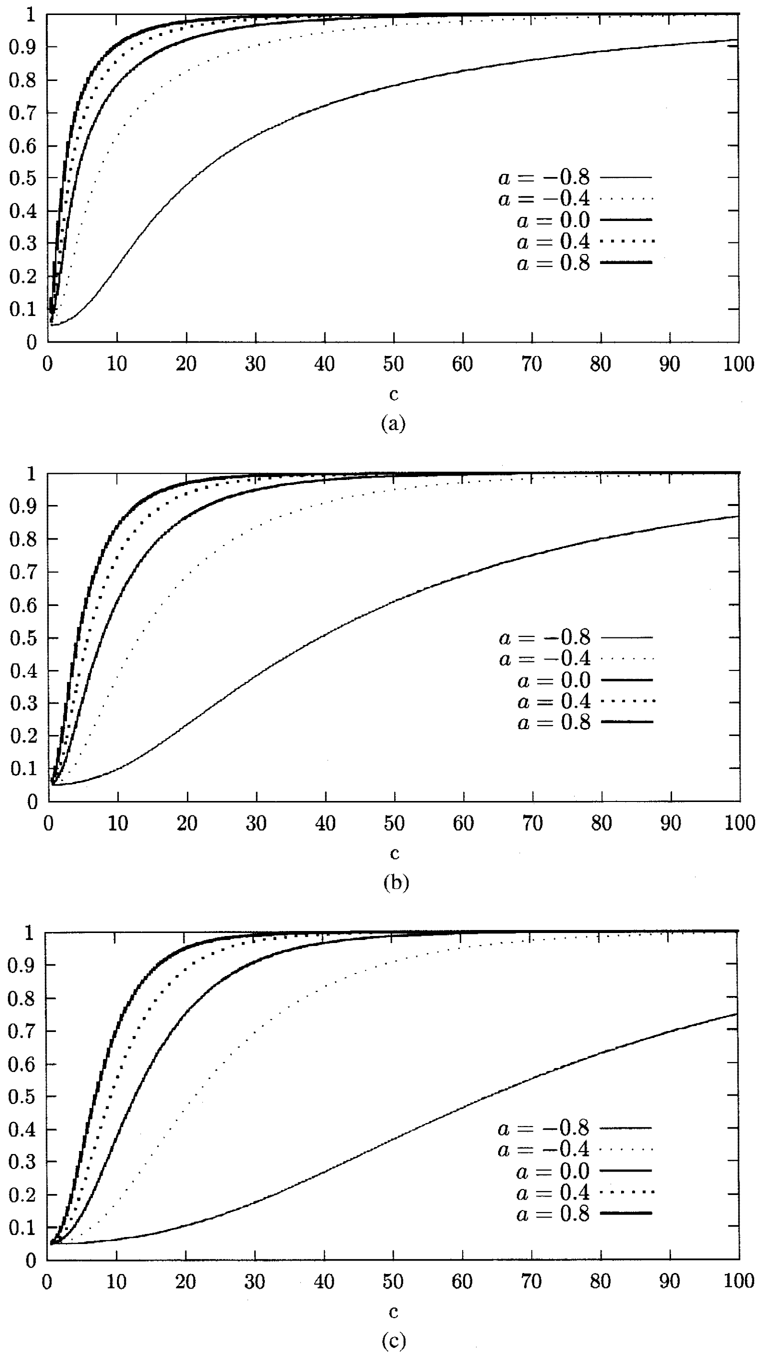

FIGURE 1. The limiting powers of S_T^{π}. (a) Case A; (b) Case B; (c) Case C.

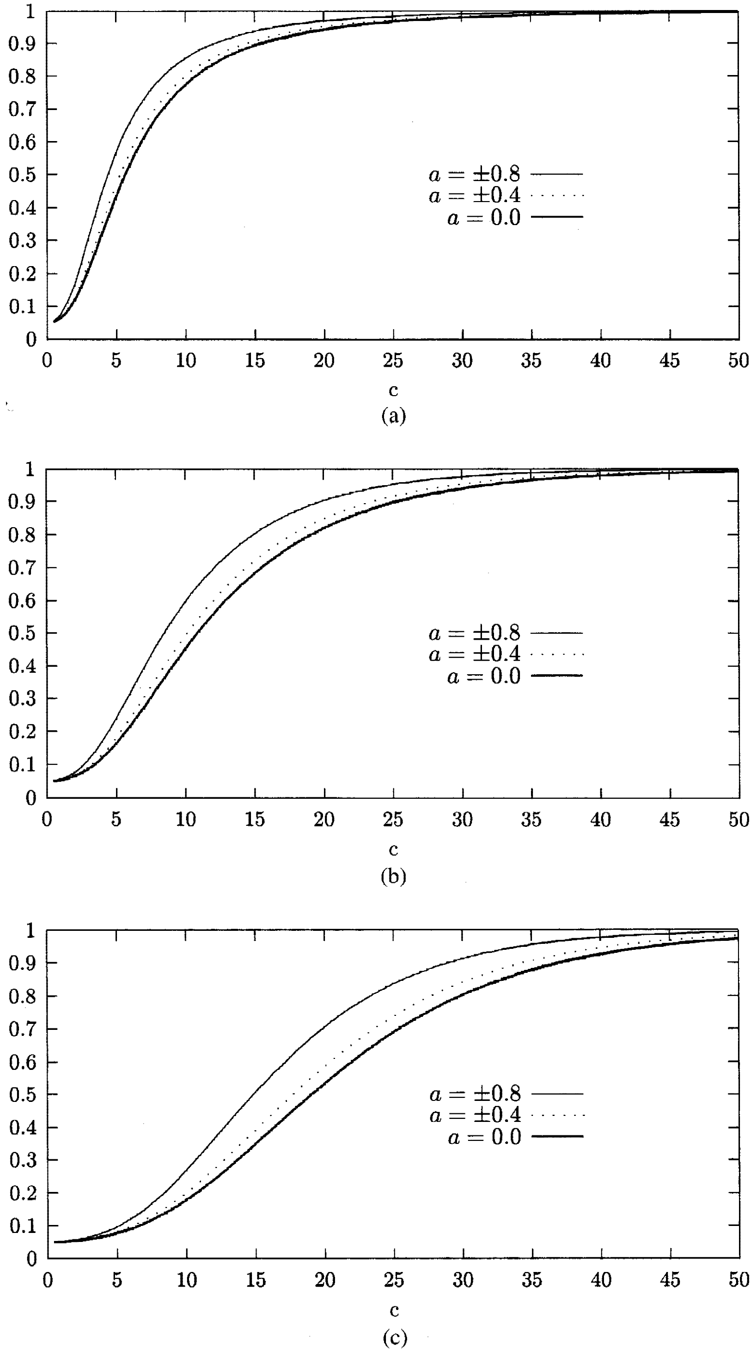

FIGURE 2. The limiting powers of $S_T^{\pi/2}$. (a) Case A; (b) Case B; (c) Case C.

statistic against a negative unit root (annual unit roots) has only trivial power under the alternative of annual unit roots (a negative unit root). This tendency has been observed in Caner (1998), Canova and Hansen (1995), and Hylleberg (1995) by Monte Carlo simulations, and our investigation of the power function supports their results theoretically. We can also show that both S_T^{π} and $S_T^{\pi/2}$ also have only trivial asymptotic power under the alternative of a unit root in the same way as Corollary 1, although we do not prove it to save space.

4. FINITE SAMPLE PERFORMANCE

In this section we investigate the finite sample performance of the test statistics in the previous section for Case C. See Caner (1998), Canova and Hansen (1995), and Hylleberg (1995) for the model with seasonal dummies.

We consider the following model as the data generating process (DGP):

$$y_t = x_t'\beta + r_t + u_t, \qquad A_m(B)r_t = \epsilon_t, \qquad u_t = au_{t-1} + v_t, \tag{12}$$

where $A_m = (1 + B)$ and $(1 + B^2)$ for $m = \pi$ and $\pi/2$, respectively, $\{\epsilon_t\} \sim NID(0,\rho)$, $\{v_t\} \sim NID(0,1)$, and they are independent. Because S_T^m is invariant to β, we set $\beta = 0$ without loss of generality. In the simulation study, we set $a = 0$, ± 0.4, ± 0.8 and $\rho = 0$ and 0.1, and the sample size is $T = 50$ and 150. For the estimation of q_m, we use the Bartlett kernel and select $\ell = \ell 1 = 3$ when $T = 50$ and $\ell = \ell 1 = 5$ when $T = 150$ as in Canova and Hansen (1995). We also consider the case when the longer lag truncation parameter is selected ($\ell = \ell 2 = 6$ when $T = 50$ and $\ell = \ell 2 = 8$ when $T = 150$). The number of replications is 1,000 in all experiments, and the level of significance is set equal to 0.05.

We also report the results of the Hylleberg et al. test for comparison. Consider the following regression:

$$\phi(B)y_{4t} = \psi_1 y_{1t-1} + \psi_2 y_{2t-1} + \psi_3 y_{3t-1} + \psi_4 y_{3t-2} + x_t'\beta + e_t,$$

where $y_{1t} = (1 + B + B^2 + B^3)y_t$, $y_{2t} = -(1 - B + B^2 - B^3)y_t$, $y_{3t} = -(1 - B^2)y_t$, $y_{4t} = (1 - B^4)y_t$, x_t consists of seasonal dummies and trends, and $\phi(B)$ is selected as the sixth-order lag polynomial as in Canova and Hansen (1995). The Hylleberg et al. test rejects the null of a negative unit root when the t-statistic for $\psi_2 = 0$ is small and rejects the null of annual unit roots when the F-statistic for $\psi_3 = \psi_4 = 0$ is large. Critical values of the Hylleberg et al. test when x_t contains seasonal dummies and trends are given in Table 1 of Smith and Taylor (1998).

Table 2 reports the case when $\rho = 0$, and then the entries in Table 2 are the size of the Canova and Hansen test and the power of the Hylleberg et al. test. As in the table, S_T^{π} tends to overreject the null hypothesis of no unit root when a goes to -1, whereas there is a tendency of underrejection for $S_T^{\pi/2}$ when $|a|$ becomes large. The power performance of the Hylleberg et al. test for annual

TABLE 2. The size and power of the Canova and Hansen test and the Hylleberg et al. test ($\rho = 0$)

	a	S_T^{π} ($\ell 1$)	S_T^{π} ($\ell 2$)	HEGY (π)	$S_T^{\pi/2}$ ($\ell 1$)	$S_T^{\pi/2}$ ($\ell 2$)	HEGY[a] ($\pi/2$)
$T = 150$	0.8	0.009	0.010	0.904	0.004	0.010	0.994
	0.4	0.026	0.027	0.881	0.031	0.034	0.975
	0	0.042	0.043	0.823	0.051	0.053	0.956
	−0.4	0.075	0.060	0.717	0.033	0.036	0.974
	−0.8	0.292	0.166	0.349	0.002	0.005	0.993
$T = 50$	0.8	0.005	0.013	0.149	0.009	0.021	0.163
	0.4	0.009	0.015	0.116	0.025	0.043	0.114
	0	0.027	0.029	0.080	0.045	0.061	0.090
	−0.4	0.082	0.049	0.068	0.032	0.053	0.104
	−0.8	0.303	0.133	0.059	0.009	0.016	0.162

[a]Hylleberg, Engle, Granger, and Yoo (1990).

unit roots seems good when $T = 150$, and so does the test for a negative unit root except for the case when $a = -0.8$, although they have very low power when the sample size is 50.

Table 3 shows the case when $\rho = 0.1$. The power of the Canova and Hansen test against a negative unit root seems to be affected by the value of a, whereas that of the test against annual unit roots is not greatly affected. As discussed in Corollary 1, the rejection frequency of S_T^{π} ($S_T^{\pi/2}$) does not increase when the DGP has only annual unit roots (a negative unit root). We can also see that the selection of the longer lag truncation parameter entails the reduction of the power, although the size of the test tends to be stable. For the Hylleberg et al. test, the case when $\rho = 0.1$ corresponds to the null hypothesis, and the rejection frequency of the Hylleberg et al. test in Table 3 is the empirical size. The rejection frequency of the Hylleberg et al. test for a negative unit root (annual unit roots) is very high when the DGP has annual unit root (a negative unit root) and $T = 150$, although some other cases have empirical size very close to the nominal size. Similar results have been already obtained by Canova and Hansen (1995) for the case without seasonal trends.

5. CONCLUDING REMARKS

We have investigated the Canova and Hansen test for the null hypothesis of no seasonal unit root against the alternative hypothesis of seasonal unit roots. Our analysis shows that the local asymptotic power depends only on the ratio of the squared local alternative to the spectral density at the tested frequency. We also showed that the local limiting power of the Canova and Hansen test against a

TABLE 3. The size and power of the Canova and Hansen test and the Hylleberg et al. test ($\rho = 0.1$)

	a	S_T^{π} $(\ell 1)$	S_T^{π} $(\ell 2)$	HEGY (π)	$S_T^{\pi/2}$ $(\ell 1)$	$S_T^{\pi/2}$ $(\ell 2)$	HEGY[a] $(\pi/2)$
				$r_t = -r_{t-1} + \epsilon_t$			
$T = 150$	0.8	0.837	0.708	0.043	0.000	0.005	0.992
	0.4	0.829	0.703	0.044	0.011	0.019	0.982
	0	0.808	0.680	0.056	0.015	0.036	0.970
	−0.4	0.730	0.621	0.107	0.012	0.020	0.985
	−0.8	0.605	0.462	0.155	0.000	0.007	0.996
$T = 50$	0.8	0.385	0.275	0.057	0.013	0.026	0.131
	0.4	0.375	0.274	0.055	0.025	0.046	0.100
	0	0.337	0.230	0.048	0.047	0.062	0.100
	−0.4	0.274	0.167	0.054	0.022	0.041	0.137
	−0.8	0.345	0.156	0.050	0.012	0.020	0.164
				$r_t = -r_{t-2} + \epsilon_t$			
$T = 150$	0.8	0.004	0.007	0.899	0.820	0.792	0.186
	0.4	0.013	0.020	0.897	0.866	0.810	0.189
	0	0.026	0.029	0.894	0.873	0.800	0.166
	−0.4	0.058	0.053	0.820	0.879	0.819	0.160
	−0.8	0.282	0.162	0.452	0.832	0.803	0.164
$T = 50$	0.8	0.003	0.009	0.133	0.179	0.202	0.076
	0.4	0.016	0.018	0.120	0.232	0.235	0.057
	0	0.026	0.029	0.091	0.288	0.247	0.056
	−0.4	0.077	0.051	0.073	0.237	0.233	0.072
	−0.8	0.309	0.121	0.070	0.188	0.186	0.092

[a]Hylleberg, Engle, Granger, and Yoo (1990).

negative unit root (annual unit roots) does not increase when the true process has annual unit roots (a negative unit root).

NOTE

1. We thank a co-editor who pointed out this relation. Part of the proof is due to him.

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APPENDIX

LEMMA A. *Suppose that the statistic S_N is defined by*

$$S_N = \frac{1}{N} z' B_N z + \frac{\gamma}{N^2} z' B_N^2 z = \frac{1}{N} \sum_{j,k=1}^{N} \left(B_N(j,k) + \frac{\gamma}{N} \sum_{l=1}^{N} B_N(j,l) B_N(l,k) \right) z_j z_k, \tag{A.1}$$

where $z = [z_1, \dots, z_N]'$, $\{z_t\} \sim i.i.d.(0,1)$, and B_N satisfies

$$\lim_{N\to\infty} \max_{j,k} \left| B_N(j,k) - K\left(\frac{j}{N}, \frac{k}{N}\right) \right| = 0,$$

with $K(s,t)$ ($\not\equiv 0$) a symmetric, continuous, and nearly definite function. Then,

$$S_N \xrightarrow{d} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} + \frac{\gamma}{\lambda_n^2} \right) \zeta_n^2,$$

where $\{\zeta_n\} \sim NID(0,1)$ *and* $\{\lambda_n\}$ *is a sequence of eigenvalues of K repeated as many times as their multiplicities, with its characteristic function is given by*

$$\lim_{N\to\infty} E(e^{i\theta S_N}) = \prod_{n=1}^{\infty}\left[1 - 2i\theta\left(\frac{1}{\lambda_n} + \frac{\gamma}{\lambda_n^2}\right)\right]^{-1/2}$$
$$= [D(i\theta + \sqrt{-\theta^2 + 2i\gamma\theta})D(i\theta - \sqrt{-\theta^2 + 2i\gamma\theta})]^{-1/2},$$

where $D(\lambda)$ *is the Fredholm determinant of K.*

Proof. See Theorem 5.13 of Tanaka (1996, p. 175).

Proof of equivalence of S_T^m to the Canova and Hansen test. The Canova and Hansen test is defined as

$$S_{CH}^m = \frac{1}{T^2}\sum_{t=1}^{T} \hat{F}_t' \Upsilon_m (\Upsilon_m \hat{\Omega}^f \Upsilon_m)^{-1} \Upsilon_m \hat{F}_t$$

for $m = \pi$ and $\pi/2$, where $\hat{F}_t = \sum_{j=1}^t f_j \tilde{y}_j$ with $f_j = [\cos(j\pi/2), \sin(j\pi/2), \cos(j\pi)]'$ and $\Upsilon_\pi = \text{diag}\{0,0,1\}$ when testing against a negative unit root and $\Upsilon_{\pi/2} = \text{diag}\{1,1,0\}$ when testing against annual unit roots.

For $m = \pi$, we have

$$S_T^\pi = \frac{1}{\hat{q}_\pi T^2}\sum_{t=1}^{T}\left(\sum_{j=t}^{T}(-1)^{j-t}\tilde{y}_j\right)^2$$
$$= \frac{1}{\hat{q}_\pi T^2}\sum_{t=1}^{T}\left(\sum_{j=1}^{t}(-1)^{j-t}\tilde{y}_j\right)^2$$
$$= \frac{1}{\hat{q}_\pi T^2}\sum_{t=1}^{T}\left(\sum_{j=1}^{t}(-1)^{j}\tilde{y}_j\right)^2,$$

which is the same as the Canova and Hansen test because $\hat{q}_\pi = \Upsilon_\pi' \hat{\Omega}^f \Upsilon_\pi$ as shown in Section 3. The second equality is established because $\sum_{j=1}^T (-1)^{j-t}\tilde{y}_j = 0$, which holds because $\tilde{y}_t$ are regression residuals of y_t on x_t, seasonal dummies.

Similarly for $m = \pi/2$, we have

$$S_T^{\pi/2} = \frac{1}{\hat{q}_{\pi/2} T^2}\sum_{t=1}^{T/2}\left\{\left(\sum_{j=t}^{T/2}(-1)^{j-t}\tilde{y}_{2j-1}\right)^2 + \left(\sum_{j=t}^{T/2}(-1)^{j-t}\tilde{y}_{2j}\right)^2\right\}$$
$$= \frac{1}{\hat{q}_{\pi/2} T^2}\sum_{t=1}^{T/2}\left\{\left(\sum_{j=1}^{t}(-1)^{j}\tilde{y}_{2j-1}\right)^2 + \left(\sum_{j=1}^{t}(-1)^{j}\tilde{y}_{2j}\right)^2\right\},$$

which is equivalent to the Canova and Hansen test if we use $4\hat{q}_{\pi/2}$ for the construction of the Canova and Hansen test as the spectral density estimator at frequency $\pi/2$ with information that the off-diagonal elements of Ω^f are zero, i.e., if we use $\Upsilon_{\pi/2}\hat{\Omega}^f\Upsilon_{\pi/2} = \text{diag}\{2\hat{q}_{\pi/2}, 2\hat{q}_{\pi/2}\}$. ■

Proof of Theorem 1. In the proof we omit the subscript m unless there is confusion. First we show that $\hat{q}_\pi$ is consistent under H_1^π for Case B. Because $\tilde{y}_t = \tilde{u}_t + \tilde{r}_t$ where $\tilde{u}_t$ and $\tilde{r}_t$ denote regression residuals of u_t and r_t on x_t, respectively, $\hat{q}_\pi$ can be expressed as

$$\hat{q}_\pi = \sum_{j=-\ell}^{\ell} w(j,\ell)\cos(j\pi)\frac{1}{T}\sum_t \tilde{u}_{t+j}\tilde{u}_t$$
$$+ \sum_{j=-\ell}^{\ell} w(j,\ell)\cos(j\pi)\frac{1}{T}\sum_t (\tilde{u}_{t+j}\tilde{r}_t + \tilde{u}_t\tilde{r}_{t+j} + \tilde{r}_{t+j}\tilde{r}_t). \qquad \textbf{(A.2)}$$

Because the first term converges to σ_π^2 in probability as discussed in Canova and Hansen (1995), it is enough to show that the second term of (A.2) converges to zero in probability.

Noting that $\tilde{r}_t = r_t - x_t(\sum_j x_j x_j')^{-1}\sum_j x_j r_j$ we have

$$E[\tilde{r}_t^2] \le 2E\left[r_t^2 + \sum_{j=1}^{T} r_j x_j'\left(\sum_{j=1}^{T} x_j x_j'\right)^{-1} x_t x_t'\left(\sum_{j=1}^{T} x_j x_j'\right)^{-1}\sum_{j=1}^{T} x_j r_j\right]. \qquad \textbf{(A.3)}$$

Under H_1^π $r_t = \sum_{j=1}^{t}(-1)^{t-j}\epsilon_j$ where $\epsilon_j \sim NID(0, c^2\sigma^2/T^2)$ because $\rho = c^2/T^2$, and then

$$E[r_t^2] = \frac{c^2\sigma^2 t}{T^2} = O(T^{-1}) \qquad \textbf{(A.4)}$$

for all t. Similarly, because x_t is seasonal dummies for Case B and $\sum x_j x_j' = N \times I_4$ because $T = 4N$, we have

$$\begin{aligned}
&E\left[\sum_{j=1}^{T} r_j x_j'\left(\sum_{j=1}^{T} x_j x_j'\right)^{-1} x_t x_t'\left(\sum_{j=1}^{T} x_j x_j'\right)^{-1}\sum_{j=1}^{T} x_j r_j\right]\\
&\quad = \frac{1}{N^2}E\left[\sum_{j=1}^{T} r_j x_j' x_t x_t' \sum_{j=1}^{T} x_j r_j\right]\\
&\quad \le \frac{1}{N^2}E\left[\left(\sum_{j=1}^{N} r_{4j-3}\right)^2 + \left(\sum_{j=1}^{N} r_{4j-2}\right)^2 + \left(\sum_{j=1}^{N} r_{4j-1}\right)^2 + \left(\sum_{j=1}^{N} r_{4j}\right)^2\right]\\
&\quad \le \frac{1}{N^2}E\left[N\sum_{j=1}^{N} r_{4j-3}^2 + N\sum_{j=1}^{N} r_{4j-2}^2 + N\sum_{j=1}^{N} r_{4j-1}^2 + N\sum_{j=1}^{N} r_{4j}^2\right]\\
&\quad = \frac{1}{N}\sum_{j=1}^{T} E[r_j^2] = \frac{1}{N}\sum_{j=1}^{T}\frac{c^2\sigma^2 j}{T^2} = O(T^{-1}) \qquad \textbf{(A.5)}
\end{aligned}$$

for all t. Then by (A.3), (A.4), and (A.5), $E[\tilde{r}_t^2] = O(T^{-1})$ for all t and by the Cauchy–Schwarz inequality,

$$\left|E\left[\frac{1}{T}\sum_{t=1}^{T-j}\tilde{r}_{t+j}\tilde{r}_t\right]\right| \le \frac{1}{T}\sum_{t=1}^{T-j} E[\tilde{r}_{t+j}^2]^{1/2}E[\tilde{r}_t^2]^{1/2} = O(T^{-1}),$$

so that $T^{-1}\sum_t \tilde{r}_{t+j}\tilde{r}_t$ is $O_p(T^{-1})$ for a fixed j. Because the range of j is $-\ell$ to ℓ and $\ell = O_p(T^{1/5})$, we have

$$\sum_{j=-\ell}^{\ell} w(j,\ell)\cos(j\pi)\frac{1}{T}\sum_t \tilde{r}_{t+j}\tilde{r}_t \xrightarrow{p} 0, \qquad \textbf{(A.6)}$$

where $\xrightarrow{p}$ denotes convergence in probability.

Similarly $T^{-1}\sum_t \tilde{u}_{t+j}\tilde{r}_t$ is $O_p(T^{-1/2})$ for fixed j because $|E[\tilde{u}_{t+j}\tilde{r}_t]| \leq E[\tilde{u}_{t+j}^2]^{1/2}E[\tilde{r}_t^2]^{1/2} = O(T^{-1/2})$, and then we have

$$\sum_{j=-\ell}^{\ell} w(j,\ell)\cos(j\pi)\frac{1}{T}\sum_t \tilde{u}_{t+j}\tilde{r}_t \xrightarrow{p} 0. \tag{A.7}$$

With (A.6) and (A.7), we conclude that $\hat{q}_\pi$ converges to σ_π^2 in probability under local alternatives.

In a similar way convergence for Cases A and C and convergence of $\hat{q}_{\pi/2}$ to $\sigma_{\pi/2}^2/4$ can be proved. According to this result, we can investigate the limiting distribution of S_T^m with q_m, the probability limit of $\hat{q}_m$, instead of $\hat{q}_m$.

Hereafter, we will use the vectorized expression such as u or the annualized vector series such as $\{U_j\}$, not the original series $\{u_t\}$. Notice the relation that $u = [U_1', \ldots, U_N']$ because we assume $N = T/4$ is an integer.

Here we decompose the annualized process $\{U_j\}$ into three terms. By the Beveridge–Nelson decomposition, we have

$$U_j = AV_j + V_{j-1}^* - V_j^*,$$

where $V_j^* = \sum_{i=0}^{\infty} A_i^* V_{j-i}$ with $A_i^* = \sum_{l=i+1}^{\infty} A_l$ and in the vectorized form,

$$u = (I_N \otimes A)v + v_{-1}^* - v^*,$$

where, e.g., $v^* = [V_1^{*\prime}, \ldots, V_N^{*\prime}]'$. Then, we have

$$y = X\beta + L\epsilon + (I_N \otimes A)v + v_{-1}^* - v^*,$$

so that

$$\begin{aligned} S_T^m &= \frac{1}{q_m T^2} y'MLL'My \\ &= \frac{1}{q_m T^2}\{L\epsilon + (I_N \otimes A)v\}'MLL'M\{L\epsilon + (I_N \otimes A)v\} \\ &\quad + \frac{2}{q_m T^2}\{L\epsilon + (I_N \otimes A)v\}'MLL'M(v_{-1}^* - v^*) \\ &\quad + \frac{1}{q_m T^2}(v_{-1}^* - v^*)'MLL'M(v_{-1}^* - v^*). \end{aligned} \tag{A.8}$$

Here we show that the last two terms are $o_p(1)$. Let us consider Case B, where $X = [I_4, \ldots, I_4]'$. Note that, in the annualized vector form, the typical jth block of $M(v_{-1}^* - v^*)$ can be expressed as

$$V_{j-1}^* - \frac{1}{N}\sum_{j=1}^{N} V_{j-1}^* - \left(V_j^* - \frac{1}{N}\sum_{j=1}^{N} V_j^*\right) = (V_{j-1}^* - V_j^*) - \frac{1}{N}(V_0^* - V_N^*),$$

and then the typical jth block of $L'M(v_{-1}^* - v^*)$ becomes

$$Q_0'(V_{j-1}^* - V_j^*) + Q_1' \sum_{k=j+1}^{N} (V_{k-1}^* - V_k^*) - \frac{1}{N} Q_0'(V_0^* - V_N^*) - \frac{1}{N} \sum_{k=j+1}^{N} Q_1'(V_0^* - V_N^*)$$

$$= Q_0'(V_{j-1}^* - V_j^*) + Q_1'(V_j - V_N) - \frac{1}{N} Q_0'(V_0^* - V_N^*) - \frac{N-j}{N} Q_1'(V_0^* - V_N^*)$$

$$\equiv \mathcal{R}_j, \quad \text{say.}$$

Because $\{A_j^*\}$ is absolutely summable, $\{V_j^*\}$ is a second-order stationary sequence, and then $\mathcal{R}_j$ is $O_p(1)$ for all j. This shows that

$$\frac{1}{q_m T^2} (v_{-1}^* - v^*)' MLL'M(v_{-1}^* - v^*) = \frac{1}{q_m T^2} \sum_{j=1}^{N} \mathcal{R}_j' \mathcal{R}_j \xrightarrow{p} 0.$$

For the second term of (A.8), note that, in general, $(a'b)^2 \leq (a'a)(b'b)$ for the vectors a and b. Then, the square of the second term divided by 4 is bounded above by the product of the first term and the third term. Because the first term will be shown to be $O_p(1)$ later and the third term converges to zero in probability as shown previously, the second term of (A.8) is $o_p(1)$.

Similarly the same result is obtained for Cases A and C, and we omit the proof.

Now we are ready to consider the first term of (A.8) as far as the limiting distribution is concerned. Under the assumption of normality, $L\epsilon + (I_N \otimes A)v \sim N(0, \sigma^2(\rho LL' + (1/\sigma^2) I_N \otimes \Sigma))$ because $\sigma^2 AA' = \Sigma$ and ϵ and v are independent. Then, defining $z = [Z_1', \ldots, Z_N']' \equiv \sigma^{-1}(\rho LL' + (1/\sigma^2) I_N \otimes \Sigma)^{-1/2}(L\epsilon + (I_N \otimes A))v \sim N(0, I_T)$, we have

$$\begin{aligned}
S_T^m &= \frac{1}{q_m T^2} \{L\epsilon + (I_N \otimes A)v\}' MLL'M \{L\epsilon + (I_N \otimes A)v\} \\
&= \frac{\sigma^2}{q_m T^2} z' \left(\rho LL' + \frac{1}{\sigma^2} I_N \otimes \Sigma\right)^{1/2} MLL'M \left(\rho LL' + \frac{1}{\sigma^2} I_N \otimes \Sigma\right)^{1/2} z \\
&\overset{d}{=} \frac{\sigma^2}{q_m T^2} z' L'M \left(\rho LL' + \frac{1}{\sigma^2} I_N \otimes \Sigma\right) MLz \\
&= \frac{1}{16 q_m N^2} z' \left\{ L'M(I_N \otimes \Sigma)ML + \frac{c^2\sigma^2}{16N^2} (L'ML)(L'ML) \right\} z \\
&= \frac{1}{16 q_m N} \sum_{j,k=1}^{N} Z_j' \left\{ \frac{1}{N} [L'(I_N \otimes \Sigma)ML](j,k) \right. \\
&\qquad \left. + \frac{c^2\sigma^2}{16N} \sum_{l=1}^{N} \frac{1}{N} [L'ML](j,l) \frac{1}{N} [L'ML](l,k) \right\} Z_k, \qquad \textbf{(A.9)}
\end{aligned}$$

where $\overset{d}{=}$ denotes equality in distribution, $Z_j = [z_{4j-3}, z_{4j-2}, z_{4j-1}, z_{4j}]' \sim NID(0, I_4)$ for $j = 1, \ldots, N$, and $[H](j,k)$, $j,k = 1, \ldots, N$, for the $T \times T$ matrix H denotes the (j,k) block of H when we partition H into $N \times N$ blocks with each block a 4×4 matrix, as we divide L in (2). For example, $[L_m](1,1) = Q_{m,0}$ and $[X](N,2) = NI_4$ for Case C. The second equality holds from the definition of z, and the third equality is due to the relation $T = 4N$ and $\rho = c^2/T^2$. The equality in distribution is established using the relation that $z'HH'z \overset{d}{=} zHH'z$ for any $T \times T$ matrix H, which holds because HH' and $H'H$ have the same eigenvalues and z is a standard normal distribution. The last equality holds because $I_N \otimes \Sigma$ and M are commutative and M is idempotent, i.e., $(I_N \otimes \Sigma)M = M(I_N \otimes \Sigma)$, and $M^2 = M$.

To derive the limiting distribution of S_T^m, we examine the terms in braces of (A.9), such as $N^{-1}[L'(I_N \otimes \Sigma)ML](j,k)$. Here we consider Case C, in which the model has seasonal dummies and trends. Note that

$$L'(I_N \otimes \Sigma)L = \begin{bmatrix} Q_0' & Q_1' & \dots & Q_1' \\ & Q_0' & \ddots & \vdots \\ & & \ddots & Q_1' \\ \mathbf{0} & & & Q_0' \end{bmatrix} \begin{bmatrix} \Sigma & & & \mathbf{0} \\ & \Sigma & & \\ & & \ddots & \\ \mathbf{0} & & & \Sigma \end{bmatrix} \begin{bmatrix} Q_0 & & & \mathbf{0} \\ Q_1 & Q_0 & & \\ \vdots & \ddots & \ddots & \\ Q_1 & \dots & Q_1 & Q_0 \end{bmatrix}$$

$$= \begin{bmatrix} Q_0'\Sigma Q_0 + (N-1)Q_1'\Sigma Q_1 & Q_1'\Sigma Q_0 + (N-2)Q_1'\Sigma Q_1 & \dots & Q_1'\Sigma Q_0 \\ Q_0'\Sigma Q_1 + (N-2)Q_1'\Sigma Q_1 & Q_0'\Sigma Q_0 + (N-2)Q_1'\Sigma Q_1 & \dots & Q_1'\Sigma Q_0 \\ \vdots & \vdots & \ddots & \vdots \\ Q_0'\Sigma Q_1 & Q_0'\Sigma Q_1 & \dots & Q_0'\Sigma Q_0 \end{bmatrix}.$$

Then the (j,k) block of $L'(I_N \otimes \Sigma)L$ becomes

$$Q_0'\Sigma Q_1 + (N-j)Q_1'\Sigma Q_1 = (N-j)Q_1'\Sigma Q_1 + O(1) \quad \text{for } j > k,$$

$$Q_1'\Sigma Q_1 + (N-k)Q_1'\Sigma Q_1 = (N-k)Q_1'\Sigma Q_1 + O(1) \quad \text{for } j < k,$$

$$Q_0'\Sigma Q_0 + (N-j)Q_1'\Sigma Q_1 = (N-j)Q_1'\Sigma Q_1 + O(1) \quad \text{for } j = k.$$

Then, in general, we can write

$$[L'(I_N \otimes \Sigma)L](j,k) = (N - \max(j,k))Q_1'\Sigma Q_1 + O(1). \tag{A.10}$$

Similarly, the jth row block of $L'(I_N \otimes \Sigma)X$ is expressed as

$$\begin{aligned}[L'(I_N \otimes \Sigma)X](j,\cdot) &= \left[Q_0'\Sigma + (N-j)Q_1'\Sigma, jQ_0'\Sigma + \sum_{i=j+1}^{N} Q_1'\Sigma\right] \\ &= \left[(N-j)Q_1'\Sigma, \frac{N^2-j^2}{2} Q_1'\Sigma\right] + [O(1), O(N)],\end{aligned} \tag{A.11}$$

and the kth column block of $X'L$ is given by

$$\begin{aligned}[X'L](\cdot,k) &= \begin{bmatrix} Q_0 + (N-k)Q_1 \\ kQ_0 + \sum_{i=k+1}^{N} Q_1 \end{bmatrix} \\ &= \begin{bmatrix} (N-k)Q_1 \\ \dfrac{N^2-k^2}{2} Q_1 \end{bmatrix} + \begin{bmatrix} O(1) \\ O(N) \end{bmatrix},\end{aligned} \tag{A.12}$$

where we define $\sum_{i=a}^{b} = 0$ for $a > b$ and

$$(X'X)^{-1} = \begin{bmatrix} \dfrac{4}{N} I_4 & -\dfrac{6}{N^2} I_4 \\ -\dfrac{6}{N^2} I_4 & \dfrac{12}{N^3} I_4 \end{bmatrix} + \begin{bmatrix} O(N^{-2}) & O(N^{-3}) \\ O(N^{-3}) & O(N^{-4}) \end{bmatrix}. \tag{A.13}$$

Then, by the equations (A.10)–(A.13), the (j,k) block of $N^{-1}L'(I_N \otimes \Sigma)ML$ can be expressed as

$$\begin{aligned}
&\frac{1}{N}[L'(I_N \otimes \Sigma)ML](j,k) \\
&\quad = \frac{1}{N}[L'(I_N \otimes \Sigma)L](j,k) - \frac{1}{N}[L'(I_N \otimes \Sigma)X(X'X)^{-1}X'L](j,k) \\
&\quad = \frac{1}{N}[L'(I_N \otimes \Sigma)L](j,k) - \frac{1}{N}[L'(I_N \otimes \Sigma)X](j,\cdot)(X'X)^{-1}[X'L](\cdot,k) \\
&\quad = \left\{1 - \max\left(\frac{j}{N},\frac{k}{N}\right) - \left[4\left(1-\frac{j}{N}\right)\left(1-\frac{k}{N}\right) - 3\left(1-\frac{j^2}{N^2}\right)\left(1-\frac{k}{N}\right)\right.\right. \\
&\qquad \left.\left. - 3\left(1-\frac{j}{N}\right)\left(1-\frac{k^2}{N^2}\right) + 3\left(1-\frac{j^2}{N^2}\right)\left(1-\frac{k^2}{N^2}\right)\right]\right\} Q_1'\Sigma Q_1 + O(N^{-1}) \\
&\quad = \left\{\min\left(\frac{j}{N},\frac{k}{N}\right) - 4\frac{jk}{N^2} + 3\frac{jk}{N^2}\left(\frac{j}{N}+\frac{k}{N}\right) - 3\frac{j^2k^2}{N^4}\right\} Q_1'\Sigma Q_1 + O(N^{-1}) \\
&\quad = K_C\left(\frac{j}{N},\frac{k}{N}\right) Q_1'\Sigma Q_1 + O(N^{-1}),
\end{aligned}$$

where the fourth equality is established using the fact that $s + t - \max(s,t) = \min(s,t)$.

Completely in the same way, we have

$$\frac{1}{N}[L'ML](j,k) = K_C\left(\frac{j}{N},\frac{k}{N}\right) Q_1'Q_1 + O(N^{-1}).$$

Then it is enough to consider

$$\begin{aligned}
S_T^{m*} = \frac{1}{16 q_m N} \sum_{j,k=1}^{N} Z_j' \left\{ K_C\left(\frac{j}{N},\frac{k}{N}\right) Q_1'\Sigma Q_1 + \frac{c^2\sigma^2}{16N} \sum_{l=1}^{N} K_C\left(\frac{j}{N},\frac{l}{N}\right) \right. \\
\left. \times K_C\left(\frac{l}{N},\frac{k}{N}\right)(Q_1'Q_1)^2 \right\} Z_k,
\end{aligned}$$

as far as the limiting distribution is concerned, because $|S_T^m - S_T^{m*}|$ converges to zero in probability.

Here note that $\{P'Z_j\}$ has the same distribution as $\{Z_j\}$, $NID(0, I_4)$, because the matrix P is orthonormal where P is defined in (9), and then, by redefining $Z_j = P'Z_j$, we have

$$S_T^{m*} = \frac{1}{16 q_m N} \sum_{j,k=1}^{N} Z_j' \left\{ K_C\left(\frac{j}{N}, \frac{k}{N}\right) P' Q_1' \Sigma Q_1 P \right.$$

$$\left. + \frac{c^2 \sigma^2}{16N} \sum_{l=1}^{N} K_C\left(\frac{j}{N}, \frac{l}{N}\right) K_C\left(\frac{l}{N}, \frac{k}{N}\right) (P' Q_1' Q_1 P)^2 \right\} Z_k.$$

(i) When $m = \pi$, we can easily calculate that $P'Q_1'Q_1P = P'Q_{\pi,1}'Q_{\pi,1}P = \mathrm{diag}\{0,16,0,0\}$ and $P'Q_1'\Sigma Q_1 P = P'Q_{\pi,1}P(P'\Sigma P)P'Q_{\pi,1}P = \mathrm{diag}\{0,4,0,0\} \times P'\Sigma P\, \mathrm{diag}\{0,4,0,0\} = \mathrm{diag}\{0,16\sigma_\pi^2,0,0\}$. Then, S_T^{m*} can be expressed as

$$S_T^{m*} = \frac{1}{16\sigma_\pi^2 N} \sum_{j,k=1}^{N} Z_j' \left\{ K_C\left(\frac{j}{N}, \frac{k}{N}\right) \mathrm{diag}\{0,16\sigma_\pi^2,0,0\} \right.$$

$$\left. + \frac{c^2\sigma^2}{16N} \sum_{l=1}^{N} K_C\left(\frac{j}{N}, \frac{l}{N}\right) K_C\left(\frac{l}{N}, \frac{k}{N}\right) \mathrm{diag}\{0,16,0,0\}^2 \right\} Z_k$$

$$= \frac{1}{N} \sum_{j,k=1}^{N} \left\{ K_C\left(\frac{j}{N}, \frac{k}{N}\right) + \frac{c^2\sigma^2}{\sigma_\pi^2 N} \sum_{l=1}^{N} K_C\left(\frac{j}{N}, \frac{l}{N}\right) K_C\left(\frac{l}{N}, \frac{k}{N}\right) \right\} z_{2,j} z_{2,k}, \tag{A.14}$$

where $z_{2,j}$ is the second element of Z_j.

Now we apply Lemma A to the last expression of (A.14). Noting that $K_C(j/N, k/N)$ and $c^2\sigma^2/\sigma_\pi^2$ correspond to $B_N(j,k)$ and γ in (A.1), respectively, we obtain Theorem 1(i) for Case C.

(ii) When $m = \pi/2$, we can easily calculate that $P'Q_1'Q_1P = P'Q_{\pi/2,1}'Q_{\pi/2,1}P = \mathrm{diag}\{0,0,4,4\}$ and $P'Q_1'\Sigma Q_1P = P'Q_{\pi/2,1}P(P'\Sigma P)P'Q_{\pi/2,1}P = \mathrm{diag}\{0,0, 4\sigma_{\pi/2}^2, 4\sigma_{\pi/2}^2\}$. Then, S_T^{m*} can be expressed as

$$S_T^{m*} = \frac{4}{16\sigma_{\pi/2}^2 N} \sum_{j,k=1}^{N} Z_j' \left\{ K_C\left(\frac{j}{N}, \frac{k}{N}\right) \mathrm{diag}\{0,0,4\sigma_{\pi/2}^2,4\sigma_{\pi/2}^2\} \right.$$

$$\left. + \frac{c^2\sigma^2}{16N} \sum_{l=1}^{N} K_C\left(\frac{j}{N}, \frac{l}{N}\right) K_C\left(\frac{l}{N}, \frac{k}{N}\right) (\mathrm{diag}\{0,0,4,4\})^2 \right\} Z_k$$

$$= \frac{1}{N} \sum_{j,k=1}^{N} \left\{ K_C\left(\frac{j}{N}, \frac{k}{N}\right) + \frac{c^2\sigma^2}{4\sigma_{\pi/2}^2 N} \sum_{l=1}^{N} K_C\left(\frac{j}{N}, \frac{l}{N}\right) K_C\left(\frac{l}{N}, \frac{k}{N}\right) \right\} z_{3,j} z_{3,k}$$

$$+ \frac{1}{N} \sum_{j,k=1}^{N} \left\{ K_C\left(\frac{j}{N}, \frac{k}{N}\right) + \frac{c^2\sigma^2}{4\sigma_{\pi/2}^2 N} \sum_{l=1}^{N} K_C\left(\frac{j}{N}, \frac{l}{N}\right) K_C\left(\frac{l}{N}, \frac{k}{N}\right) \right\} z_{4,j} z_{4,k}, \tag{A.15}$$

where $z_{3,j}$ and $z_{4,j}$ are the third and fourth elements of Z_j.

Because $z_{3,j}$ and $z_{4,j}$ are independent, we can apply Lemma A to the two terms in (A.15) separately, and then we obtain Theorem 1(ii) for Case C.

For Case A, $M = I_T$ so that

$$\frac{1}{N}[L'ML](j,k) = \frac{1}{N}[L'L](j,k) = \left(1 - \max\left(\frac{j}{N}, \frac{k}{N}\right)\right) Q_1' Q_1 + O(N^{-1})$$
$$= K_A\left(\frac{j}{N}, \frac{k}{N}\right) Q_1' Q_1 + O(N^{-1}),$$
$$\frac{1}{N}[L'(I_N \otimes \Sigma)ML](j,k) = K_A\left(\frac{j}{N}, \frac{k}{N}\right) Q_1' \Sigma Q_1 + O(N^{-1}).$$

Similarly to Case C, we have, for Case B,

$$\frac{1}{N}[L'ML](j,k) = \left\{\left(1 - \max\left(\frac{j}{N}, \frac{k}{N}\right)\right) - \left(1 - \frac{j}{N}\right)\left(1 - \frac{k}{N}\right)\right\}$$
$$\times Q_1' Q_1 + O(N^{-1})$$
$$= \left\{\min\left(\frac{j}{N}, \frac{k}{N}\right) - \frac{jk}{N^2}\right\} Q_1' Q_1 + O(N^{-1})$$
$$= K_B\left(\frac{j}{N}, \frac{k}{N}\right) Q_1' Q_1 + O(N^{-1}),$$
$$\frac{1}{N}[L'(I_N \otimes \Sigma)ML](j,k) = K_B\left(\frac{j}{N}, \frac{k}{N}\right) Q_1' \Sigma Q_1 + O(N^{-1}).$$

Then, completely in the same way as Case C, we can establish Theorem 1 for Cases A and B. ∎

Proof of Corollary 1. The proof is similar to that of Theorem 1. Completely in the same way as (A.9), it can be shown that

$$S_T^m \overset{d}{=} \frac{1}{16 q_m N} \sum_{j,k=1}^{N} Z_j' \left\{ \frac{1}{N}[L_m'(I_N \otimes \Sigma)ML_m](j,k) + \frac{c^2\sigma^2}{16N} \sum_{l=1}^{N} \frac{1}{N}[L_m' ML_i](j,l)\, \frac{1}{N}[L_i' ML_m](l,k) \right\} Z_k$$
$$\overset{p}{=} \frac{1}{16 q_m N} \sum_{j,k=1}^{N} Z_j' \left\{ K_{\cdot}\left(\frac{j}{N}, \frac{k}{N}\right) Q_{m,1}' \Sigma Q_{m,1} + \frac{c^2\sigma^2}{16N} \sum_{l=1}^{N} K_{\cdot}\left(\frac{j}{N}, \frac{l}{N}\right) K_{\cdot}\left(\frac{l}{N}, \frac{k}{N}\right) (Q_{m,1}' Q_{i,1})(Q_{i,1}' Q_{m,1}) \right\} Z_k,$$

under H_1^i for $i, m = \pi, \pi/2$ $(i \neq m)$, but we can easily check that $Q_{\pi,1}' Q_{\pi/2,1} = 0$ and then the second term in braces vanishes, so that S_T^m converges in distribution to the null distribution (10). ∎