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<td>Issue Date</td>
<td>2005-01</td>
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<tr>
<td>Type</td>
<td>Technical Report</td>
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<tr>
<td>Text Version</td>
<td>publisher</td>
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<tr>
<td>URL</td>
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Fixed Size Confidence Regions for Parameters of Stationary Processes Based on a Minimum Contrast Estimator

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Abstract
For parameters of stationary processes with zero mean and spectral density, sequential procedures are proposed for constructing fixed size confidence ellipsoidal regions for unknown parameters using a minimum contrast estimator. The confidence ellipsoids are shown to be asymptotically consistent and the associated stopping rules are shown to be asymptotically efficient as the size of the region becomes small when the assumed parametric model is correct. Monte Carlo simulations are given to investigate the performance of our proposed sequential procedures.

1 Introduction
It is well documented in literature that sequential sampling methods provide a useful way of constructing confidence intervals or regions for parameters with a fixed size and a prescribed coverage probability. Chow and Robbins (1965) proposed a sequential sampling rule for constructing a fixed-width confidence interval for an unknown mean with a prescribed probability and developed its asymptotic theory. This sampling rule is referred to as the “Chow-Robbins procedure.” For details, refer to Chapter 8 of Ghosh, Mukhopadyay and Sen (1997). Their ideas have been used to develop sequential fixed size confidence regions for parameters associated with dependent and independent observations. For independent, identically distributed observations, we refer the reader to Srivastava (1967), Khan (1969), Yu (1989), Woodroofe (1982), and Chang and Martinsek (1992).

With regard to time series, Sriram (1987) developed a point and interval estimation for the mean of a first order autoregressive (AR(1)) model. Fakhre-Zakeri and Lee (1992, 1993) later considered a sequential point and fixed-width confidence interval estimation for the mean of a scalar- or vector-valued linear process. Sequential procedures dealing

In this article, we assume that the observations are stationary processes with parametric spectral density $f_\theta(\lambda)$, where $\theta$ is an unknown parameter. In order to estimate $\theta$, we use a minimum contrast estimator, $\hat{\theta}_n^{(MCE)}$, which minimizes the criterion $D(f_\theta, \hat{f}_n) = \int_{-\pi}^{\pi} K\{f_\theta(\lambda)/\hat{f}_n(\lambda)\}d\lambda$ with respect to $\theta$, where $\hat{f}_n(\lambda)$ is a non-parametric spectral estimator of $f_\theta(\lambda)$, and $K(\cdot)$ is an appropriate function. It was shown that under appropriate conditions, the main order term of $\sqrt{n}(\hat{\theta}_n^{(MCE)} - \theta)$ can be written as $F = \sqrt{n} \int \Psi(\lambda)\{\hat{f}_n(\lambda) - f(\lambda)\}d\lambda$, where $\Psi(\lambda)$ is an integrable function. Although the nonparametric spectral estimator deviates from $f(\lambda)$ by a probability order that is greater than $n^{-1/2}$, Taniguchi (1987) showed that the integrable functionals obey the $\sqrt{n}$-consistent asymptotics, and that $\hat{\theta}_n^{(MCE)}$ is asymptotically efficient if $f = f_\theta$. Therefore, it can be seen that the integral functional $F$ is the key quantity. The sequential estimation problem of this integral functional has been studied by Shiohama and Taniguchi (2001). The minimum contrast estimator has the following desirable property. For various spectra $f_\theta(\lambda)$, by appropriately selecting the function $K(\cdot)$ in $D(f_\theta, \hat{f}_n)$ we can obtain the non-iterative efficient estimators of $\theta$ in explicit forms, whereas with the exception of autoregressive models, the (quasi) maximum likelihood estimations procedure requires iterative methods. For details, refer to Taniguchi (1987) and Taniguchi and Kakizawa (2000).

In Section 2, we introduce the minimum contrast estimator (MCE) and construct sequential fixed size confidence regions for $\theta$ based on it. We then state the main theorem which establishes the asymptotic consistency and efficiency of our sequential procedure. Proofs are provided in Section 3. Section 4 comprises a brief discussion on estimation with fixed proportional accuracy and estimation of a particular linear combination of the components of $\theta$. Section 5 contains several Monte Carlo simulations that demonstrate the performances of our sequential procedure based on the MCE. In this paper, we denote the set of all integers by $\mathbb{Z}$. 

2
2 Stopping Rule and Main Theorem

Let \( \{X_t, t \geq 0\} \) be a scalar-valued linear process of the form
\[
X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \quad t \in \mathbb{Z},
\]
(2.1)
where \( \{\varepsilon_t\} \) is a sequence of i.i.d. random variables with \( E\{\varepsilon_t\} = 0, E\{\varepsilon_t^2\} = \sigma^2 \) and \( E\{\varepsilon_t^{2p}\} < \infty, \) for \( p > 2. \) Then the process \( \{X_t; t \in \mathbb{Z}\} \) is a second-order stationary process with spectral density \( f(\lambda). \) Let \( F \) be the space of spectral densities defined by
\[
F = \left\{ f; f(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} a_j e^{-ij\lambda} \right|^2, \text{ there exist } C < \infty \text{ and } \delta > 0 \text{ such that} \right. \\
\left. \sum_{j=0}^{\infty} (1 + j^2)|a_j| \leq C, \sum_{j=0}^{\infty} |a_j z^j| \leq \delta, \text{ for all } |z| \leq 1 \right\}.
\]

Denote \( I_n(\lambda) \), the periodogram constructed from a realization \( \{X_1, \ldots, X_n\} \), by
\[
I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{it\lambda} \right|^2.
\]

We estimate \( f(\lambda) \) by the weighted averages of the periodogram \( I_n(\lambda) \), with a spectral window \( W_n(\lambda) \) as the weight, i.e.,
\[
\hat{f}_n(\lambda) = \int_{-\pi}^{\pi} W_n(\lambda - \mu) I_n(\mu) d\mu.
\]
(2.2)

The following conditions are imposed on \( W_n(\cdot) \).

(A.1) (i) \( W_n(\lambda) \) can be expressed as
\[
W_n(\lambda) = \frac{1}{2\pi} \sum_{l=-M}^{M} w \left( \frac{l}{M} \right) e^{-il\lambda}.
\]
(ii) \( w(x) \) is a continuous, even function with \( w(0) = 1, \) and satisfies
\[
\left\{ \begin{array}{l}
|w(x)| \leq 1, \\
\int_{-\infty}^{\infty} w(x)^2 dx < \infty, \lim_{x \to 0} \frac{1-w(x)}{|x|^4} = \kappa_2 < \infty.
\end{array} \right.
\]
(iii) \( M = M(n) \) satisfies
\[
n^{1/4}M + M/n^{1/2} \to 0 \text{ as } n \to \infty.
\]
Concrete examples of $W_n(\cdot)$ satisfying (A.1) can be found in Hannan (1970), Brillinger (1981), and Robinson (1983). We then define the criterion that measures the nearness of $f_\theta$ to $f$ as

$$D(f_\theta, f) = \int_{-\pi}^{\pi} K\{f_\theta(\lambda)/f(\lambda)\}d\lambda.$$ 

The following are examples of $K(\cdot)$:

(i) $K(x) = \log x + x^{-1}$,
(ii) $K(x) = -\log x + x$,
(iii) $K(x) = (\log x)^2$,
(iv) $K(x) = x \log x - x$,
(v) $(x^\alpha - 1)^2, \ 0 < \alpha < \infty$.

We impose the following assumptions on $K(\cdot)$ and $f_\theta(\lambda)$.

(A.2) (i) $K(x)$ is a three times continuously differentiable function in $(0, \infty)$ and has a unique minimum at $x = 1$.

(ii) The spectral model $f_\theta(\lambda)$ is three times continuously differentiable with respect to $\theta$, and every component of the second derivative $\partial^2 f_\theta/\partial \theta \partial \theta'$ is continuous in $\lambda$.

In order to estimate the unknown $\theta$, since $f(\lambda)$ is unknown, we estimate $f(\lambda)$ by a nonparametric estimator (2.2) satisfying (A.1). Therefore, a semiparametric estimator $\hat{\theta}_n^{(MCE)}$ of $\theta$ is defined as

$$\hat{\theta}_n^{(MCE)} = \arg\min_{\theta \in \Theta} D(f_\theta, \hat{f}_n(\lambda)).$$

(2.3)

Suppose that Assumptions (A.1) and (A.2) hold and $f = f_\theta$, then

$$\sqrt{n}(\hat{\theta}_n^{(MCE)} - \theta) \xrightarrow{L} N(0, F(\theta)^{-1}),$$

(2.4)

where

$$F(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} \log f_\theta(\lambda) \frac{\partial}{\partial \theta'} \log f_\theta(\lambda) d\lambda,$$

(2.5)

which is referred to as the Fisher information matrix in time series analysis; refer to Taniguchi (1987) and Taniguchi and Kakizawa (2001). If we select an appropriate $K(\cdot)$,
the minimum contrast estimator (2.3) provides explicit, non-iterative, and efficient estimators for various spectral parameterizations.

Suppose that the spectral density \( f_\theta(\lambda) \) is parameterized as

\[
f_\theta(\lambda) = S\{A_\theta(\lambda)\},
\]

(2.6)

where \( A_\theta(\lambda) = \sum_j \theta_j \exp(ij\lambda) \) and \( S(\cdot) \) is a continuously three times differentiable bijective function. In order to obtain non-iterative estimators, the following relation should be imposed;

\[
K \left[ \frac{S(A_\theta(\lambda))}{f(\lambda)} \right] = c_1(\lambda)A_\theta(\lambda)^2 + c_2(\lambda)A_\theta(\lambda) + c_3(\lambda) + c_4 \log S\{A_\theta(\lambda)\},
\]

(2.7)

where \( c_i(\lambda), i = 1, 2, 3, \) are functions that are independent of \( \theta \), and \( c_4 \) is a constant that is independent of \( \theta \) and \( \lambda \). If we estimate an innovation-free parameter \( \theta = (\theta_1, \ldots, \theta_q)' \), then from Theorem 6 of Taniguchi (1987), we observe that the non-iterative estimator is given by

\[
\hat{\theta}_n^{(MCE)} = \hat{R}^{-1}\hat{r},
\]

(2.8)

where

\[
\hat{R} = \left[ R(j-l) \right] = \int_{-\pi}^{\pi} G_1(\hat{f}_n(\lambda)) \cos(j-l)\lambda d\lambda,
\]

(2.9)

and

\[
\hat{r} = [\hat{r}(l)] = \int_{-\pi}^{\pi} G_2(\hat{f}_n(\lambda)) \cos l\lambda d\lambda.
\]

(2.10)

In this case, \( G_i(\cdot) \), \( i = 1, 2, \) satisfies a uniform Lipschitz condition (of order 1) in \([-\pi, \pi]\).

For AR models with spectral density \( f_\theta(\lambda) = \sigma^2/2\pi|\sum_{j=0}^{p} \theta_j e^{ij\lambda}|^2 \), where \( \theta_0 = 1 \) and \( \sum_{j=0}^{q} \theta_j z^j \neq 0 \) for \( |z| \leq 1 \), select \( K_{AR}(x) = \log x + \frac{1}{x} \); therefore, the non-iterative estimator is obtained by selecting \( G_1(x) = G_2(x) = x \). For MA models with spectral density \( f_\theta(\lambda) = \sigma^2/2\pi|\sum_{j=0}^{p} \theta_j e^{ij\lambda}|^2 \), where \( \theta_0 = 1 \) and \( \sum_{j=0}^{q} \theta_j z^j \neq 0 \) for \( |z| \leq 1 \), select \( K_{MA}(x) = -\log x + x \); therefore, the non-iterative estimator is obtained by selecting \( G_1(x) = G_2(x) = x^{-1} \). In the case where \( f_\theta(\lambda) = \sigma^2 \exp \left[ \sum_{j=0}^{q} \theta_j \cos(j\lambda) \right] \), \( \theta_0 = 1 \) (refer to Bloomfield (1973)), select \( K_E(x) = (\log(x))^2 \); therefore, the non-iterative estimator is obtained by selecting \( G_1(x) = 1/2 \) and \( G_2(x) = \log x \).

Based on the asymptotic normality result for \( \hat{\theta}_n^{(MCE)} \) in (2.4), it follows that

\[
n(\hat{\theta}_n^{(MCE)} - \theta)' F(\hat{\theta}_n^{(MCE)})(\hat{\theta}_n^{(MCE)} - \theta) \xrightarrow{\mathcal{L}} \chi^2(q), \quad \text{as} \quad n \to \infty,
\]

(2.11)
where
\[
F(\hat{\theta}_n^{(MCE)}) = \frac{1}{4\pi} \left[ \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} \log f_\theta(\lambda) \frac{\partial}{\partial \theta'} \log f_\theta(\lambda) d\lambda \right]_{\theta = \hat{\theta}_n^{(MCE)}}.
\] (2.12)

For any \(d > 0\), let
\[
R_n = \left\{ \theta \in \mathbb{R}^q : (\theta - \hat{\theta}_n^{(MCE)})' F(\hat{\theta}_n^{(MCE)}) (\theta - \hat{\theta}_n^{(MCE)}) \leq d^2 \lambda(F(\hat{\theta}_n^{(MCE)})) \right\},
\] (2.13)
where \(\lambda(F(\hat{\theta}_n^{(MCE)}))\) is the smallest eigenvalue of \(F(\hat{\theta}_n^{(MCE)})\). Then, \(R_n\) defines an ellipsoid with a maximum axis equal to \(2d\) \((d > 0)\), and it is in this sense that the size of the ellipsoid is fixed. Moreover, for any \(\alpha \in (0, 1)\), \(n_0(d)\) is determined by
\[
n_0(d) = \text{smallest integer } \geq a^2/d^2 \lambda(F(\theta)),
\] (2.14)
where \(a^2\) satisfies \(P[\chi^2(q) \leq a^2] = 1 - \alpha\), and \(\lambda(F(\theta))\) is the smallest eigenvalue of the covariance matrix \(F(\theta)\). From (2.13), for \(\theta \in \Theta\), we have
\[
\lim_{d \to 0} P(\theta \in R_{n_0(d)}) = 1 - \alpha.
\] (2.15)

This result in (2.15) shows that for a small value of \(d\), the sample size \(n_0(d)\) yields an ellipsoidal confidence region of a fixed size and a prescribed coverage probability. However, the sample size \(n_0(d)\) cannot be used in practice because it depends on unknown parameters. In order to overcome this, we define a stopping rule
\[
T_d = \inf \left\{ n \geq m, n \geq a^2/d^2 \lambda(F(\hat{\theta}_n^{(MCE)})) \right\},
\] (2.16)
where \(m\) is the initial sample size. The confidence ellipsoid \(R_{T_d}\) has the length of the major axis equal to \(2d\). Moreover, we have the following theorems whose proofs are provided in Section 3.

**Theorem 2.1** Suppose that Assumptions (A.1) and (A.2) hold, and \(\theta \in \Theta\). Then, for the stopping rule \(T_d\) defined in (2.16), the following holds.

\[
(i) \quad T_d/n_0(d) \to 1 \quad a.s. \quad as \quad d \to 0,
\] (2.17)

where \(d_0(d)\) is as in (2.14), and

\[
(ii) \quad \sqrt{T_d}(\hat{\theta}_d^{(MCE)} - \theta) \xrightarrow{\mathcal{D}} N(0, F(\theta)^{-1})
\] (2.18)

\[
(iii) \quad \lim_{d \to 0} P[\theta \in R_{T_d}] = 1 - \alpha \quad (asymptotic \ consistency).
\] (2.19)
Theorem 2.2 Suppose that Assumptions (A.1) and (A.2) hold, and \( \theta \in \Theta \). Then, for the stopping rule \( T_d \) and \( n_0(d) \) defined in (2.16) and (2.14), respectively, the following holds.

(i) \( \{T_d/n_0(d); 0 < d < 1\} \) is uniformly integrable \( (2.20) \) and

(ii) \( \lim_{d \to 0} E(T_d/n_0(d)) = 1 \) (asymptotic efficiency). \( (2.21) \)

The third part of Theorem 2.1 states that the coverage probability of the sequential fixed size confidence ellipsoid is asymptotically, as the size of the ellipsoid approaches zero, the desired value \( 1 - \alpha \). Theorem 2.2 asserts that this is achieved with an expected sample size that is asymptotically equivalent to the nonrandom sample size that would have been used, had \( \lambda(F(\theta)) \) been known.

3 Proofs

In this section, we present proofs for Theorems 2.1 and 2.2. The proofs for these Theorems are based on the following lemmas. Let \( \theta = (\theta_1, \ldots, \theta_q)' \) and \( \hat{\theta}_n^{(MCE)} = (\hat{\theta}_{n,1}^{(MCE)}, \ldots, \hat{\theta}_{n,q}^{(MCE)})' \). ||{·}||_p denotes the \( L_p \)-norm, i.e., \( ||{·}||_p = [E|{·}|^p]^{1/p} \).

Lemma 3.1 Suppose that (A.1) and (A.2) hold. If \( f = f_\theta \), where \( \theta \) is the innovation free parameter, then

\[
\max_{1 \leq i \leq q} \frac{||\hat{\theta}_n^{(MCE)} - \theta_i||_p}{M \cdot n^{-1/2}} = O(M \cdot n^{-1/2}). \tag{3.1}
\]

Proof. To prove (3.1), it suffices to show that for any constant vector \( \alpha = (\alpha_1, \ldots, \alpha_q)' \)

\[
\|\alpha'(\hat{\theta}_n^{(MCE)} - \theta)\|_p = O(M \cdot n^{-1/2}). \tag{3.2}
\]

From (2.8), note that

\[
\alpha'(\hat{\theta}_n^{(MCE)} - \theta) = \alpha'(\hat{R}^{-1}\hat{r} - R^{-1}r) = \alpha'(R^{-1}(\hat{r} - r) + (\hat{R}^{-1} - R^{-1})\hat{r}) = \alpha'(R^{-1}(\hat{r} - r) + \hat{R}^{-1}(\hat{R} - R)R^{-1}\hat{r}), \tag{3.3}
\]

where

\[
R = [R(j - l)] = \int_{-\pi}^{\pi} G_1(f_\theta(\lambda)) \cos(j - l) \lambda d\lambda \tag{3.4}
\]
and
\[ r = \lfloor r(l) \rfloor = \int_{-\pi}^{\pi} G_2(f_\theta(\lambda)) \cos l\lambda d\lambda. \quad (3.5) \]

Using the Minkowski inequality, we observe
\[ \|\alpha'(\hat{\theta}_n^{(MCE)} - \theta)\|_p \leq \|\alpha'R^{-1}(\hat{r} - r)\|_p + \|\alpha'\hat{R}^{-1}(\hat{R} - R)R^{-1}\hat{r}\|_p = L_1 + L_2 \quad \text{(say)} . \quad (3.6) \]

We first evaluate \( L_1 \) in (3.6). From the Minkowski inequality,
\[ \|L_1\|_p = \left\| \sum_{i,j=1}^{q} \alpha_i R_{ij}^{-1}(\hat{r}(j) - r(j)) \right\|_p \leq \sum_{i,j=1}^{q} |\alpha_i| \|R_{ij}^{-1}\| \|\hat{r}(j) - r(j)\|_p , \quad (3.7) \]

where \( R_{ij}^{-1} \) is the \((i, j)\)th element of \( R^{-1} \). Note that \( G_2(\cdot) \) satisfies the Lipschitz condition of order 1, (2.10), and (3.5); we observe that for some constants \( K_1 > 0 \) and \( K_2 > 0 \),
\[ \|\hat{r}(j) - r(j)\|_p = \left\| \int_{-\pi}^{\pi} (G_2(\hat{f}_n(\lambda) - G_2(f_\theta(\lambda))) \cos j\lambda d\lambda \right\|_p \leq \left\| \int_{-\pi}^{\pi} K_1 |\hat{f}_n(\lambda) - f_\theta(\lambda)| \cos j\lambda d\lambda \right\|_p \leq K_1 \left\| \sup_{|\lambda| \leq \pi} |\hat{f}_n(\lambda) - f_\theta(\lambda)| \int_{-\pi}^{\pi} \cos j\lambda d\lambda \right\|_p \leq K_2 \left\| \sup_{|\lambda| \leq \pi} |\hat{f}_n(\lambda) - f_\theta(\lambda)| \right\|_p . \quad (3.8) \]

From the equation (4.7) by Shiohama and Taniguchi (2004), we have
\[ \left\| \sup_{|\lambda| \leq \pi} |\hat{f}_n(\lambda) - f_\theta(\lambda)| \right\|_p = (M \cdot n^{-1/2}) . \quad (3.9) \]

Hence \( L_1 = O(M \cdot n^{-1/2}) \). \( L_2 \) can be evaluated as follows:
\[ \|L_2\|_p = \left\| \sum_{i,j,k,l=1}^{q} \alpha_i R_{ij}^{-1}(\hat{R}(j - k) - R(j - k))R_{kl}^{-1}\hat{r}(l) \right\|_p \leq \sum_{i,j,k,l=1}^{q} |\alpha_i| \|R_{kl}^{-1}\| \left\| \hat{R}_{ij}^{-1}\hat{r}(l)(\hat{R}(j - k) - R(j - k)) \right\|_p \leq \sum_{i,j,k,l=1}^{q} |\alpha_i| \|R_{kl}^{-1}\| \left\| \hat{R}_{ij}^{-1}\hat{r}(l) \right\|_{2p} \left\| \hat{R}(j - k) - R(j - k) \right\|_{2p} . \quad (3.10) \]
As before we observe
\[ \left\| \hat{R}(j-k) - R(j-k) \right\|_{2p} = O(M \cdot n^{-1/2}). \] (3.11)

Since
\[ \left\| \hat{R}_{ij}^{-1}(l) \right\|_{2p} = O(1), \] (3.12)

the desired result can be obtained. \( \square \)

**Lemma 3.2** Suppose that (A.1) and (A.2) hold. If \( f = f_\theta \), where \( \theta \) is the innovation-free parameter, then we have
\[ \max_{1 \leq i,j \leq k} \left\| F_{ij}(\hat{\theta}_n^{(MCE)}) - F_{ij}(\theta) \right\|_{p/2} = O(M \cdot n^{-1/2}), \] (3.13)

where \( F_{ij}(\hat{\theta}_n^{(MCE)}) \) and \( F_{ij}(\theta) \) are the \((i, j)\)th elements of \( F(\hat{\theta}_n^{(MCE)}) \) and \( F(\theta) \), respectively.

**Proof.** On the basis of the mean-value theorem, we have
\[ F_{ij}(\hat{\theta}_n^{(MCE)}) - F_{ij}(\theta) = \frac{\partial}{\partial \theta} F_{ij}(\theta^*)(\hat{\theta}_n^{(MCE)} - \theta), \] (3.14)

where \( \| \hat{\theta}_n^{(MCE)} - \theta^* \| \leq \| \hat{\theta}_n^{(MCE)} - \theta \| \). On the basis of Theorem 3 by Taniguchi (1987), we have \( \hat{\theta}_n^{(MCE)} \rightarrow_p \theta \), which implies that \( \theta^* \rightarrow_p \theta \); hence,
\[ \left\| F_{ij}(\hat{\theta}_n^{(MCE)}) - F_{ij}(\theta) \right\|_{p/2} = \left\| \frac{\partial}{\partial \theta} F_{ij}(\theta^*)(\hat{\theta}_n^{(MCE)} - \theta) \right\|_{p/2} \leq \left\| \frac{\partial}{\partial \theta} F_{ij}(\theta^*) \right\|_p \left\| (\hat{\theta}_n^{(MCE)} - \theta) \right\|_p \]. (3.15)

From (A.2) we have \( \| \partial/\partial \theta F_{ij}(\theta^*) \|_p = O(1) \). Hence, from Lemma 3.1, we obtain (3.13). \( \square \)

**Lemma 3.3** Under the same assumptions as those in Lemma 3.2, we have
\[ \| \lambda(F(\hat{\theta}_n^{(MCE)})) - \lambda(F(\theta)) \|_{p/2} = O(M \cdot n^{-1/2}). \] (3.16)

In particular, for any \( \varepsilon > 0 \),
\[ P(\| \lambda(F(\hat{\theta}_n^{(MCE)})) - \lambda(F(\theta)) \| > \varepsilon) = O(M^p \cdot n^{-p/2}). \] (3.17)
Proof of Theorem 2.1 In order to prove (i), we observe from (3.17) and the Borel-Cantelli lemma that
\[ \lambda(F(n^{MCE})) \rightarrow \lambda(F(\theta)) \quad \text{a.s. as } n \rightarrow \infty. \] (3.18)
Let \( f(n) = n\lambda(F(n^{MCE})) / \lambda(F(\theta)) \) and \( t = a^2 / d\lambda(F(\theta)) = n_0(d) \rightarrow \infty \) as \( d \rightarrow 0 \). Then the conditions of Lemma 1 of Chow and Robbins (1965) are satisfied, and hence
\[ \lim_{t \rightarrow \infty} T_d / t = \lim_{d \rightarrow 0} T_d / n_0(d) = 1 \quad \text{a.s.} \]
It is clear that (ii) implies (iii). So, only (ii) needs to be proved. From Theorem 5 of Taniguchi (1987) we have
\[ \sqrt{n}(\hat{\theta}^{\{MCE\}} - \theta) = \sqrt{n} \int_{-\pi}^{\pi} \rho_{f_\theta} \{ \hat{f}_n(\lambda) - f_\theta(\lambda) \} d\lambda + o_p(1), \] (3.19)
where
\[ \rho_{f_\theta} = \left[ \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} \log f_\theta(\lambda) \frac{\partial}{\partial \theta} \log f_\theta(\lambda) \right]^{-1} \frac{\partial}{\partial \theta} f_\theta^{-1}(\lambda). \] (3.20)
This implies that the limiting distribution of \( \sqrt{n}(\hat{\theta}^{\{MCE\}} - \theta) \) is described by the integral functional of the spectral density. Let \( \xi_n = \sqrt{n} \int_{-\pi}^{\pi} \rho_{f_\theta} \{ \hat{f}_n(\lambda) - f_\theta(\lambda) \} d\lambda \). To show (ii), we need to show that the sequence \( \{ \xi_n, n \geq 1 \} \) is uniformly continuous in probability, that is,
\[ P \left\{ \max_{0 \leq k \leq n_0} \| \xi_{n+k} - \xi_n \| \geq \varepsilon \right\} < \varepsilon \quad \text{for all } n \geq 1, \] (3.21)
where \( \| \cdot \| \) is the Euclidian norm. Essentially,
\[
\begin{align*}
\| \xi_{n+k} - \xi_n \| & = \left\| \sqrt{n+k} \int_{-\pi}^{\pi} \rho_{f_\theta} \{ \hat{f}_{n+k}(\lambda) - f_\theta(\lambda) \} d\lambda - \sqrt{n} \int_{-\pi}^{\pi} \rho_{f_\theta} \{ \hat{f}_n(\lambda) - f_\theta(\lambda) \} d\lambda \right\| \\
& = \left\| \sqrt{n+k} \int_{-\pi}^{\pi} \rho_{f_\theta} \hat{f}_{n+k}(\lambda) d\lambda - \sqrt{n+k} \int_{-\pi}^{\pi} \rho_{f_\theta} \hat{f}_n(\lambda) d\lambda + \sqrt{n+k} \int_{-\pi}^{\pi} \rho_{f_\theta} \hat{f}_n(\lambda) d\lambda - \sqrt{n} \int_{-\pi}^{\pi} \rho_{f_\theta} \hat{f}_n(\lambda) d\lambda \right\| \\
& \leq \left\| (\sqrt{n+k} - \sqrt{n}) \int_{-\pi}^{\pi} \rho_{f_\theta} \{ \hat{f}_n(\lambda) - f_\theta(\lambda) \} d\lambda \right\| + \left\| (\sqrt{n+k} - \sqrt{n}) \int_{-\pi}^{\pi} \rho_{f_\theta} \{ \hat{f}_{n+k}(\lambda) - \hat{f}_n(\lambda) \} d\lambda \right\|
\end{align*}
\]
It can be observed that
\[ \sqrt{n} \left\| \int_{-\pi}^{\pi} \rho_{f_\theta} \{ \hat{f}_n(\lambda) - f_\theta(\lambda) \} d\lambda \right\| = o_p(1). \]
We also observe that
\[
\max_{0 \leq k \leq n \delta} \left\| \sqrt{n + k} \int_{-\pi}^{\pi} \rho_{f_0} \{ \hat{f}_{n+k}(\lambda) - \hat{f}_n(\lambda) \} d\lambda \right\| = O_p \left( \sqrt{n} \right) + \int_{-\pi}^{\pi} \rho_{f_0} \{ \hat{f}_{n+k}(\lambda) - f_\theta(\lambda) \} d\lambda - \int_{-\pi}^{\pi} \rho_{f_0} \{ \hat{f}_n(\lambda) - f_\theta(\lambda) \} d\lambda \]
\[
\left\| \right\| = O_p(\sqrt{n}) \sqrt{1 + \delta} \max_{0 \leq k \leq n \delta} \int_{-\pi}^{\pi} \rho_{f_0} \{ \hat{f}_{n+k}(\lambda) - f_\theta(\lambda) \} d\lambda - \int_{-\pi}^{\pi} \rho_{f_0} \{ \hat{f}_n(\lambda) - f_\theta(\lambda) \} d\lambda \]
\[
\left\| \right\| (3.22)
\]
which, along with (3.22), implies (3.21). Therefore, using Anscombe’s theorem we obtain
\[
\sqrt{T_d} \int_{-\pi}^{\pi} \rho_{f_0} \{ \hat{f}_{T_d}(\lambda) - f_\theta(\lambda) \} d\lambda \overset{d}{\rightarrow} N(0, F(\theta)^{-1}) \quad \text{as } d \to 0,
\]
(3.23)
which implies (ii).
}\]

On the basis of Theorem 2.1, \( d^2 T_d (F(\theta))/a^2 \to 1 \) a.s. as \( d \to 0 \). Hence, to prove the asymptotic efficiency, it is sufficient to show that \( \{d^2 T_d : d \in (0, 1)\} \) is uniformly integrable.

**Proof of Theorem 2.2.** Let \( \delta > 0 \) and \( K_d = [a^2/d^{-2} \lambda^{-1}(F(\theta))d(1+\delta)] + 1 \). Then, for \( k \geq K_d \) and some \( \eta > 0 \), it can be shown that
\[
P[T_d \geq k] \leq P \left[ \left| \lambda^{-1}(F(\hat{\theta}_k^{(MCE)})) - \lambda^{-1}(F(\theta)) \right| \geq \eta \right]
\]
\[
= O(M^p \cdot k^{-p/2}),
\]
(3.24)
where (3.17) is used to obtain the last equation. This implies that \( \sum_{k \geq 1} P(T_d > k) < \infty \). Based on this and on the arguments by Woodroofe (1982), it follows that
\[
\{d^2 T_d : d \in (0, 1)\} \text{ is uniformly integrable.}
\]
(3.25)
Hence, \( \lim_{d \to 0} E(T_d/n_0(d)) = 1 \).

4 Some Related Fixed Size Confidence Sets

*Fixed proportional accuracy confidence ellipsoids*

Suppose \( \theta_i, i = 1, \ldots, q \), are nonzero and at least one of the parameter values is near the origin, then a smaller confidence ellipsoid can be constructed for \( \theta \) which gives us an improvement in the accuracy of the estimates of small coordinates. One approach is to
construct an ellipsoidal region such that the statistical distance between $\hat{\theta}_n^{(MCE)}$ and $\theta$ is less than a certain fraction of the true value of $\theta(1) = \min_{1 \leq j \leq q} |\theta_j|$. This yields the following ellipsoidal region.

$$\Gamma_n = \left\{ z : (z - \hat{\theta}_n^{(MCE)})'F(\hat{\theta}_n^{(MCE)})(z - \hat{\theta}_n^{(MCE)}) \leq d^2 \lambda(F(\hat{\theta}_n^{(MCE)}))\hat{\theta}_n(1) \right\}$$

for $d > 0$, where $\hat{\theta}_n(1) = \min_{1 \leq j \leq q} |\hat{\theta}_{n,j}|$. $\Gamma_n$ defines an ellipsoid having the length of the major axis equal to $2d\sqrt{\hat{\theta}_n(1)}$.

For any given $\alpha \in (0, 1)$ and $d > 0$, it is desired to have

$$P[\theta \in \Gamma_n] \approx 1 - \alpha.$$  \hfill (4.2)

Since $\hat{\theta}_n^{(MCE)} \to \theta$ almost surely, $\hat{\theta}_n(1) \to \theta(1)$ a.s. as $n \to \infty$, and therefore,

$$\frac{(\hat{\theta}_n^{(MCE)} - \theta)'F(\hat{\theta}_n^{(MCE)})(\hat{\theta}_n^{(MCE)} - \theta)}{\hat{\theta}_n(1)} \xrightarrow{d} \chi^2(q)/\theta(1).$$  \hfill (4.3)

Hence, to satisfy (4.2), we define a sample size

$$t_0(d) = \text{smallest integer } \geq a^2/[d^2\lambda(F(\theta))\theta(1)],$$

where $a^2$ and $\lambda(F(\theta))$ are defined as in (2.14). Since both $\lambda(F(\theta))$ and $\theta(1)$ are unknown, it is impossible to decide the sample size in advance. This suggests a stopping time

$$N_d = \inf\{n \geq m : n \geq a^2/[d^2\lambda(F(\hat{\theta}_n^{(MCE)}))\hat{\theta}_n(1)]\}.$$  \hfill (4.5)

Then we obtain the following theorem.

**Theorem 4.1** Suppose that Assumptions (A.1) and (A.2) hold, and $\theta \in \Theta$. Then, for the stopping rule $N_d$ defined in (4.5), the following holds.

(i) $N_d/t_0(d) \to 1$ a.s. as $d \to 0$,

(ii) $\lim_{d \to 0} P[\theta \in \Gamma_{N_d}] = 1 - \alpha$,

where $t_0(d)$ is as in (4.4) and

(iii) $\{N_d/t_0(d); 0 < d < 1\}$ is uniformly integrable,

(iv) $\lim_{d \to 0} E[N_d/t_0(d)] = 1$.  \hfill (4.9)
Confidence interval for a linear combination of \( \theta \)

In practice, we may only be interested in a particular linear combination of the components of \( \theta \), rather than the entire vector. That is, for some \( C \in \mathbb{R}^q, \|C\| \neq 0 \), we can construct a fixed-width confidence interval for \( C'\theta \). It follows from the asymptotic normality of \( \hat{\theta}_{n(MCE)} \) that, as \( n \to \infty \),

\[
\sqrt{n}(C'\hat{\theta}_{n(MCE)} - C'\theta) \xrightarrow{d} N(0, C'F(\theta)^{-1}C). \tag{4.10}
\]

If \( F(\theta) \) were known, then for a given \( d > 0, \alpha \in (0,1) \), and the sample size is determined by

\[
h_0(d) = \text{smallest integer } \geq \frac{z_{\alpha/2}^2}{[d^2C'F(\theta)^{-1}C]}. \tag{4.11}
\]

From (4.10), we have

\[
\lim_{d \to 0} P(C'\theta \in [C'\hat{\theta}_{n(MCE)}(H_d) - d, C'\hat{\theta}_{n(MCE)}(H_d) + d]) = 1 - \alpha, \tag{4.12}
\]

where \( z_{\alpha/2}^2 \) satisfies \( \Phi(z_{\alpha/2}) - \Phi(-z_{\alpha/2}) = 1 - \alpha \). However, since \( F(\theta) \) is unknown, the sample size \( h_0(d) \) cannot be used. As observed previously, (4.11) suggests the stopping rule

\[
H_d = \inf\{n \geq m : n \geq \frac{z_{\alpha/2}^2}{[d^2C'F(\hat{\theta}_{n(MCE)}(H_d))^{-1}C]}\}. \tag{4.13}
\]

We have the following theorem.

**Theorem 4.2** Suppose that Assumptions (A.1) and (A.2) hold, and \( \theta \in \Theta \). Then, for the stopping rules \( H_d \) and \( h_0(d) \) defined in (4.13) and (4.11), respectively, the following hold.

\[
(i) \quad H_d/h_0(d) \to 1 \quad \text{a.s. as } d \to 0, \tag{4.14}
\]

\[
(ii) \quad \lim_{d \to 0} P\left[C'\theta \in [C'\hat{\theta}_{H_d(MCE)} - d, C'\hat{\theta}_{H_d(MCE)} + d]\right] = 1 - \alpha. \tag{4.15}
\]

Furthermore,

\[
(iii) \quad \{H_d/h_0(d); 0 < d < 1\} \quad \text{is uniformly integrable}, \tag{4.16}
\]

\[
(iv) \quad \lim_{d \to 0} E[H_d/h_0(d)] = 1. \tag{4.17}
\]

Theorems 4.1 and 4.2 can be proved using arguments similar in the proofs for Theorems 2.1 and 2.2, and are therefore omitted for brevity.
5 Simulations

In this section, we present some Monte Carlo simulations to verify that our sequential procedures are asymptotically consistent and efficient. We consider the standard Gaussian AR(1), MA(1), and AR(2) models.

1. AR(1): \( X_t = \theta X_{t-1} + \varepsilon_t, \quad \theta = 0.1, 0.2, \ldots, 0.9; \)
2. MA(1): \( X_t = \varepsilon_t + \theta \varepsilon_{t-1}, \quad \theta = 0.1, 0.2, \ldots, 0.9; \)
3. AR(2): \( X_t = \theta_1 X_{t-1} + \theta_2 X_{t-2} + \varepsilon_t, \) where \( (\theta_1, \theta_2) = (1.1, -0.24), (0.6, -0.4), (0.2, -0.35), (0.2, 0.35), (0.05, 0.35), \) and \( (1.5, -0.75). \)

The innovations \( \{\varepsilon_t\} \) are standard normal distributions. For the simulation study we set \( n_0(d) = 100, 200, 300, 400, 500, \) and the coverage probability as 0.90. The initial sample size \( m \) is chosen as \( m = 2 \) for AR(1) and AR(2) models, and \( m = 50 \) for MA(1) models. For each choice of parameters, 1000 replications of the series were generated. The results are presented in Tables 1, 2, and 3.

As observed from Tables 1 and 2, the expected sample sizes and the coverage probabilities depend on the value of \( \theta \). For smaller values of \( \theta \), the agreement between the asymptotic theory and the simulations is better regardless of the value of \( d \). On the other hand, the bias in \( T_d \) increases substantially with an increase in \( \theta \), while the relative discrepancy decreases with a decrease in \( d \). For AR(1) models, the rate of convergence of the coverage probability is not strongly affected by \( \theta \), while that for MA(1) models deteriorates with an increase in \( \theta \). It is also noted that for large values of \( \theta \), the standard deviations of the expected sample size for MA(1) models are larger than those for AR(1) models.

The situation for AR(2) models appears quite similar to what has been observed for AR(1) models. Let \( \pi_1 \) and \( \pi_2 \) be the roots of the characteristic equation \( 1 - \theta_1 z - \theta_2 z^2 = 0. \) The absolute values of \( \pi_1 \) and \( \pi_2 \) are also provided in Table 3. In case of AR(2) models with absolute values of roots close to the unit circle, the behavior of sequential procedure exhibits a poor performance.
Table 1. Simulation results of 90% confidence interval for AR(1) model.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n_0(d)$</th>
<th>$T_d$ (s.d.*$)$</th>
<th>$T_d/n_0(d)$</th>
<th>c.p.*</th>
<th>$d$</th>
<th>$n_0(d)$</th>
<th>$T_d$ (s.d.*$)$</th>
<th>$T_d/n_0(d)$</th>
<th>c.p.*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = 0.1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.164</td>
<td>100</td>
<td>99.69 (2.23)</td>
<td>0.9969</td>
<td>.917</td>
<td>0.161</td>
<td>100</td>
<td>100.47 (3.80)</td>
<td>1.0047</td>
<td>.904</td>
</tr>
<tr>
<td>0.116</td>
<td>200</td>
<td>199.97 (2.65)</td>
<td>0.9998</td>
<td>.914</td>
<td>0.114</td>
<td>200</td>
<td>200.53 (5.37)</td>
<td>1.0026</td>
<td>.913</td>
</tr>
<tr>
<td>0.094</td>
<td>300</td>
<td>299.97 (3.48)</td>
<td>0.9999</td>
<td>.901</td>
<td>0.093</td>
<td>300</td>
<td>300.08 (7.14)</td>
<td>1.0003</td>
<td>.890</td>
</tr>
<tr>
<td>0.082</td>
<td>400</td>
<td>400.03 (3.87)</td>
<td>1.0001</td>
<td>.908</td>
<td>0.081</td>
<td>400</td>
<td>400.71 (7.87)</td>
<td>1.0018</td>
<td>.914</td>
</tr>
<tr>
<td>0.073</td>
<td>500</td>
<td>499.80 (4.50)</td>
<td>0.9996</td>
<td>.912</td>
<td>0.072</td>
<td>500</td>
<td>500.51 (8.86)</td>
<td>1.0010</td>
<td>.901</td>
</tr>
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<td>$\theta = 0.3$</td>
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</tr>
<tr>
<td>0.157</td>
<td>100</td>
<td>101.09 (5.66)</td>
<td>1.0109</td>
<td>.910</td>
<td>0.151</td>
<td>100</td>
<td>102.26 (7.54)</td>
<td>1.0226</td>
<td>.921</td>
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<tr>
<td>0.111</td>
<td>200</td>
<td>201.73 (8.01)</td>
<td>1.0086</td>
<td>.907</td>
<td>0.107</td>
<td>200</td>
<td>202.65 (11.50)</td>
<td>1.0132</td>
<td>.907</td>
</tr>
<tr>
<td>0.091</td>
<td>300</td>
<td>302.34 (10.19)</td>
<td>1.0078</td>
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<td>0.087</td>
<td>300</td>
<td>302.91 (13.94)</td>
<td>1.0097</td>
<td>.901</td>
</tr>
<tr>
<td>0.078</td>
<td>400</td>
<td>401.80 (11.96)</td>
<td>1.0045</td>
<td>.914</td>
<td>0.075</td>
<td>400</td>
<td>402.61 (16.44)</td>
<td>1.0065</td>
<td>.907</td>
</tr>
<tr>
<td>0.070</td>
<td>500</td>
<td>502.00 (13.82)</td>
<td>1.0040</td>
<td>.897</td>
<td>0.067</td>
<td>500</td>
<td>502.99 (18.44)</td>
<td>1.0060</td>
<td>.906</td>
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<tr>
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</tr>
<tr>
<td>0.142</td>
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<td>104.44 (10.23)</td>
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<td>0.132</td>
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<td>107.78 (12.59)</td>
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<td>.874</td>
</tr>
<tr>
<td>0.100</td>
<td>200</td>
<td>203.93 (14.74)</td>
<td>1.0196</td>
<td>.909</td>
<td>0.093</td>
<td>200</td>
<td>207.24 (19.35)</td>
<td>1.0362</td>
<td>.896</td>
</tr>
<tr>
<td>0.082</td>
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<tr>
<td>0.071</td>
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<td>406.76 (22.41)</td>
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<td>.909</td>
<td>0.068</td>
<td>400</td>
<td>407.99 (29.15)</td>
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<td>.883</td>
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<tr>
<td>0.064</td>
<td>500</td>
<td>506.65 (24.13)</td>
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<td>.908</td>
<td>0.059</td>
<td>500</td>
<td>508.28 (32.79)</td>
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<td>0.117</td>
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<td>111.95 (16.01)</td>
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<td>222.82 (31.27)</td>
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<td>0.053</td>
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<td>.897</td>
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*s.d., standard deviation; c.p., coverage probability.*
Table 2. Simulation results of 90% confidence interval for MA(1) model.

<table>
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<th>$d$</th>
<th>$n_0(d)$</th>
<th>$T_d$ (s.d.$^*$)</th>
<th>$T_d/n_0(d)$</th>
<th>c.p.$^*$</th>
<th>$d$</th>
<th>$n_0(d)$</th>
<th>$T_d$ (s.d.$^*$)</th>
<th>$T_d/n_0(d)$</th>
<th>c.p.$^*$</th>
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<td>0.164</td>
<td>100</td>
<td>99.62 (2.75)</td>
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<td>0.161</td>
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<td>499.55 (4.60)</td>
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<td>201.96 (12.13)</td>
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<td>0.091</td>
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<td>301.08 (11.28)</td>
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<td>401.66 (18.01)</td>
<td>1.0042</td>
<td>.885</td>
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<td>0.070</td>
<td>500</td>
<td>501.00 (14.41)</td>
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<td>.887</td>
<td>0.067</td>
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<td>0.142</td>
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<td>101.88 (14.14)</td>
<td>1.0188</td>
<td>.835</td>
<td>0.132</td>
<td>100</td>
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<td>1.0396</td>
<td>.804</td>
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<tr>
<td>0.100</td>
<td>200</td>
<td>203.80 (16.51)</td>
<td>1.0190</td>
<td>.892</td>
<td>0.093</td>
<td>200</td>
<td>206.09 (22.72)</td>
<td>1.0305</td>
<td>.874</td>
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<td>0.082</td>
<td>300</td>
<td>303.54 (20.25)</td>
<td>1.0118</td>
<td>.883</td>
<td>0.076</td>
<td>300</td>
<td>310.76 (25.03)</td>
<td>1.0359</td>
<td>.882</td>
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<td>400</td>
<td>403.67 (22.64)</td>
<td>1.0092</td>
<td>.889</td>
<td>0.068</td>
<td>400</td>
<td>408.46 (30.05)</td>
<td>1.0212</td>
<td>.892</td>
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<tr>
<td>0.064</td>
<td>500</td>
<td>505.28 (25.19)</td>
<td>1.0106</td>
<td>.914</td>
<td>0.059</td>
<td>500</td>
<td>507.92 (33.09)</td>
<td>1.0158</td>
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<tr>
<td>0.117</td>
<td>100</td>
<td>106.92 (25.49)</td>
<td>1.0692</td>
<td>.728</td>
<td>0.098</td>
<td>100</td>
<td>111.41 (37.91)</td>
<td>1.1141</td>
<td>.530</td>
</tr>
<tr>
<td>0.083</td>
<td>200</td>
<td>211.36 (33.20)</td>
<td>1.0568</td>
<td>.681</td>
<td>0.070</td>
<td>200</td>
<td>217.90 (56.10)</td>
<td>1.0895</td>
<td>.590</td>
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<td>0.068</td>
<td>300</td>
<td>313.84 (38.40)</td>
<td>1.0461</td>
<td>.859</td>
<td>0.057</td>
<td>300</td>
<td>325.51 (67.35)</td>
<td>1.0850</td>
<td>.650</td>
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<tr>
<td>0.059</td>
<td>400</td>
<td>419.07 (39.83)</td>
<td>1.0477</td>
<td>.852</td>
<td>0.049</td>
<td>400</td>
<td>432.80 (75.80)</td>
<td>1.0820</td>
<td>.674</td>
</tr>
<tr>
<td>0.053</td>
<td>500</td>
<td>518.10 (46.63)</td>
<td>1.0362</td>
<td>.883</td>
<td>0.044</td>
<td>500</td>
<td>537.25 (78.29)</td>
<td>1.0745</td>
<td>.750</td>
</tr>
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</tr>
<tr>
<td>0.072</td>
<td>100</td>
<td>133.05 (64.05)</td>
<td>1.3305</td>
<td>.529</td>
<td>0.051</td>
<td>200</td>
<td>233.29 (102.83)</td>
<td>1.1665</td>
<td>.374</td>
</tr>
<tr>
<td>0.051</td>
<td>200</td>
<td>233.29 (102.83)</td>
<td>1.1665</td>
<td>.374</td>
<td>0.041</td>
<td>300</td>
<td>337.05 (132.64)</td>
<td>1.1235</td>
<td>.335</td>
</tr>
<tr>
<td>0.036</td>
<td>400</td>
<td>425.08 (158.88)</td>
<td>1.0627</td>
<td>.286</td>
<td>0.032</td>
<td>500</td>
<td>538.13 (181.26)</td>
<td>1.0763</td>
<td>.299</td>
</tr>
</tbody>
</table>

*s.d., standard deviation; c.p., coverage probability.
Table 3. Simulation results of 90% confidence interval for AR(2) models.

<table>
<thead>
<tr>
<th>d</th>
<th>n_0(d)</th>
<th>( T_d ) (s.d.*)</th>
<th>( \bar{T}_d/n_0(d) )</th>
<th>c.p.*</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 = 1.1, \theta_2 = -0.24,</td>
<td></td>
<td></td>
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<tr>
<td>0.286</td>
<td>100</td>
<td>92.45 (5.85)</td>
<td>0.9245</td>
<td>0.662</td>
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<tr>
<td>0.202</td>
<td>200</td>
<td>190.15 (8.00)</td>
<td>0.9508</td>
<td>0.912</td>
</tr>
<tr>
<td>0.165</td>
<td>300</td>
<td>288.42 (9.10)</td>
<td>0.9614</td>
<td>0.854</td>
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<tr>
<td>0.143</td>
<td>400</td>
<td>387.02 (11.33)</td>
<td>0.9676</td>
<td>0.888</td>
</tr>
<tr>
<td>0.128</td>
<td>500</td>
<td>486.87 (12.94)</td>
<td>0.9737</td>
<td>0.901</td>
</tr>
</tbody>
</table>

| \( \theta = 0.6, \theta_2 = -0.4, | | | | |
| 0.235 | 100   | 101.52 (7.62) | 1.0152 | 0.776 |
| 0.166 | 200   | 201.36 (11.50) | 1.0068 | 0.906 |
| 0.136 | 300   | 302.10 (14.93) | 1.0070 | 0.898 |
| 0.118 | 400   | 402.12 (17.97) | 1.0053 | 0.888 |
| 0.105 | 500   | 502.06 (20.70) | 1.0041 | 0.891 |

| \( \theta_1 = 0.2, \theta_2 = -0.35, | | | | |
| 0.215 | 100   | 101.17 (8.01) | 1.0117 | 0.725 |
| 0.152 | 200   | 201.81 (12.61) | 1.0091 | 0.905 |
| 0.124 | 300   | 302.68 (15.59) | 1.0089 | 0.903 |
| 0.108 | 400   | 401.32 (18.47) | 1.0033 | 0.896 |
| 0.096 | 500   | 502.06 (19.65) | 1.0052 | 0.907 |

| \( \theta_1 = 0.2, \theta_2 = 0.35, | | | | |
| 0.230 | 100   | 101.68 (11.21) | 1.0168 | 0.749 |
| 0.163 | 200   | 202.61 (16.64) | 1.0130 | 0.909 |
| 0.133 | 300   | 302.25 (21.18) | 1.0075 | 0.894 |
| 0.115 | 400   | 402.05 (25.00) | 1.0051 | 0.897 |
| 0.103 | 500   | 503.00 (26.95) | 1.0060 | 0.903 |

| \( \theta_1 = 0.05, \theta_2 = 0.35, | | | | |
| 0.209 | 100   | 107.61 (9.06) | 1.0761 | 0.748 |
| 0.148 | 200   | 208.14 (15.17) | 1.0407 | 0.900 |
| 0.120 | 300   | 308.85 (19.53) | 1.0295 | 0.893 |
| 0.104 | 400   | 407.45 (24.70) | 1.0186 | 0.897 |
| 0.093 | 500   | 508.01 (27.89) | 1.0160 | 0.909 |

| \( \theta_1 = 1.5, \theta_2 = -0.75, | | | | |
| 0.193 | 100   | 151.41 (7.62) | 1.5141 | 0.445 |
| 0.137 | 200   | 241.99 (11.49) | 1.2099 | 0.611 |
| 0.112 | 300   | 331.24 (28.77) | 1.1041 | 0.751 |
| 0.097 | 400   | 441.86 (34.84) | 1.1047 | 0.707 |
| 0.087 | 500   | 549.12 (34.86) | 1.0982 | 0.714 |

*s.d., standard deviation; c.p., coverage probability.

References


