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SENSITIVITY ANALYSIS OF STATIONARY STATES IN OPTIMAL GROWTH: A DIFFERENTIABLE APPROACH*

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Abstract

This paper provides a sensitivity analysis of stationary states in a discrete time model of capital accumulation. We classify the stationary states in terms of the concept of regularity, and investigate the properties of the regular stationary states that are robust to any small perturbation in economies. We prove that the set of regular stationary economies is an open dense subset of the space of economies. More specifically, if the space of economies is identified with the set of discount rates, then the regular stationary economies form a subset of full measure. Moreover, if the space of economies is identified with the set of pairs of a discount rate and a return function, then the economies that generate a regular stationary state exist generically.

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I. *Introduction*

In optimal growth theory, difficulties arise in the sensitivity analysis of stationary states when the stationary states are not locally unique given the parameters of a model. In particular, in the presence of the continuum of stationary states, the effect of a parametric change in economies on the stationary states is quite ambiguous, which poses a limitation of the analysis in various contexts of applications. The purpose of this paper is to characterize the economies that generate finite numbers of stationary states in a discrete time model of capital accumulation. In this paper, we classify the stationary states in terms of the concept of "regularity", developed by general equilibrium theory with a differentiable approach, along the

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lines of Balasko (1988), Debreu (1970), Dierker (1982), and Smale (1974a), and investigate the properties of regular stationary states that are robust to any small perturbation in parameters.

The attention here is focused on the regular stationary economies because first, "almost all" stationary economies are regular, and second, regular stationary economies generate locally finite stationary states. We show that the set of regular stationary economies is an open dense subset of the space of economies. In other words, if a regular stationary economy is subjected to any small perturbation, it remains regular, but if an economy is not regular, an arbitrarily small perturbation can make it regular. More specifically, if the space of economies is identified with the set of discount rates, then the regular stationary economies form a subset of full measure. Since the critical stationary economies then have zero measure, the probability of choosing a critical stationary economy at random from the space of economies is zero. Moreover, if the space of economies is identified with the set of pairs of a discount rate and a return function, then the economies that generate a regular stationary state exist "generically" in the space of economies.

The most relevant work to the analysis developed here is Magill, and Scheinkman (1979) and McKenzie (1986). The former is involved in the sensitivity analysis of stationary states with a differentiable approach in continuous time and the latter develops an exhaustive study of comparative statics of stationary states in discrete time. While the generic property established by Magill, and Scheinkman (1979) is restricted to the class of C^2-return functions that have the symmetric second order derivatives, we treat a class of C^2-return functions without the symmetric condition. With additional assumptions on the second order derivatives of the return function, McKenzie (1986) obtains sharper results on the effect of a parametric change in discount rates on the stationary states than the results of this paper.

This paper is organized as follows. Section II collects the preliminary results for dynamic programming and examines the differentiability of the value function that is employed throughout the paper. Section III contains the main result. We first introduce mathematical definitions and terminologies from differentiable topology. We then establish the generic property of regular stationary states that are subjected to any small perturbation of discount rates, and present a characterization of the regular stationary states in terms of the linearized Euler equation. We also analyze the generic property of regular stationary states when a discount rate and a return function are perturbed simultaneously. A conclusion is provided in Section IV.

II. *Description of the Model*

In this section, we present the model and a brief summary of the fundamental results of optimal growth theory, which is required for the analysis in the succeeding sections. A basic reference for the most of the results stated is Stokey et al. (1989, Chapter 4).

1. Preliminary Results

Let X denote the set of feasible capital stocks, which is a subset of the n-dimensional Euclidean space with positive orthant, $\mathbb{R}^n_+$. The technology is described by the set-valued mapping Γ: $X \to 2^X$ from X to itself. We write $y \in \Gamma(x)$ to mean that given the current investment of the capital stock x, the capital stock y is available in the next period. The graph of Γ is denoted by $A = \{(x, y) \in X \times X \mid y \in \Gamma(x)\}$. Let u: $X \times X \to \mathbb{R}$ be a return function and $\delta \in (0, 1)$ be a discount rate. Time is indexed by $t = 0, 1, \ldots$.

The model of optimal growth with discounting the future is described by the following problem with the subsequent Assumptions 1 to 6:

$$v_\delta(x) = \max \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t) \text{ for each } t \text{ and } x_0 = x \in \text{ int } X \text{ given.}$$

Assumption 1. X is nonempty and convex.

Assumption 2. Γ is continuous and $\Gamma(x)$ is compact for any $x \in X$.

Assumption 3. A is a convex subset of $X \times X$ with int $A \neq \phi$.

Assumption 4. u is continuous on A and twice continuously differentiable on int A.

Assumption 5. There exists some $\alpha > 0$ such that $(x, y) \mapsto u(x, y) + \dfrac{\alpha}{2} \| y \|^2$ is a concave function on A.

Assumption 6. There exists some $M > 0$ such that $-{}^{\mathrm{T}}(a, b) \mathrm{D}^2 u(x, y)(a, b) \leq 1$ for $(x, y) \in$ int A and $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ implies $-{}^{\mathrm{T}} a \mathrm{D}_{11} u(x, y) a \leq M$.

Assumptions 1 to 4 are standard. Assumptions 1 and 3 together guarantee the convex technology that rules out increasing returns to scale. Assumptions 2 and 4 together assure the existence of an optimal path. Assumptions 5 and 6 ensure the differentiability of the policy function and the twice differentiability of the value function.

The optimization problem we are facing with can be equivalently described as one of dynamic programming. Let $C^0(X, \mathbb{R})$ be the space of bounded continuous functions on X with the sup norm. Define the mapping T_δ: $C^0(X, \mathbb{R}) \to C^0(X, \mathbb{R})$ by the formula

$$T_\delta v(x) = \max_{y \in \Gamma(x)} [u(x, y) + \delta v(y)].$$

Then $T_\delta v(x)$ is well defined, and by the contraction mapping theorem, for any $\delta \in (0, 1)$ there exists a unique $v_\delta \in C^0(X, \mathbb{R})$ such that $T_\delta v_\delta = v_\delta$. The bounded continuous function v_δ is referred as the *value function* and the above functional equation induces the *Bellman equation*

$$v_\delta(x) = \max_{y \in \Gamma(x)} [u(x, y) + \delta v_\delta(y)].$$

The dynamic behavior of the model is described by the policy function h_δ: $X \to X$ defined as the unique solution $y \in \Gamma(x)$ attaining the maximum in the Bellman equation, i.e., $v_\delta(x) = u(x, h_\delta(x)) + \delta v_\delta(h_\delta(x))$.

Under our assumptions, the value function v_δ is concave on X and differentiable at every $x \in$ int X. Applying the theorem of Benveniste and Scheinkman (1979), the derivative of the value function is given by

$$\mathrm{D} v_\delta(x) = \mathrm{D}_1 u(x, h_\delta(x)). \tag{1}$$

A recent contribution to the theory of dynamic programming in infinite horizon is the resolution of C^1-differentiability of the policy function, which is provided by Santos (1991) under α_y-concavity condition (Assumption 5) and Assumption 6 on the return function. Throughout this paper, we utilize C^1-differentiability of the policy function for each fixed δ. For an earlier study that uses the Lipschitz continuous policy function, see Montrucchio (1987).

We assume that the policy function satisfies the interiority condition in the following sense.

Assumption 7. $(x, h_\delta(x)) \in \operatorname{int} A$ for any $x \in \operatorname{int} X$.

Theorem 1 (Santos). *If Assumptions 1 to 7 hold, then the policy function h_δ is differentiable at every $x \in \operatorname{int} X$.*

Theorem 1 and equation (1) together imply that the value function v_δ has the second order derivative at every $x \in \operatorname{int} X$. Note that the policy function is a solution of the equation

$$\mathrm{D}_2 u(x, h_\delta(x)) + \delta \mathrm{D} v_\delta(h_\delta(x)) = 0. \tag{2}$$

By combining equations (1) and (2), we have the Euler difference equation

$$\mathrm{D}_2 u(x, h_\delta(x)) + \delta \mathrm{D}_1 u(h_\delta(x), h_\delta{}^2(x)) = 0 \quad \text{for any } x \in \operatorname{int} X, \tag{3}$$

where $h_\delta^2(x) = h_\delta(h_\delta(x))$. Fixed points of h_δ are said to be *stationary state* for δ. The set of stationary states for δ is defined by

$$\Sigma^\delta = \{x \in X \mid h_\delta(x) = x\}.$$

In view of (3), we obtain

$$\mathrm{D}_2 u(x, x) + \delta \mathrm{D}_1 u(x, x) = 0 \quad \text{for any } x \in \Sigma^\delta.$$

III. *Genericity of Regular Stationary Economy*

In this section, we first introduce some fundamental definitions and terminologies from differentiable topology, which is used in the sequel. A basic reference is Hirsch (1976). We then define a regular stationary state and perturb a discount rate as a parameter. We also investigate the relationship between the regular stationary states and the characteristic equation derived from the Euler equation evaluated at the regular stationary states. We finally perturb a discount rate and a return function simultaneously as parameters and establish the generic property of regular stationary states.

1. Mathematical Remarks

We say that a property holds *generically* if there exists an open dense subset of U of a topological space X such that the property holds for every $x \in U$.

Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^k$, $U \subset X$ be an open set and $f \in C^1(U, Y)$. A point $x \in U$ is called a *regular point* for f if $\mathrm{D}f(x): X \to Y$ is surjective. A point $x \in U$ is a *critical point* for f if x is not a regular point for f. A point $y \in Y$ is a *regular value* of f if each point of $f^{-1}(y)$ is a regular point for f. A point $y \in Y$ is a *critical value* of f if it is not a regular value of f; that is, $f^{-1}(y)$ contains a critical point for f. From these definitions, it follows that x is a critical point for f if and only if rank $\mathrm{D}f(x) < k$.

Let f: $M \to N$ be a C^1-mapping of manifolds M and N, and let $W \subset N$ be a submanifold of N. f is *transversal* to W at a point $x \in M$ if either $f(x) \notin W$ or $f(x) \in W$ and the tangent space $T_{f(x)}(N)$ is spanned by $T_{f(x)}(W)$ and the image $df_x(T_x(M))$, i.e., $df_x(T_x(M)) + T_{f(x)}(W) = T_{f(x)}(N)$. We write $f \pitchfork W$ if f is transversal to W at every $x \in M$.

2. Perturbation in Discount Rates

In the standard one-sector growth model, it is well known that an increase in a discount rate increases the level of capital stock in a unique stationary state. To investigate the relationship between myopia and stationary states in a general reduced model of capital accumulation, we need an appropriate differentiable structure for sensitivity analysis.

Regular Stationary State

The space of economies is identified with the set of discount rates, an open interval (0, 1) of the real line. A pair $(\delta, x) \in (0, 1) \times X$ is said to be a *stationary equilibrium* if $h_\delta(x) = x$. We denote the set of stationary equilibria by S, i.e.,

$$S = \{(\delta, x) \in (0, 1) \times X \mid h_\delta(x) = x\}$$
$$= \{(\delta, x) \in (0, 1) \times X \mid \mathrm{D}_2 u(x, x) + \delta \mathrm{D}_1 u(x, x) = 0\}.$$

Definition 1. $x \in \Sigma^\delta$ is a *regular* (resp. *critical*) *stationary state* for $\delta \in (0, 1)$ if the matrix $\mathrm{D}h_\delta(x) - I$ is nonsingular (resp. singular).

Assumption 8. The $n \times n$ matrices $\mathrm{D}_{22} u(x, x) + \delta \mathrm{D}^2 v_\delta(x) + \delta \mathrm{D}_{12} u(x, x)$ and $\mathrm{D}_{22} u(x, x) + \mathrm{D}_{21} u(x, x) + \delta \mathrm{D}_{11} u(x, x) + \delta \mathrm{D}_{12} u(x, x)$ are nonsingular for any $(\delta, x) \in S$.

The nonsingularity of the second matrix in Assumption 8 imposed also by McKenzie (1986) to establish the monotonicity of the stationary state with respect to the discount rate.

Define the mapping F: $(0, 1) \times X \to \mathbb{R}^n$ by

$$F(\delta, x) = \mathrm{D}_2 u(x, x) + \delta \mathrm{D}_1 u(x, x)$$

and put $F_\delta(x) = F(\delta, x)$. Note that $F_\delta(x) = 0$ if and only if $x \in \Sigma^\delta$ is a stationary state for $\delta \in (0, 1)$.

By the following lemma, the differentiable structure of stationary states can be described in terms of the differentiable mapping F_δ, as is described by the policy function.

Lemma 1. *Suppose that Assumptions 1 to 8 hold. Then $x \in \Sigma^\delta$ is a regular (resp. critical) stationary state for $\delta \in (0, 1)$ if and only if $F_\delta(x) = 0$ and $\mathrm{D}F_\delta(x)$ is nonsingular (resp. singular).*

Proof. Differentiating (1), observing that $h_\delta(x) = x$, we have

$$\mathrm{D}^2 v_\delta(x) = \mathrm{D}_{11} u(x, x) + \mathrm{D}_{12} u(x, x) \mathrm{D}h_\delta(x). \tag{4}$$

Differentiating (2) at the stationary state yields

$$(D_{22}u(x, x)+\delta D^2 v_\delta(x))Dh_\delta(x)=-D_{21}u(x, x). \tag{5}$$

By combining (4) and (5), we obtain

$$DF_\delta(x)=-(D_{22}u(x, x)+\delta D^2 v_\delta(x)+\delta D_{12}u(x, x))(Dh_\delta(x)-I). \tag{6}$$

Since $(D_{22}u(x, x)+\delta D^2 v_\delta(x)+\delta D_{12}u(x, x))$ is nonsingular, (6) reduces to rank $DF_\delta(x)=$ rank $(Dh_\delta(x)-I)$. □

Since the rank of the $(n+1)\times n$ matrix

$$DF(\delta, x)=\begin{pmatrix} D_1 u(x, x) \\ D_{22}u(x, x)+D_{21}u(x, x)+\delta D_{11}u(x, x))+\delta D_{12}u(x, x) \end{pmatrix}$$

is n for any $(\delta, x)\in S$ by Assumption 8, the set of stationary equilibria

$$S=\{(\delta, x)\in(0, 1)\times X \mid F(\delta, x)=0\}$$

is a one-dimensional manifold of C^1-class by the regular value theorem (Hirsch 1976, Theorem 3.2).

Let Π: $S \to (0, 1)$ be a projection mapping from the set of stationary equilibria to the space of economies. It is evident that Π: $S \to (0, 1)$ is a smooth mapping on the manifolds.

Definition 2. $\delta\in(0, 1)$ is a *regular* (resp. *critical*) *stationary economy* if it is a regular (resp. critical) value of the projection mapping Π: $S \to (0, 1)$. The set of regular (resp. critical) stationary economies is denoted by $\mathscr{E}_R$ (resp. $\mathscr{E}_C$).

The following theorem states that on the set of regular economies, the stationary states are locally finite and can be represented by a differentiable mapping from $\mathscr{E}_R$ to $\mathbb{R}^n_+$. This is an immediate consequence of the regular value theorem, which is a generalized version of the implicit function theorem (see Hirsch, 1976, p. 22).

Theorem 2. *If Assumptions 1 to 8 hold, then* $\Pi^{-1}(\delta)$ *is a smooth submanifold of* S *for any* $\delta\in\mathscr{E}_R$.

The natural tool for proving the next theorem is developed in the theory of regular economies introduced by Debreu (1970). The basic ideas are also explained in Balasko (1988) and Dierker (1982).

Theorem 3. *Suppose that Assumptions 1 to 8 hold. Then* $x\in\Sigma^\delta$ *is a regular (resp. critical) stationary state for* $\delta\in(0, 1)$ *if and only if* $\delta\in(0, 1)$ *is a regular (resp. critical) stationary economy.*

Proof. Let us define mappings F: $(0, 1)\times X \to \mathbb{R}^n$ by $F(\delta, x)=D_2 u(x, x)+\delta D_1 u(x, x)$ and Φ: $(0, 1)\times X \to \mathbb{R}^n\times(0, 1)$ by the formula $\Phi(\delta, x)=(F(\delta, x), \delta)$. For any $(\delta, x)\in S$, we must have $\Phi(\delta, x)=(0, \delta)$. Hence, the restriction of Φ to S, $\Phi \mid S$: $S \to \{0\}\times(0, 1)$ can be identified with the projection mapping Π: $S \to (0, 1)$ defined above. Moreover, we obtain $F\circ\Phi^{-1}(0, \delta)=(F(\delta, x), \delta)=(0, \delta)$. Thus $F\circ\Phi^{-1}$ is an identity mapping on $\{0\}\times(0, 1)$, i.e., $F\circ\Phi^{-1}=\iota_{\{0\}\times(0, 1)}$. Differentiating Φ at $(\delta, x)\in S$ yields an $(n+1)\times(n+1)$ matrix

$$\mathrm{D}\Phi(\delta, x)=\left(\begin{array}{c|c} \left(\dfrac{\partial F^1(\delta, x)}{\partial \delta}, \dots, \dfrac{\partial F^n(\delta, x)}{\partial \delta}\right) & 1 \\ \hline \left(\dfrac{\partial F^j(\delta, x)}{\partial x_i}\right)_{1\le i, j\le n} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \end{array}\right)=\begin{pmatrix} \mathrm{D}_1 F(\delta, x) & 1 \\ \mathrm{D}_2 F(\delta, x) & \mathbf{0}\end{pmatrix}.$$

Therefore, by Assumption 8, det $\mathrm{D}\Phi(\delta, x)=\det \mathrm{D}_2 F(\delta, x)\neq 0$ on S. Thus $(\delta, x)\in S$ if and only if δ is a regular value of Π, which is a consequence of Lemma 1. □

Theorem 4 (Sard). *Let M and N be manifolds of dimensions m and n respectively, and let f: $M \to N$ be a C^r-mapping. If $r>\max\{0, m-n\}$, then the set of critical values of f has Lebesgue measure zero in N and the set of regular values of f is dense in N.*

For a proof, see Hirsch (1976, pp. 68-72).

The following result is an immediate consequence of Sard's theorem applied to the projection mapping Π.

Theorem 5. *Suppose that Assumptions 1 to 8 hold. Then the set of critical (resp. regular) stationary economies $\mathscr{E}_C$ (resp. $\mathscr{E}_R$) has Lebesgue measure zero (resp. full Lebesgue measure) in* $(0, 1)$ *and $\mathscr{E}_R$ (resp. $\mathscr{E}_C$) is a dense (resp. nowhere dense) subset of* $(0, 1)$.

The interpretation of regular economies due to Kehoe (1985) is also valid in our framework. Theorem 5 states that the size of the set of critical stationary economies is small, both from the point of view of measure theory, and from that of topology. This property is important. It is always possible to give a probabilistic interpretation for the Lebesgue measure of a set. Therefore, the probability that a randomly chosen stationary economy will be critical is equal to zero, so that a critical stationary economy is rather exceptional.

Having measure zero, however, does not ensure that a set is topologically small. In fact, a set can have measure zero and still be very large from an alternative point of view. For example, though the set of rational numbers has measure zero in the space of economies as a countable set, it is dense. Theorem 5 ensures that this type of difficulty disappears with sets of measure zero. Moreover, the fact that the set of critical stationary economies is small in terms of both measure theory and topology does not necessarily imply that the cardinality of the set of critical stationary economies is small. In fact, a curious example is the Cantor set of a closed interval. The Cantor set is of measure zero and nowhere dense in $\mathbb{R}$, but it has sufficiently many elements; that is, it has the cardinality of continuum.

Nondegenerateness of Regular Stationary States

Let $x_0\in\operatorname{int} X$ be given. A sequence $\{x_t\}$ of X is said to be an *optimal path* for δ if it is generated by the policy function h_δ, i.e., $h_\delta^t(x_0)=x_t$ for each t, where $h_\delta^t(x_0)=h_\delta(h_\delta^{t-1}(x_0))$ for $t\geq 1$ and $h_\delta^0(x_0)\equiv x_0$ for $t=0$. We obtain the Euler difference equation

$$\mathrm{D}_2 u(x_t, x_{t+1})+\delta \mathrm{D}_1 u(x_{t+1}, x_{t+2})=0 \quad \text{for each } t.$$

Linearization around a stationary state yields a linear difference equation in vector form with coefficient matrices

$$\delta U_{12} y_{t+2}+(\delta U_{11}+U_{22})\, y_{t+1}+U_{21} y_t=0, \tag{7}$$

where $y_t = x_t - x$, $U_{11} = D_{11}u(x, x)$, $U_{12} = D_{12}u(x, x)$, $U_{21} = D_{21}u(x, x)$ and $U_{22} = D_{22}u(x, x)$. Thus the characteristic equation of (7) is

$$\varphi(\lambda) = \det\ (\lambda^2 \delta U_{12} + \lambda(\delta U_{11} + U_{22}) + U_{21}) = 0.$$

Assumption 9. U_{21} is nonsingular.

Assumption 9 ensures the nonzero characteristic roots for $\varphi(\lambda) = 0$, which is also imposed by McKenzie (1986). Unlike Magill and Scheinkman (1979), and McKenzie (1986), however, we do not impose the symmetric condition $U_{12} = {}^{T}U_{21}$ where ${}^{T}U_{21}$ is the transpose of U_{21}, which is a stringent condition.

The following definition is a counterpart in discrete time with the definition in continuous time in Magill and Scheinkman (1979).

Definition 3. A stationary equilibrium $(\delta, x) \in S$ is said to be *nondegenerate* (resp. *degenerate*) if $\lambda_i \neq 1$ for each $i = 1, \ldots, 2n$ (resp. $\lambda_i = 1$ for some i) where λ_i is a root of the characteristic equation

$$\varphi(\lambda_i) = \det\ (\lambda_i^2 \delta U_{12} + \lambda_i(\delta U_{11} + U_{22}) + U_{21}) = 0.$$

The following theorem characterizes the local property of regular (resp. critical) stationary states in terms of the linear difference equation (7) generated by the Euler equation.

Theorem 6. *Suppose that Assumptions 1 to 9 hold. Then a stationary equilibrium* $(\delta, x) \in S$ *is nondegenerate (resp. degenerate) if and only if* $x \in \Sigma^{\delta}$ *is a regular (resp. critical) stationary state.*

Proof. It is evident from the definitions that

$$\varphi(1) = \det\ (\delta U_{12} + \delta U_{11} + U_{22} + U_{21}).$$

Thus, $\varphi(1) \neq 0$ if and only if $(\delta, x) \in S$ is nondegenerate. Since $\varphi(1) = \det\ DF_{\delta}(x)$, thus by Lemma 1, $(\delta, x) \in S$ is nondegenerate if and only if $x^{\delta} \in \Sigma^{\delta}$ is a regular stationary state. □

Theorem 6 states the fact that the nondegenerateness of stationary equilibria is a generic property with regard to perturbing discount rates.

3. Perturbation in Return Functions

Define the subset of return functions in $C^2(A, \mathbb{R})$ by

$$\mathscr{U} = \{u \in C^2(A, \mathbb{R}) \mid u \text{ satisfies Assumptions 5, 6 and 8}\}.$$

The class $\mathscr{U}$ of return functions guarantees the twice-continuous differentiability of the value function. Note that $\mathscr{U}$ is endowed with the relative topology from $C^2(\mathrm{A}, \mathbb{R})$ with C^2-norm.

Assumption 10. int $\mathscr{U}$ is nonempty.

The space of economies is identified with $(0, 1) \times \mathscr{U}$.

The following theorem states that pairs of a return function and a discount rate that generate regular stationary states exist *generically* in $\mathscr{U} \times (0, 1)$. The technique of the proof is essentially the same as (1974a).

Theorem 7. *Suppose that Assumptions 1 to 8 and 10 hold. Then there exists an open dense subset* $\mathscr{E}^*$ *of* $(0, 1)\times\mathscr{U}$ *such that* $(\delta, u)\in\mathscr{E}^*$ *and* $\mathrm{D}_2 u(x, x)+\delta\mathrm{D}_1 u(x, x)=0$ *imply that* x *is a regular stationary state for* δ.

Proof. Let us define F: $(0, 1)\times\mathscr{U}\times X \to \mathbb{R}^n$ by

$$F(\delta, u)(x)=\mathrm{D}_2 u(x, x)+\delta\mathrm{D}_1 u(x, x).$$

Note that $F(\delta, u)\in C^1(X, \mathbb{R}^n)$ for any $(\delta, u)\in(0, 1)\times U$. By virtue of Lemma 1, the proof is complete if we show that the set $\{(\delta, u)\in(0, 1)\times\mathscr{U} \mid F(\delta, u)\pitchfork\{0\}\}$ is open dense in $(0, 1)\times \mathscr{U}$. This fact follows immediately from the transversality theorem (see Hirsch, 1976, p. 74), which states that the set $\{f\in C^1(X, \mathbb{R}^n) \mid f\pitchfork\{0\}\}$ is an open dense subset of $C^1(X, \mathbb{R}^n)$. □

The interpretation of the genericity result in Theorem 7 requires some caution. Note that the genericity is restricted to the class $\mathscr{U}$ of return functions that guarantees the twice-continuous differentiability of the value function. Theorem 7 does not assure the genericity in the set of return functions in $C^2(A, \mathbb{R})$. Indeed, there may exist some return function in $C^2(A, \mathbb{R})$ that is not in $\mathscr{U}$ such that it generates a regular stationary state, and there may exist some return function in $C^2(A, \mathbb{R})$ that is not in $\mathscr{U}$ such that it generates a critical stationary state. It is an open problem to characterize the stationary states outside the return functions in $\mathscr{U}$ in our framework.

IV. *Conclusion*

We have established the generic property of regular stationary states in the presence of the perturbation of discount rates and return functions. In this paper, however, the perturbation of production technologies is completely ignored because of its difficulties. In general equilibrium with production, Smale (1974b) described a production set with the smooth boundary by using smooth functions and proved the generic property of price equilibria. The technique used by Smale (1974b) also appears effective in optimal growth theory.

To illustrate this, let g^k: $X\times X \to \mathbb{R}$ be a C^1-function for $k=1, ..., m$, and let $g=(g^1, ..., g^m)$. When the production set is given by

$$A_g=\{(x, y)\in X\times X \mid g^k(x, y)\leq 0, k=1, ..., m\},$$

we can define the set-valued mapping Γ_g: $X \to 2^X$ describing the production technology by

$$\Gamma_g(x)=\{y\in X \mid (x, y)\in A_g\}.$$

If g is taken over all C^1-functions with some properties, we can parameterize production technologies in analytical manner.

It is an open question whether the generic property of regular stationary states holds in the presence of the perturbation of production technologies. Such a generalization requires further research.

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