On the supremum and the quadratic variation of real-valued continuous local submartingales

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1 Introduction

The paper of Azéma-Gundy-Yor [1] was the first to characterize the uniform integrability of real-valued continuous martingales in terms of the tails of their supremum and quadratic variation. The existence of the two limits in the paper was considered by Elworthy-Li-Yor [3] and Galtchouk-Novikov [4] by using the Tauberian theorem. Takaoka [6] relaxed their assumptions to fully include the result of Azéma-Gundy-Yor [1] as well.

Theorem A ([6]) *Let $M = (M_t)_{t\in[0,\infty)}$ be a real-valued continuous local martingale, with $M_0 = 0$, on a certain filtered probability space satisfying the usual conditions. Assume $M_\infty \overset{\text{def}}{=} \lim_{t\to\infty} M_t$ exists a.s. and $E\big[\,|M_\infty|\,\big] < \infty$. Then the two limits*

$$\ell \overset{\text{def}}{=} \lim_{\lambda\to\infty} \lambda\, P\big[\,\sup_t |M_t| > \lambda\,\big] \quad \text{and} \quad \sigma \overset{\text{def}}{=} \lim_{\lambda\to\infty} \lambda\, P\big[\,\langle M\rangle_\infty^{1/2} > \lambda\,\big]$$

exist in $\mathbb{R}_+ \cup \{\infty\}$ and satisfy

$$\ell \;=\; \sqrt{\frac{\pi}{2}}\,\sigma \;=\; \sup_{U\in\mathcal{T}(M)} E\big[\,|M_U|\,\big] - E\big[\,|M_\infty|\,\big], \tag{1.1}$$

where $\mathcal{T}(M)$ is the set of all stopping times U such that the stopped process $(M_{t\wedge U})_t$ is of class D. Furthermore, M is a uniformly integrable martingale if and only if $\ell = \sigma = 0$.

If $(M_t^-)_t$ is of class D, then (1.1) is also equal to $-E\big[\,M_\infty\,\big]$. Here $x^- \overset{\text{def}}{=}$ $\max\{-x, 0\}$.

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In this article, we further generalize Theorem A to the case of local submartingales. This theorem also refines Theorem 1" of Elworthy-Li-Yor [3]. As a corollary, we show a property on stochastic integration with respect to martingales, which cannot be derived directly from Theorem A.

The rest of this paper is organized as follows. In Section 2 we state our main theorem and its corollary. The proof of the main theorem is given in Section 3. The Appendix is concerned about the final remark of Section 2.

The author would like to thank Professor F. Delbaen for his remarks (see the final remark of Section 2 and the Appendix).

2 Results

Theorem 1 *Let $X = (X_t)_{t\in[0,\infty)}$ be a real-valued continuous local submartingale, with $X_0 = 0$, on a certain filtered probability space satisfying the usual conditions. Assume the following two properties:*

- $X_\infty \stackrel{\text{def}}{=} \lim_{t\to\infty} X_t$ *exists a.s. and* $E\big[\,|X_\infty|\,\big] < \infty$;
- *the process* $(X_t^-)_t$ *is of class D.*

Then the two limits

$$\ell \stackrel{\text{def}}{=} \lim_{\lambda\to\infty} \lambda\, P\big[\, \sup_t X_t > \lambda \,\big] \quad \text{and} \quad \sigma \stackrel{\text{def}}{=} \lim_{\lambda\to\infty} \lambda\, P\big[\, \langle X\rangle_\infty^{1/2} > \lambda \,\big]$$

exist in $\mathbb{R}_+ \cup \{\infty\}$ *and satisfy*

$$\ell \;=\; \sqrt{\frac{\pi}{2}}\,\sigma \;=\; \sup_{U\in\mathcal{T}(X)} E[\,X_U\,] \,-\, E[\,X_\infty\,], \tag{2.1}$$

where $\mathcal{T}(X)$ is the set of all stopping times U such that the stopped process $(X_{t\wedge U})_t$ is of class D. Furthermore, X is of class D if and only if $\ell = \sigma = 0$.

Remarks. **(i)** By setting $X_t = |M_t|$, we can consider Theorem A as a special case of Theorem 1.

(ii) The expression on the right-hand side of (2.1) is less complicated than it looks. Indeed, let $X = M^X + A^X$ be the Doob-Meyer decomposition of the local submartingale X. Then, as will be shown in the proof, we have that

$$\sup_{U\in\mathcal{T}(X)} E[\,X_U\,] \;=\; E[\,A^X_\infty\,]$$

and that

$$\sup_{U\in\mathcal{T}(X)} E[\,X_U\,] \,-\, E[\,X_\infty\,] \;=\; -E[\,M^X_\infty\,].$$

Corollary 2 *Let* $(M_t)_{t\in[0,\infty)}$ *be a real-valued uniformly integrable martingale with continuous paths and* $(H_t)_{t\in(0,\infty)}$ *a predictable, bounded process. Assume either that*

$$E\Big[\,\Big(\int_0^\infty H_s\,dM_s\Big)^+\,\Big]\;<\;\infty \tag{2.2}$$

or that

$$E\Big[\,\Big(\int_0^\infty H_s\,dM_s\Big)^-\,\Big]\;<\;\infty. \tag{2.3}$$

Then the process $\int_0^\cdot H_s\,dM_s$ *is again a uniformly integrable martingale.*

Proof of the Corollary. Without loss of generality we assume (2.3). We apply Theorem 1 to the local submartingale $\big(\int_0^\cdot H_s\,dM_s\big)^-$ to see that it is of class D. Thus the process $\big(\int_0^t H_s\,dM_s\big)_{t\in[0,\infty]}$ is a supermartingale with its terminal element and in particular

$$E\Big[\,\Big|\int_0^\infty H_s\,dM_s\Big|\,\Big]\;<\;\infty.$$

We again apply Theorem 1 to the local submartingale $\big|\int_0^\cdot H_s\,dM_s\big|$ to get the desired result. $\square$

Remark. Delbaen [2] has pointed out that the corollary itself holds even without the continuity assumption on M. With his permission, the proof is given in the Appendix of the present paper.

3 Proof of the main theorem

The following proof of Theorem 1 is somewhat simpler than that of the main theorem of Takaoka [6].

Lemma 3 *For every sequence* $(T_n)_{n=1}^\infty$ *in* $\mathcal{T}(X)$ *increasing to* ∞ *a.s., we have*

$$\lim_{n\to\infty} E\big[\,X_{T_n}\,\big]\;=\;\sup_n E\big[\,X_{T_n}\,\big]\;=\;\sup_{U\in\mathcal{T}(X)} E\big[\,X_U\,\big]\;=\;\sup_{U\in\mathcal{T}} E\big[\,X_U\,\big],$$

where $\mathcal{T}$ *is the set of all stopping times.*

Proof. For every stopping time U, observe that $U\wedge T_n\in\mathcal{T}(X)$ and that $X_U=\lim_{n\to\infty} X_{U\wedge T_n}$ a.s. Hence, by Fatou's lemma,

$$E[\,X_U\,]\;\le\;\varliminf_{n\to\infty} E\big[\,X_{U\wedge T_n}\,\big]\;\le\;\varliminf_{n\to\infty} E\big[\,X_{T_n}\,\big]. \qquad \square$$

Proof of Theorem 1. We divide the proof into five steps.

Step 1. We first show the existence of the limit ℓ and the equality $\ell = \sup_{U\in\mathcal{T}(X)} E[\,X_U\,] - E[\,X_\infty\,]$. For $\lambda > 0$, define the stopping time

$$T_\lambda \stackrel{\text{def}}{=} \inf\big\{\,t\,:\,X_t > \lambda\,\big\}; \qquad (\inf\emptyset \stackrel{\text{def}}{=} \infty)$$

then $T_\lambda \in \mathcal{T}(X)$ and

$$E\big[\,X_{T_\lambda}\,\big] \;=\; \lambda\,P\big[\,\sup_t\,X_t\,>\lambda\,\big] \,+\, E\big[\,X_\infty\,;\,\sup_t X_t\,\le\lambda\,\big].$$

Here the left-hand side increases with λ, and the second term on the right-hand side converges to $E[\,X_\infty\,]\,(<\infty)$ as $\lambda\to\infty$. Therefore the limit of the first term on the right-hand side also exists. The desired equality also follows from this together with the above Lemma.

Step 2. For the proof of the equivalence $(\,X$ is of class $D\,) \Leftrightarrow (\ell = 0)$, we do the following observation:

$$\begin{aligned}
\ell = 0 \quad &\Leftrightarrow \quad \sup_{U\in\mathcal{T}(X)} E[\,X_U\,] \,=\, E[\,X_\infty\,] \qquad \text{(by Step 1)}\\
&\Leftrightarrow \quad \lim_{n\to\infty} E[\,X_{T_n}\,] \,=\, E[\,X_\infty\,] \;\text{ for every sequence }\; T_n \;\text{ in } \mathcal{T}(X)\\
&\qquad\quad \text{increasing to }\;\infty\;\text{ a.s.} \qquad \text{(by Lemma 3)}\\
&\Leftrightarrow \quad \lim_{n\to\infty} E[\,|X_{T_n}|\,] \,=\, E[\,|X_\infty|\,] \;\text{ for every sequence }\; T_n \;\text{ in } \mathcal{T}(X)\\
&\qquad\quad \text{increasing to }\;\infty\;\text{ a.s.} \qquad (\text{since } X^- \text{ is of class } D)\\
&\Leftrightarrow \quad \lim_{n\to\infty} X_{T_n} \,=\, X_\infty \;\text{ in } L^1 \;\text{ for every sequence }\; T_n \;\text{ in } \mathcal{T}(X)\\
&\qquad\quad \text{increasing to }\;\infty\text{ a.s.} \qquad (\text{since }\; E[\,|X_\infty|\,] < \infty\,)\\
&\Leftrightarrow \quad X \;\text{ is of class } D.
\end{aligned}$$

Step 3. In this step we make some preparations for the proof of the existence of σ and the equality $\sqrt{\frac{\pi}{2}}\,\sigma \;=\; \sup_{U\in\mathcal{T}(X)} E[\,X_U\,] - E[\,X_\infty\,]$. Consider the Doob-Meyer decomposition of the local submartingale X :

$$X_t \;=\; M_t \,+\, A_t,$$

where both the local martingale M and the non-decreasing process A have continuous paths. Note that $\langle X\rangle \,=\, \langle M\rangle$. We see that

$$\forall U \in \mathcal{T}(X), \qquad E[\,X_U\,] \,=\, E[\,A_U\,],$$

and hence

$$E[\,A_\infty\,] \;=\; \sup_{U\in\mathcal{T}(X)} E[\,X_U\,]. \tag{3.1}$$

For $x > 0$, define the stopping time

$$S_x \stackrel{\text{def}}{=} \inf\big\{\,t\,:\,M_t < -x\,\big\}. \qquad (\inf\emptyset \stackrel{\text{def}}{=} \infty)$$

Since $(M_{t\wedge S_x})_t$ is a continuous local martingale bounded below, it is proved in Elworthy-Li-Yor [3] and Galtchouk-Novikov [4] that

$$\sqrt{\tfrac{\pi}{2}} \lim_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_{S_x}^{1/2} > \lambda \,\big] = -E[\, M_{S_x} \,] \tag{3.2}$$

Also, observe that

$$\begin{aligned} -E\big[\, M_{S_x} \,\big] &= x\, P\big[\, \inf_t M_t < -x \,\big] - E\big[\, M_\infty \,;\, \inf_t M_t \ge -x \,\big] \\ &= x\, P\big[\, \sup_t M_t^- > x \,\big] + E\big[\, A_\infty \,;\, \inf_t M_t \ge -x \,\big] \\ &\quad - E\big[\, X_\infty \,;\, \inf_t M_t \ge -x \,\big]. \end{aligned} \tag{3.3}$$

Step 4. In this step we consider the case $\ell = \infty$, and in Step 5 we will deal with the case $\ell < \infty$. If $\ell = \infty$, then it follows from (3.1) that $E[\, A_\infty \,] = \infty$. We also see that

$$\text{the RHS of (3.3)} \;\ge\; E\big[\, A_\infty \,;\, \inf_t M_t \ge -x \,\big] - E\big[\, |X_\infty| \,\big],$$

which tends to ∞ as $x \to \infty$. This together with (3.2) gives

$$\begin{aligned} &\sqrt{\tfrac{\pi}{2}} \liminf_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_\infty^{1/2} > \lambda \,\big] \\ &\ge \sqrt{\tfrac{\pi}{2}} \lim_{x\to\infty} \lim_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_{S_x}^{1/2} > \lambda \,\big] \\ &= \infty. \end{aligned}$$

Thus the limit σ exists in this case and equals ∞.

Step 5. For this step we assume $\ell < \infty$. Then it follows from (3.1) that

$$E[\, A_\infty \,] < \infty$$

and hence the process $(M_t^-)_t$ is of class D. We apply the argument of Steps 1 & 2 to the local submartingale $(M_t^-)_t$ and see that the first term of the right-hand side of (3.3) tends to 0 as $x \to \infty$. This together with (3.2) implies that

$$\begin{aligned} \sqrt{\tfrac{\pi}{2}} \lim_{x\to\infty} \lim_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_{S_x}^{1/2} > \lambda \,\big] &= E[\, A_\infty \,] - E[\, X_\infty \,] \\ &= \sup_{U\in\mathcal{T}(X)} E[\, X_U \,] - E[\, X_\infty \,]. \end{aligned}$$

For the rest of this step, we will prove

$$\limsup_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_\infty^{1/2} > \lambda \,\big] \le \lim_{x\to\infty} \lim_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_{S_x}^{1/2} > \lambda \,\big].$$

Note that this implies the existence of the limit σ since it is trivial that

$$\liminf_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_\infty^{1/2} > \lambda \,\big] \ge \lim_{x\to\infty} \lim_{\lambda\to\infty} \lambda P\big[\, \langle M\rangle_{S_x}^{1/2} > \lambda \,\big].$$

It suffices to show that, for each fixed $0 < a < 1$,

$$\limsup_{\lambda\to\infty} \lambda P\big[\langle M\rangle_\infty^{1/2} > \lambda\big] \le \frac{1}{a} \lim_{x\to\infty} \lim_{\lambda\to\infty} \lambda P\big[\langle M\rangle_{S_x}^{1/2} > \lambda\big].$$

For $x > 0$, we have

$$P\big[\langle M\rangle_\infty^{1/2} > \lambda\big] \le P\big[\langle M\rangle_{S_x}^{1/2} \le a\lambda,\ \langle M\rangle_\infty^{1/2} > \lambda\big] + P\big[\langle M\rangle_{S_x}^{1/2} > a\lambda\big]$$

and hence

$$\begin{aligned}\limsup_{\lambda\to\infty} \lambda P\big[\langle M\rangle_\infty^{1/2} > \lambda\big] \quad &\le \quad \frac{1}{a} \lim_{\lambda\to\infty} \lambda P\big[\langle M\rangle_{S_x}^{1/2} > \lambda\big] \\ &\quad + \sup_\lambda \lambda P\big[\langle M\rangle_{S_x}^{1/2} \le a\lambda,\ \langle M\rangle_\infty^{1/2} > \lambda\big].\end{aligned}$$

Thus it suffices to show

$$\lim_{x\to\infty} \sup_\lambda \lambda P\big[\langle M\rangle_{S_x}^{1/2} \le a\lambda,\ \langle M\rangle_\infty^{1/2} > \lambda\big] = 0.$$

Fix $x > 0$ for the moment. For $t \ge 0$, define

$$N_t^{(x)} \stackrel{\text{def}}{=} M_{S_x+t} - M_{S_x} \qquad \text{and} \qquad \mathcal{G}_t^{(x)} \stackrel{\text{def}}{=} \mathcal{F}_{S_x+t}.$$

Note that $\big(N_t^{(x)}\big)_t$ is a continuous local martingale w.r.t. the filtration $\big(\mathcal{G}_t^{(x)}\big)_t$. Also, observe that

$$\begin{aligned}&\sup_\lambda \lambda P\big[\langle M\rangle_{S_x}^{1/2} \le a\lambda,\ \langle M\rangle_\infty^{1/2} > \lambda\big] \\ &\quad\le \sup_\lambda \lambda P\big[\langle N^{(x)}\rangle_\infty^{1/2} > \sqrt{1-a^2}\,\lambda\big] \\ &\quad= \tfrac{1}{\sqrt{1-a^2}} \sup_\lambda \lambda P\big[\langle N^{(x)}\rangle_\infty^{1/2} > \lambda\big] \\ &\quad\le \tfrac{C}{\sqrt{1-a^2}} \sup_\lambda \lambda P\big[\sup_t |N_t^{(x)}| > \lambda\big], \end{aligned} \tag{3.4}$$

where the last inequality follows from the well-known good λ inequality (see e.g. §IV.4 of Revuz-Yor [5]), with the constant C universal; in particular, C does not depend on x. Since

$$\forall\lambda > 0, \qquad \lambda P\big[\sup_t |N_t^{(x)}| > \lambda\big] \le E\big[|N_{\tau_\lambda}^{(x)}|\big], \qquad \big(\tau_\lambda \stackrel{\text{def}}{=} \inf\{t : |N_t^{(x)}| > \lambda\}\big)$$

it follows that

$$\begin{aligned}(3.4) \quad &\le \quad \tfrac{C}{\sqrt{1-a^2}} \sup_{U\in\mathcal{T}(N^{(x)})} E\big[|N_U^{(x)}|\big] \\ &\qquad \big(\text{where } \mathcal{T}(N^{(x)}) \text{ is defined the same way as } \mathcal{T}(X)\big) \\ &= \quad \tfrac{C}{\sqrt{1-a^2}}\ 2 \sup_{U\in\mathcal{T}(N^{(x)})} E\big[N_U^{(x)\,-}\big] \\ &\le \quad \tfrac{C}{\sqrt{1-a^2}}\ 2 \sup_{U\in\mathcal{T}} E\big[M_U^- : \inf_t M_t < -x\big],\end{aligned}$$

where, as in Lemma 3, $\mathcal{T}$ denotes the set of all stopping times. The last expression converges to 0 as $x \to \infty$, since the process $(M_t^-)_t$ is of class D i.e. the random variables $\{M_U^-\}_{U \in \mathcal{T}}$ are uniformly integrable. $\square$

4 Appendix

As mentioned in the final remark of Section 2, Delbaen [2] has pointed out that the assertion of Corollary 2 holds even without the continuity assumption on M. With his permission, we here give his statement and its proof.

Theorem B (Delbaen [2]) *Let $(M_t)_{t \in [0,\infty)}$ be a real-valued uniformly integrable martingale, possibly with jumps, and $(H_t)_{t \in (0,\infty)}$ a predictable, bounded process. Assume either that*

$$E\Big[\,\big(\int_{0+}^{\infty} H_s\, dM_s\big)^+\Big] \;<\; \infty \tag{4.1}$$

or that

$$E\Big[\,\big(\int_{0+}^{\infty} H_s\, dM_s\big)^-\Big] \;<\; \infty. \tag{4.2}$$

Then the process $\int_{0+}^{\cdot} H_s\, dM_s$ is again a uniformly integrable martingale.

Proof Define the local martingale $N_t \overset{\text{def}}{=} \int_{0+}^{t} H_s\, dM_s$. It should be noted that, even without the assumptions (4.1) and (4.2), the general theory of martingales gives the following two properties:

- $N_\infty \overset{\text{def}}{=} \lim_{t\to\infty} N_t$ exists a.s. ;
- $\lim_{\lambda\to\infty} \lambda\, P\big[\,\sup_t |N_t| > \lambda\,\big] \;=\; 0.$

Now, without loss of generality let us assume (4.2) as well as $M_0 = 0$ and $|H| \le 1$. For each $n \in \mathbb{N}$, define the stopping time

$$\tau_n \overset{\text{def}}{=} \inf\big\{\, t \,:\, N_t^- > n \;\text{ or }\; |M_t| > n \,\big\}. \qquad (\inf \emptyset \overset{\text{def}}{=} \infty)$$

Then the local submartingale N^- stopped by τ_n satisfies $E\big[\,\sup_t N^-_{t\wedge\tau_n}\,\big] < \infty$ since

$$\begin{aligned}
\sup_t N^-_{t\wedge\tau_n} \;&\le\; n + |\Delta N_{\tau_n}| \\
&\le\; n + |\Delta M_{\tau_n}| \\
&\le\; 2n + |M_{\tau_n}|.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\left| N^-_{\tau_n} - N^-_\infty \right| \quad &= \quad \left| \{ N^-_{\tau_n} - N^-_\infty \} \, 1_{\{\tau_n < \infty\}} \right| \\[1em]
&\leq \quad N^-_\infty \, 1_{\{\tau_n < \infty\}} \\
&\quad + \left\{ 2n \, + \, |M_{\tau_n}| \right\} 1_{\{\tau_n < \infty, \; |M_{\tau_n}| \geq n\}} \\
&\quad + 3n \; 1_{\{\tau_n < \infty, \; |M_{\tau_n}| < n\}} \\[1em]
&\leq \quad N^-_\infty \, 1_{\{\tau_n < \infty\}} \\
&\quad + \left\{ 2n \; 1_{\{\sup_t |M_t| \geq n\}} \; + \; |M_{\tau_n}| \; 1_{\{\tau_n < \infty\}} \right\} \\
&\quad + 3n \; 1_{\{\sup_t N^-_t \geq n\}}
\end{aligned}
$$

and it follows that

$$\lim_{n \to \infty} N^-_{\tau_n} \; = \; N^-_\infty \qquad \text{in the } L^1 \text{ sense.}$$

Thus the local submartingale N^- is of class D, the process $(N_t)_{t \in [0,\infty]}$ is a supermartingale with its terminal element and in particular

$$E\left[\left| N_\infty \right| \right] \; < \; \infty.$$

We repeat the same argument with N^- replaced by the local submartingale $\left| N \right|$ to get the desired result. $\square$

References

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