EXACT SOLUTIONS OF A MODEL FOR ASSET PRICES BY K. TAKAOKA

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Abstract. We are concerned with a model for asset prices introduced by Koichiro Takaoka, which extends the well known Black-Scholes model. For the pricing of contingent claims, partial differential equation (PDE) is derived in a special case under the typical delta hedging strategy. We present an exact pricing formula by way of solving the equation.

1. Introduction

Over the last forty years, mathematical finance has developed into a research area rich in both the theoretical structure and powerful real-world applications. A principal breakthrough, as is well known, was made in 1973 by F. Black and M. Scholes [3] and R.C. Merton [10]. These articles were based on stochastic methodology, which involve, among other things, a Wiener process, Ito's Lemma, and the Markov property of diffusions. Celebrated Black-Scholes pricing formula for European call options has played a fundamental role since then. For more information and other background materials, we refer to [2][8][9][11][14] for instance and the references cited therein.

We recall that the basic Black-Scholes model consists of one random security and a risk-less cash account bond, whose prices, denoted respectively by S_t and B_t, are assumed to follow

$$\begin{aligned} S_t &= S_0 \exp(\sigma W_t + \mu t), \\ B_t &= e^{rt}. \end{aligned} \tag{1}$$

Here W_t stands for the one-dimensional Brownian motion with $W_0 = 0$ and S_0 is a nonnegative constant; fixed constants r, σ, and μ are the risk-less interest rate, the stock volatility, and the stock drift, respectively.

It has been criticized, however, that discrepancies are observed between real markets and what the Black-Scholes depicts. Major deficiencies stem from the assumption that the volatility σ is kept fixed in (1), and much effort has been paid to remedy such situation. See [2][6][9] for example and also [5].

On the other hand, Koichiro Takaoka, in his recent work [13], introduced a new model in order to well describe the behavior of real continuous process. His model assumes

$$\begin{aligned} S_t &= S_0 e^{rt} \int_0^\infty \exp(\sigma(W_t + Ct) - \frac{\sigma^2}{2} t) f(\sigma) d\sigma, \\ B_t &= e^{rt}, \end{aligned} \tag{2}$$

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where C denotes a constant. A nonnegative weight function $f(\sigma)$ is defined on $[0, \infty)$ and assumed to satisfy

$$\int_0^\infty f(\sigma)d\sigma = 1, \qquad \int_0^\infty \sigma f(\sigma)d\sigma < \infty.$$

We remark that the second condition on f is imposed from technical reasons. If we allow $f(\sigma) = \delta(\sigma - \bar\sigma)$ where δ represents the Dirac delta function, then we have $S_t = e^{rt}\exp(\bar\sigma(W_t + Ct) - (\bar\sigma^2 t/2))$ and we recover the Black-Scholes model (1) with the volatility $\bar\sigma$ by adjusting C properly; in other words, our new model (2) extends the basic Black-Scholes model (1).

The extended model (2) has various nice properties; the process S_t induces a complete market, it is Markov, and so on. A closed pricing formula for the European call option for example is also valid; however, exact explicit formula needs further specialization.

In this paper we deal with the case $f(\sigma) = 2^{-1}(\delta(\sigma - \bar\sigma + \varepsilon) + \delta(\sigma - \bar\sigma - \varepsilon))$ $(|\bar\sigma| > |\varepsilon|)$ so that (2) is reduced to

$$\begin{aligned} S_t &= S_0 e^{rt}\exp(\bar\sigma(W_t + Ct) - \frac{\bar\sigma^2 + \varepsilon^2}{2}t)\cosh(\varepsilon(W_t + (C - \bar\sigma)t)), \\ B_t &= e^{rt}. \end{aligned} \tag{3}$$

We are concerned with the exact pricing formula for contingent claims under the process (3). T. Sakaguchi [12] made a preliminary treatment of this problem based on the theory of partial differential equations (PDEs); in the current paper we put forward an investigation along this direction. The main observation of this article, which is stated in §3, claims that there is a class of explicit exact solutions for these PDEs. These explicit solutions are interpreted to be written on newly defined stochastic variable. We also show that the pricing formula is possible without introducing such variable, which is formulated in somewhat abstract way.

The organization of the paper is as follows: In §2 we recall the PDE which the price of contingent claims obeys. §3 is devoted to deduce the explicit exact formula under this framework. We solve the PDE with single varible of (3) in §4 and provide an abstract pricing formula. Discussions are given in §5.

2. Prices for contingent claims

In this section we wish to derive partial differential equations which contingent claims follow under the process governed by (2). Although the derivation is rather well known we present it here for the readers' convenience.

First we introduce an auxiliary stochastic variable defined by

$$A_t = S_0 e^{rt}\exp(\sigma(W_t + Ct) - \frac{\sigma^2 + \varepsilon^2}{2}t)\sinh(\varepsilon(W_t + (C - \sigma)t)) \tag{4}$$

Here and in (3) we drop the bar on σ. Then it is easy to verify that the next stochastic differential equation holds.

$$\begin{aligned} dS_t &= ((r + \sigma C)S_t + \varepsilon C A_t)dt + (\sigma S_t + \varepsilon A_t)dW_t \\ dA_t &= ((r + \sigma C)A_t + \varepsilon C S_t)dt + (\sigma A_t + \varepsilon S_t)dW_t. \end{aligned} \tag{5}$$

Let us denote by $V(t,S_t,A_t)$ the no-arbitrage price of an option. As is performed in §3.5 of [14] we construct a portfolio of one option and a number Δ_1, Δ_2 of the underlying asset S_t, A_t, respectively. The value Π of this contingent claim is

$$\Pi := V(t,S_t,A_t) - \Delta_1 S_t - \Delta_2 A_t.$$

We compute, removing the subscript t,

$$\begin{aligned} d\Pi &= dV - \Delta_1 dS - \Delta_2 dA \\ &= \frac{\partial V}{\partial t}dt + \Big(\frac{\partial V}{\partial S} - \Delta_1\Big)dS + \Big(\frac{\partial V}{\partial A} - \Delta_2\Big)dA \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\sigma S + \varepsilon A)^2 dt + \frac{\partial^2 V}{\partial S \partial A}(\sigma S + \varepsilon A)(\sigma A + \varepsilon S)dt + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}(\sigma A + \varepsilon S)^2 dt, \end{aligned}$$

where the use of $(dW)^2 = dt$ has been made. Putting $\Delta_1 = \partial V/\partial S$ and $\Delta_2 = \partial V/\partial A$, that is, we employ a standard delta hedging strategy, we discover that the portfolio Π follows the random-free process, which should be agreed with risk-less assets with a growth of $r\Pi dt$. As a result we arrive at a partial differential equation

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}(\sigma S + \varepsilon A)^2\frac{\partial^2 V}{\partial S^2} + (\sigma S + \varepsilon A)(\sigma A + \varepsilon S)\frac{\partial^2 V}{\partial S \partial A} + \frac{1}{2}(\sigma A + \varepsilon S)^2\frac{\partial^2 V}{\partial A^2} \\ - r\Big(V - S\frac{\partial V}{\partial S} - A\frac{\partial V}{\partial A}\Big) = 0 \qquad \text{in } t < T,\ S > 0,\ A \in \boldsymbol{R}. \end{aligned} \tag{6}$$

We note that a similar equation can be found in (2.4) of [1].

Now our intention is to solve (6) in any way with suitable maturity as well as boundary conditions.

3. Exact explicit pricing formula

We intend to show that there exists at least one exact explicit solution for (6) under appropriate conditions. For this purpose we try to find a solution of the form $V = V(t,Z)$ with $Z := A/S$. We observe that (6) becomes

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}(\sigma + \varepsilon Z)^2\Big(Z^2\frac{\partial^2 V}{\partial Z^2} + 2Z\frac{\partial V}{\partial Z}\Big) - (\sigma + \varepsilon Z)(\sigma Z + \varepsilon)\Big(Z\frac{\partial^2 V}{\partial Z^2} + \frac{\partial V}{\partial Z}\Big) \\ + \frac{1}{2}(\sigma Z + \varepsilon)^2\frac{\partial^2 V}{\partial Z^2} - rV = 0 \qquad \text{in } t < T,\ |Z| < 1, \end{aligned}$$

and hence we obtain

$$\frac{\partial V}{\partial t} + \frac{\varepsilon^2}{2}(1 - Z^2)^2\frac{\partial^2 V}{\partial Z^2} - \varepsilon(1 - Z^2)(\varepsilon Z + \sigma)\frac{\partial V}{\partial Z} - rV = 0. \tag{7}$$

At this point we make the following transformations

$$U(\tau, Z) := e^{k\tau}V(t,Z), \quad \tau := \frac{\varepsilon^2}{2}(T - t), \qquad k := \frac{r}{\varepsilon^2/2}, \quad \eta := \frac{\sigma}{\varepsilon},$$

which implies that

$$\frac{\partial U}{\partial \tau} = (1 - Z^2)^2 \frac{\partial^2 U}{\partial Z^2} - 2(1 - Z^2)(Z + \eta) \frac{\partial U}{\partial Z} \qquad \text{in } t < T,\ |Z| < 1.$$

Now we further make a change of variables

$$u(\tau, x) := U(\tau, Z), \qquad Z =: \tanh(x + 2\eta\tau).$$

We then finally discover that the typical heat equation emerges.

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{in } \tau > 0,\ -\infty < x < \infty.$$

It is standard to express the solution in terms of the heat kernel for a wide class of initial and boundary conditions. As a result, the next theorem, which is our principal observation of the current paper, is easily seen to follow.

Theorem 1. *For any bounded continuous maturity data $H(Z)$ for $|Z| < 1$, there exists a unique solution $V(t, Z)$ to (7) with $V|_{t=T} = H$. The explicit form of the solution is given by*

$$\begin{aligned} V(t, Z) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi\varepsilon^2(T-t)}} \int_{-\infty}^{\infty} H(\tanh y) \exp\Big[-\frac{(\tanh^{-1} Z - \varepsilon\sigma(T-t) - y)^2}{2\varepsilon^2(T-t)}\Big]\, dy \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi\varepsilon^2(T-t)}} \int_{-1}^{1} \frac{H(Y)}{1 - Y^2} \exp\Big[-\frac{(\tanh^{-1} Z - \varepsilon\sigma(T-t) - \tanh^{-1} Y)^2}{2\varepsilon^2(T-t)}\Big]\, dY. \end{aligned}$$

As an example of the theorem we resolve (7) supplemented by European call type condition $V|_{t=T} := \max\{Z - K, 0\}$ with $0 < K < 1$. Since $u(0, x) = \max\{\tanh x - K, 0\}$ in this case, we assert that

$$\begin{aligned} u(\tau, x) &= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \max\{\tanh y - K, 0\} \exp\Big[-\frac{(x-y)^2}{4\tau}\Big]\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{(\tanh^{-1} K - x)/\sqrt{2\tau}}^{\infty} (\tanh(\sqrt{2\tau} y + x) - K) \exp\Big[-\frac{y^2}{2}\Big]\, dy \\ &= \frac{1}{\sqrt{4\pi\tau}} \int_{\tanh^{-1} K}^{\infty} \exp\Big[-\frac{(x-y)^2}{4\tau}\Big]\, dy - KN\Big(\frac{x - \tanh^{-1} K}{\sqrt{2\tau}}\Big), \end{aligned}$$

where $N(d) := (2\pi)^{-1/2} \int_{-\infty}^{d} e^{-y^2/2} dy = (2\pi)^{-1/2} \int_{-d}^{\infty} e^{-y^2/2} dy$ is the cumulative function for a normal distribution. Returning to the original variable we deduce that

$$\begin{aligned} V(t, Z) = {} & \frac{e^{-r(T-t)}}{\sqrt{2\pi\varepsilon^2(T-t)}} \int_{K}^{1} \frac{Y}{1 - Y^2} \exp\Big[-\frac{(\tanh^{-1} Z - \varepsilon\sigma(T-t) - \tanh^{-1} Y)^2}{2\varepsilon^2(T-t)}\Big]\, dY \\ & - KN\Big(\frac{\tanh^{-1} Z - \varepsilon\sigma(T-t) - \tanh^{-1} K}{\sqrt{2\varepsilon^2(T-t)}}\Big). \end{aligned}$$

4. Existence theorem

Here we include the pricing formula for contingent claims written on S. To be specific we derive a PDE with single variable S and solve the equation.

Since the mapping $x \mapsto S_0 e^{rt} \exp(\sigma(x+Ct) - 2^{-1}(\sigma^2+\varepsilon^2)t)\cosh(\varepsilon(x+(C-\sigma)t))$ is monotone if $|\sigma/\varepsilon| > 1$, which is ascertained by the assumption, we are able to represent W_t by S_t as the inverse function of (3); namely,

$$W_t = F^{-1}(t, S_t),$$

where $F(t, W_t) := S_0 e^{rt} \exp(\sigma(W_t+Ct) - 2^{-1}(\sigma^2+\varepsilon^2)t)\cosh(\varepsilon(W_t+(C-\sigma)t))$. This inversion technique is suggested by the referee. We then deduce that

$$\begin{aligned} A_t &= S_0 e^{rt} \exp(\sigma(F^{-1}(t,S_t)+Ct) - \frac{\sigma^2+\varepsilon^2}{2}t)\sinh(\varepsilon(F^{-1}(t,S_t)+(C-\sigma)t)) \\ &=: G(t, S_t), \end{aligned}$$

which yields

$$dS_t = ((r+\sigma C)S_t + \varepsilon C G(t,S_t))dt + (\sigma S_t + \varepsilon G(t,S_t))dW_t.$$

Similar reasoning as in §2 applied to the no-arbitrage value $V(t, S_t)$ then leads to

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma S + \varepsilon G(t,S))^2 \frac{\partial^2 V}{\partial S^2} + r\Big(S\frac{\partial V}{\partial S} - V\Big) = 0 \qquad \text{in } t < T,\ S > 0. \tag{8}$$

Taking into account that $G(t, S_t)$ is sufficiently regular, we see that (8) is well-posed for uniformly continuous maturity data with moderate, say polynomial, growth condition. We refer to [4][7] for the details. In a summary we obtain the next theorem.

Theorem 2. *For uniformly continuous data $H(S)$ $(S > 0)$ with polynomial growth, there exists a unique solution $V(t, S)$ to (8) with $V|_{t=T} = H$.*

5. Discussions

We have determined a class of exact solutions to a model for asset prices introduced by Takaoka [13], which extends the basic Black-Scholes model [3]. Although our results are of mathematical nature, we believe that our established formula is worth publishing, since exact explicit resolution is somehow hardly known to be common in mathematical finance. In addition there would exist a possible usage as test problems in order to check the effectiveness of new numerical schemes.

One reason why our exact solutions written on Z have little interpretation as economics is that the hedging strategy is too elaborate to perform in practice. The deltas Δ_1 and Δ_2 defined in §2 are artificial and they are not expressed as the asset price S only. On the other hand the delta $\Delta := \partial V/\partial S$ of §4 is given by the single variable S; however, the computation involves an inverse function F^{-1} and the explicit form is a little obscure, although the numerical implementation is possible.

There remain many questions unanswered. To examine the existence of other family of exact solutions under different kind of conditions is an important step. Moreover and in particular free boundary type conditions, which are peculiar to American type or other types options, are challenging issues. These would be our future topics for researches and works are now in progress.

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