BIFURCATIONS OF STEADY STATES FOR THE EGUCHI-OKI-MATSUMURA MODEL OF PHASE SEPARATION

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Abstract. We are concerned with bifurcations of steady states for a model system of phase separation, which is introduced by Eguchi-Oki-Matsumura (EOM). The system consists of coupled two evolution equations and admits steady state solutions with different energies. We analyze the bifurcation phenomena of these steady states with respect to the principal parameter, which is related to the temperature.

1. Introduction

Phase separation phenomena observed in a variety of physical sciences are challenging as well as interesting topics for research since mid-20th century. For example, we recall certain polymer mixtures and binary alloys such as Fe-Al. Mathematical analysis of these fascinating pattern dynamics customarily rests on evolution models, which include, among other things, the famous Cahn-Hilliard equation [3]. These models now form an important class of equations, so-called the equations of pattern formation [23]. Much attention has been paid and extensive studies have been performed so far concerning these equations both from the mathematical and computational point of view. We refer to [1][5][6][9][10][11][12][16][17][20][22][24] and the references cited therein.

In 1984, Eguchi, Oki, and Matsumura (EOM), with the aim of a better understanding as to such phenomena, introduced a coupled system of evolution equations for the local concentration and the local degree of order [8]. Based on the so-called continuum approach, this motion law is derived from the first principles of thermodynamics of irreversible process under certain assumptions on the free energy; the system extends and links the Cahn-Hilliard equation and the Ginzburg-Landau equation. The principal parameter involved in the EOM system is related to the temperature (see §2 for details). As the temperature decreases from above to below the critical temperature, this parameter increases from negative to positive and gradually unstabilizes the system. Since phase separation phenomena are observed to emerge below the critical temperature, in the EOM model the investigation within positive principal parameters is important.

We have to recall at this point that a similar coupled system is introduced by J.W. Cahn and A. Novick-Cohen [4] by a different methodology; namely, the so-called discrete approach is employed in [4]. The derived dynamics is then called as an Allen-Cahn/Cahn-Hilliard system. Compared to the EOM system, the former may involve degenerate mobility and thus poses various challenging problems. Existence of solutions, the long time behavior, possible triple-junction motion, and etc. are investigated so far [2][7][19][21].

Returning to the EOM system, in our previous work which is restricted to the one space dimension case, we have established that there are infinitely many steady state solutions

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according to the largeness of the principal temperature and the total concentration [14]. See Theorem 1 in §2. Therefore the stability analysis, or preferably, the bifurcation analysis on these steady state solutions in terms of the principal parameter is indispensable to well understanding of the model itself, and as a result, we may deepen our knowledge on the mechanisms of phase separation.

The aim of the present article is thus as follows: We undertake the bifurcation analysis on steady state solutions of the EOM system with respect to the principal parameter primarily from the analytical point of view. Our main results are addressed in Theorem 2 in §2, whose proofs are given in §3 and §4. Our observations substantially illustrates a complicated structure of bifurcations and the important role of the principal parameter.

2. Eguchi-Oki-Matsumura model

First we recall the Eguchi-Oki-Matsumura model for phase separation, and our findings concerning this dynamics.

In one space dimension, under a suitable scaling of parameters, the EOM system in [8] is expressed as follows [14][15].

$$
(1) \qquad \begin{cases} u_t = -\varepsilon^2 u_{xxxx} + ((a+v^2)u)_{xx} & \text{in } 0 < x < l, \quad t > 0 \\ v_t = v_{xx} + (b - u^2 - v^2)v & \text{in } 0 < x < l, \quad t > 0 \\ u_x = u_{xxx} = v_x = 0 & \text{at } x = 0 \text{ and } l, \quad t > 0 \\ u|_{t=0} = u_0, \ v|_{t=0} = v_0 & \text{on } 0 \le x \le l, \end{cases}
$$

where $u = u(x,t)$, $v = v(x,t)$ denote the local concentration and the local degree of order, respectively. The constants a, b are assumed to be positive. Here $b \in \boldsymbol{R}$ is the principal parameter explained in §1; phase separation are regarded as what occur within positive b. The parameter ε depends on the ratio of the surface energy between the original u and v. It is easily noticed that the equation for u is Cahn-Hilliard type while the one for v is Ginzburg-Landau type. A direct calculation shows that the total concentration of u is conserved under the evolution; namely, for any solution (u,v) to (1), there holds

$$
\frac{1}{l}\int_0^l u(x,t)\,dx = m,
$$

where m is a positive constant. Here and hereafter, the solution are understood to be classical. Given initial data u_0, v_0 should satisfy required compatibility conditions:

$$
(u_0)_x = (u_0)_{xxx} = (v_0)_x = 0 \qquad \text{at } x = 0,\, l, \quad \text{and } \frac{1}{l}\int_0^l u_0(x)\,dx = m.
$$

System (1) takes its rise in a gradient flow of the total free energy

$$
(2) \qquad F[u,v] := \int_0^l \Big(\frac{\varepsilon^2}{2}u_x^2 + \frac{1}{2}v_x^2 + \frac{a}{2}u^2 + \frac{1}{4}v^4 - \frac{b}{2}v^2 + \frac{1}{2}u^2v^2\Big)\,dx.
$$

Indeed one easily show that

$$
\frac{d}{dt}F[u,v](t) = -\int_0^l \{-\varepsilon^2 u_{xxx} + ((a+v^2)u)_x\}^2\,dx - \int_0^l v_t^2\,dx \le 0,
$$

which means that F serves as Lyapunov functional of the system.

Here one realizes the reason why b plays a critical role in the EOM model system from the form of the potential energy in the free energy (2). Precisely stated, we write the potential as

$$f(u,v) := \frac{a}{2}u^2 + \frac{1}{4}v^4 - \frac{b}{2}v^2 + \frac{1}{2}u^2v^2, \tag{3}$$

so that $F[u,v] = \int_0^l (2^{-1}\varepsilon^2 u_x^2 + 2^{-1}v_x^2 + f(u,v))dx$. It is easy to see that the origin $(u,v) = (0,0)$ is saddle point with $f(0,0) = 0$ and $(u,v) = (0, \pm\sqrt{b})$ are local minimum with $f(0,\pm\sqrt{b}) = -b^2/4$. As b increases, the potential (3) thus accordingly looks like double well.

Utilizing the non-increasing property of the total free energy (2), we show in [14] that every element of the ω-limit set of (1) consists of steady state solutions. We recall that steady state solutions are given by $u_t = v_t \equiv 0$ in (1); that is,

$$\begin{cases} -\varepsilon^2 u_{xxxx} + ((a+v^2)u)_{xx} = 0 & \text{in } 0 < x < l \\ v_{xx} + (b - u^2 - v^2)v = 0 & \text{in } 0 < x < l \\ u_x = u_{xxx} = v_x = 0 & \text{at } x = 0 \text{ and } l \\ (1/l)\int_0^l u\,dx = m. \end{cases} \tag{4}$$

Furthermore, the solution (u,v) to system (4) is constructed as a critical point of the functional (2) in the function space

$$\mathcal{A} := \{(u,v) \in (H^1(0,l))^2 \mid \frac{1}{l}\int_0^l u\,dx = m\}.$$

We recall that $(u,v) = (m,0)$ and $(m, \pm\sqrt{b-m^2})$ if $b > m^2$ are referred to as trivial solutions. One of main achievements in [14] is that the structure of steady state solutions is rather rich enough; to be specific, the next properties is established in [13][14].

Theorem 1. ([14]) *There is at least one monotone non-trivial steady state solution of (4) if we assign suitably large b and m^2. Moreover, for any integer $k \geq 2$ and for appropriately chosen large b and m^2 depending on k, (4) has at least one non-monotone non-trivial steady state solution, each of whose derivatives changes sign exactly $(k-1)$-times.*

The focus of the current paper is to investigate how this compound structure of steady states develop as the principal parameter b increases; in other words, we are concerned with a bifurcation problem in terms of b. As a first step the case of trivial steady state is treated. We believe that this is an essential issue for better appreciation of the phase separation model governed by the EOM system.

Our main contributions now read as follows.

Theorem 2. *(I) For any integer $j = 1, 2, \cdots$, $(u,v;b) = (m, 0, m^2 + (j\pi/l)^2)$ is a bifurcation point of the steady state system (4) for the Eguchi-Oki-Matsumura model. There exists a one-parameter family of nontrivial steady state solutions $(m + \bar{u}(x,s), s\cos(j\pi x/l) + \bar{v}(x,s); b(s))$ $|s| < \delta$ with sufficiently small δ such that*

$$\bar{u}(x,0) = \bar{v}(x,0) = \bar{u}_s(x,0) = \bar{v}_s(x,0) = 0$$
$$b(s) = m^2 + \left(\frac{j\pi}{l}\right)^2 + \lambda_j(s),$$

where $\lambda_j(s)$ is a smooth function of s verifying

$$\lambda_j(s) = \Big(\frac{3}{4} - \frac{m^2}{4}\frac{1}{(2j\pi/l)^2\varepsilon^2 + a}\Big)s^2 + O(s^3) \qquad as\ s \to 0.$$

(II) Suppose $b > m^2 > a/2$. For any integer j satisfying

$$j < \frac{\sqrt{2}l(2m^2-a)/\pi}{\sqrt{2}m + \sqrt{a + 2\varepsilon^2(2m^2-a)}}, \tag{5}$$

$(u, v; b) = (m, \pm\sqrt{b(j;+) - m^2}; b(j;+))$, $(m, \pm\sqrt{b(j;-) - m^2}; b(j;-))$ are bifurcation points of (4), where

$$\begin{aligned} b(j;\pm) = \frac{1}{4}\Big\{ & 8m^2 - 2a - (1+2\varepsilon^2)\Big(\frac{j\pi}{l}\Big)^2 \\ & \pm\sqrt{(1-2\varepsilon^2)^2\Big(\frac{j\pi}{l}\Big)^4 - 4\{2m^2 + a + 2\varepsilon^2(2m^2-a)\}\Big(\frac{j\pi}{l}\Big)^2 + 4(2m^2-a)^2}\Big\}. \end{aligned}$$

Then there exists a one-parameter family of nontrivial steady state solutions

$$\begin{aligned} u(x;s) &= m - (\alpha_j^2 + 2X^2)s\cos\frac{j\pi x}{l} + \bar{u}(x,s) \\ v(x;s) &= \pm X \pm 2mXs\cos\frac{j\pi x}{l} + \bar{v}(x,s) \\ b(s) &= m^2 + (X + \lambda_j(s))^2, \end{aligned}$$

for $|s| < \delta$ with appropriately small δ such that

$$\bar{u}(x,0) = \bar{v}(x,0) = \bar{u}_s(x,0) = \bar{v}_s(x,0) = 0.$$

Here $\lambda_j(s)$ is a smooth function of s with

$$\begin{aligned} \lambda_j(s) = \frac{X}{4m^2(j\pi/l)^2 - ((j\pi/l)^2 + 2X^2)^2}\Big[& -3m^2\{2m^2X^2 - ((j\pi/l)^2 + 2X^2)^2\} \\ & + \frac{1}{12(j\pi/l)^2\{(1+2\varepsilon^2)X^2 + a + 5\varepsilon^2(j\pi/l)^2\}}\Big\{24m^2X^2(m^2 - (j\pi/l)^2 - 2X^2)\cdot \\ & \cdot\{4X^4 + 8(j\pi/l)^2X^2 - (j\pi/l)^2(4m^2 - 3(j\pi/l)^2)\} \\ & + \{4m^2((j\pi/l)^2 - X^2) - ((j\pi/l)^2 + 2X^2)^2\}D(m,\varepsilon,a,j)\Big\}\Big]s^2 + O(s^3) \qquad as\ s \to 0, \end{aligned}$$

where $D(m,\varepsilon,a,j)$ denotes

$$\begin{aligned} & D(m,\varepsilon,a,j) \\ & := -2(14m^2 + (j\pi/l)^2(1-2\varepsilon^2) + 2a)X^4 + \{8m^4 - 4((1+7\varepsilon^2)(j\pi/l)^2 + 3a)m^2 \\ & \qquad + ((1+8\varepsilon^2)(j\pi/l)^2 + 2a)((-1+2\varepsilon^2)(j\pi/l)^2 + 2a)\}X^2 \\ & \quad + (j\pi/l)^2(4\varepsilon^2(j\pi/l)^2 + a)(4m^2 - (1-2\varepsilon^2)(j\pi/l)^2 + 2a), \end{aligned} \tag{6}$$

and X is given as any one of real two roots of

$$2X^4 - (4m^2 - 2a - (1+2\varepsilon^2)(j\pi/l)^2)X^2 + (a + \varepsilon^2(j\pi/l)^2)(j\pi/l)^2 = 0.$$

In the part II, we remark that the next condition is fulfilled.

$$b(i;+) \neq b(j;-) \qquad \text{for every } i,j = 1,2,\cdots, \text{ satisfying (5)}, \tag{7}$$

which is crucial to the application of the general theorem of bifurcation.

The signification of the above theorem is illuminating; from the steady state $(u,v) = (m,0)$, there bifurcate one parameter families of nontrivial steady state solutions for unbounded sequence of b. While from the steady state $(u,v) = (m, \pm\sqrt{b-m^2})$, the bifurcation points of b, which are indexed by j satisfying (5), are bounded from above.

3. Trivial steady state – Part 1

We begin with the proof of Theorem 2 (I) in the case of trivial steady state solution $(u,v) \equiv (m,0)$ in (4). We perform a translation $u \to u+m$, $v \to v$ so that (4) turns into

$$\begin{cases} -\varepsilon^2 u_{xxxx} + ((a+v^2)(u+m))_{xx} = 0 & \text{in } 0<x<l \\ v_{xx} + (b-m^2-u^2-2mu-v^2)v = 0 & \text{in } 0<x<l \\ u_x = u_{xxx} = v_x = 0 & \text{at } x=0 \text{ and } l \\ (1/l)\int_0^l u\,dx = 0. \end{cases} \tag{8}$$

Trivial steady state solution $(u,v) = (m,0)$ is then given by $(u,v) = (0,0)$. We linearize system (4) or (8) around this steady state to obtain

$$L_b \begin{pmatrix} U \\ V \end{pmatrix} := \begin{pmatrix} -\varepsilon^2 U_{xxxx} + aU_{xx} \\ V_{xx} + (b-m^2)V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{9}$$

where $U = U(x)$, $V = V(x)$ are defined on $0 \le x \le l$ with verifying $U_x = V_x = 0$ at $x = 0, l$ and $\int_0^l U dx = 0$. It is easy to see that the $U(x) \equiv 0$ and $V(x) = s\cos(j\pi x/l)$ $(s \in \boldsymbol{R})$ if and only if $b_j := m^2 + (j\pi/l)^2$ $(j = 1,2,\cdots)$ which are bifurcation points.

We want to apply a standard method of bifurcation theory. Confer Chapter 3 in [18] for example. We define a Hilbert space

$$E := \mathrm{Span}\{\cos\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \cdots\} \times \mathrm{Span}\{1, \cos\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \cdots\} \ni \begin{pmatrix} U \\ V \end{pmatrix}.$$

The linear operator L_b maps E into E and there holds

$$\mathrm{Ker} L_{b_j} = \mathrm{Span}\begin{pmatrix} 0 \\ \cos(j\pi x/l) \end{pmatrix} = \mathrm{Coker} L_{b_j}.$$

Therefore a well-known bifurcation theorem (see Theorem 3.2.2. in [18], where other conditions are also certified) implies that there exists a bifurcating solution to (8) of the form

$$\begin{aligned} u(x;s) &= \sum_{n=1}^{\infty} p_n(s)\cos\frac{n\pi x}{l} \\ v(x;s) &= s\cos\frac{j\pi x}{l} + \sum_{n=0, n\neq j}^{\infty} q_n(s)\cos\frac{n\pi x}{l} \\ b(s) &= b_j + \lambda_j(s), \end{aligned} \tag{10}$$

where $\{p_n(s)\}$, $\{q_n(s)\}$ are smooth functions with respect to a parameter s $(|s| \le \delta$ with small $\delta)$ and fulfill

$$p_n(0) = q_n(0) = p_n'(0) = q_n'(0) = 0.$$

We wish to determine the behavior of $\lambda_j(s)$ as $s \to 0$, which is also a smooth function of s. To do so the use of Lyapunov-Schmidt bifurcation equation (see for instance (2.18) in [18]) is made. Of course if we admit the expression (10) of solutions then this is merely a condition on certain component. At any rate the equation in this case is

$$\int_0^l \cos\frac{j\pi x}{l}\Big\{v_{xx} + \Big(\Big(\frac{j\pi}{l}\Big)^2 + \lambda_j(s) - u^2 - 2mu - v^2\Big)v\Big\}dx = 0.$$

Since $p_n(s) = O(s^2)$ and $q_n(s) = O(s^2)$ $(n \neq j)$ as $s \to 0$, we find that

$$\lambda_j(s) = \frac{3}{4}s^2 + mp_{2j}(s) + O(s^3) \qquad \text{as } s \to 0.$$

The profile of $p_{2j}(s)$ as $s \to 0$ can be obtained from the equation for u in (8). Multiply by $\cos(2j\pi x/l)$ and integrate it over the interval $(0, l)$ so that we are led to

$$\int_0^l \cos\frac{2j\pi x}{l}\{-\varepsilon^2 u_{xx} + (a + v^2)(u + m)\}dx = 0,$$

from which we infer that

$$p_{2j}(s) = -\frac{m}{2}\frac{s^2}{(2j\pi/l)^2\varepsilon^2 + a} + O(s^3) \qquad \text{as } s \to 0.$$

In summary, at the bifurcation point $(u, v; b) = (0, 0; m^2 + (j\pi/l)^2)$ of (8) there is a one-parameter family of nontrivial solution of the form (10) with $|s| \le \delta$ such that

$$\lambda_j(s) = \Big(\frac{3}{4} - \frac{m^2}{4}\frac{1}{(2j\pi/l)^2\varepsilon^2 + a}\Big)s^2 + O(s^3) \qquad \text{as } s \to 0.$$

This completes the proof of the first part of the theorem.

4. Trivial steady state – Part 2

Next we turn our attention to the case of steady state $(u, v) \equiv (m, \pm\sqrt{b - m^2})$ in (4). We carry out a preliminary translation $u \to u + m$, $v \to v \pm \sqrt{b - m^2}$ so that (4) is reduced to

$$(11) \quad \begin{cases} -\varepsilon^2 u_{xxxx} + ((a + b - m^2 + v^2 \pm 2\sqrt{b - m^2}v)(u + m))_{xx} = 0 & \text{in } 0 < x < l \\ v_{xx} - (u^2 + 2mu + v^2 \pm 2\sqrt{b - m^2}v)(v \pm \sqrt{b - m^2}) = 0 & \text{in } 0 < x < l \\ u_x = u_{xxx} = v_x = 0 & \text{at } x = 0 \text{ and } l \\ (1/l)\int_0^l u\,dx = 0. \end{cases}$$

Trivial steady state solution $(u, v) = (m, \pm\sqrt{b - m^2})$ is then given by $(u, v) = (0, 0)$. This tiny modifications will help us to avoid possible complications and to argue transparently. The linearization of (4) or (11) around this solution becomes

$$(12) \qquad L_{b-m^2}\begin{pmatrix} U \\ V \end{pmatrix} := \begin{pmatrix} -\varepsilon^2 U_{xxxx} + (a + b - m^2)U_{xx} \pm 2m\sqrt{b - m^2}V_{xx} \\ V_{xx} - 2(b - m^2)V \mp 2m\sqrt{b - m^2}U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where U, V fulfill the same conditions as in (9). System (12) is coupled and the bifurcation analysis is not so immediate.

We first deduce that $\int_0^l V\,dx = 0$ on an integration of the equation for V. We thus integrate twice the equation for U with null constant. The result can be expressed as

$$\frac{d^2}{dx^2}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} (a+b-m^2)/\varepsilon^2 & \pm 2m\sqrt{b-m^2}/\varepsilon^2 \\ \pm 2m\sqrt{b-m^2} & 2(b-m^2) \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}. \tag{13}$$

We intend to examine the sign of the characteristic exponents $\mu_\pm$ for the coefficient matrix of (13). To simplify the notation we put $X = \sqrt{b-m^2}$.

We compute

$$\mu_\pm := \frac{1}{2\varepsilon^2}\{a+(1+2\varepsilon^2)X^2 \pm \sqrt{(a+(1+2\varepsilon^2)X^2)^2 - 8\varepsilon^2X^2(a+X^2)+16m^2\varepsilon^2X^2}\}. \tag{14}$$

It is clear that $\mu_\pm$ are real and $\mu_+ > 0$. We need a criterion such that for certain $j = 1, 2, \cdots$,

$$\mu_- = -\Big(\frac{j\pi}{l}\Big)^2 =: -\alpha_j^2. \tag{15}$$

We first assert that in order to attain (15) the condition

$$-X^2(a+X^2)+2m^2X^2 > 0 \Longleftrightarrow 2m^2 > a+X^2 \tag{16}$$

is required. Furthermore after a calculation of (14)(15), that is,

$$\sqrt{(a+(1+2\varepsilon^2)X^2)^2 - 8\varepsilon^2X^2(a+X^2)+16m^2\varepsilon^2X^2} = a+(1+2\varepsilon^2)X^2+2\varepsilon^2\alpha_j^2,$$

we infer that X^2 should satisfy

$$2X^4 - (4m^2-2a-(1+2\varepsilon^2)\alpha_j^2)X^2 + (a+\varepsilon^2\alpha_j^2)\alpha_j^2 = 0. \tag{17}$$

The quadratic equation (17) is easily solved:

$$X(j;\pm)^2 = \frac{1}{4}\{4m^2-2a-(1+2\varepsilon^2)\alpha_j^2 \pm \sqrt{(4m^2-2a-(1+2\varepsilon^2)\alpha_j^2)^2 - 8(a+\varepsilon^2\alpha_j^2)\alpha_j^2}, \tag{18}$$

which are real and positive provided the next condition holds true.

$$4m^2-2a-(1+2\varepsilon^2)\alpha_j^2 > 2\alpha_j\sqrt{2(a+\varepsilon^2\alpha_j^2)}. \tag{19}$$

We note that if (19) is verified then the requirement (16) automatically follows with regard to the solution $X(j;\pm)^2$ of (18). It is also noticed that the condition (19) serves as a restriction for j; precisely stated we estimate

$$\begin{aligned}\frac{\alpha_j^2}{2} &< \frac{(2m^2-a)^2}{2m^2+a+2\varepsilon^2(2m^2-a)+2\sqrt{2am^2+4\varepsilon^2m^2(2m^2-a)}} \\ &= \frac{(2m^2-a)^2}{(\sqrt{2}m+\sqrt{a+2\varepsilon^2(2m^2-a)})^2}.\end{aligned} \tag{20}$$

The fact that the other option is discarded by the positivity of the left hand side of (19) is easily ascertained by

$$\begin{aligned}(1+2\varepsilon^2)\alpha_j^2 &> \frac{4m^2(1+2\varepsilon^2)^2+2a(1-4\varepsilon^4)+4(1+2\varepsilon^2)\sqrt{2am^2+4\varepsilon^2m^2(2m^2-a)}}{(1-2\varepsilon^2)^2} \\ &> 4m^2+2a.\end{aligned}$$

In any case we have a rough bound $\alpha_j^2 \le O(m^2)$ as $m^2 \to \infty$.

Consequently for any integer j with (20), $b = m^2 + X(j;+)^2$ and $b = m^2 + X(j;-)^2$ are bifurcation points. Here we note that the next property holds true.

$$X(i;+) \neq X(j;-) \qquad \text{for any integer } i, j = 1, 2, \cdots, \text{ satisfying (20)}, \tag{21}$$

which corresponds to (7) and whose demonstration will be provided at the end of the prof. Therefore we find that

$$\begin{aligned} \mathrm{Ker} L_{b-m^2} &= \mathrm{Span}\begin{pmatrix} -\alpha_j^2 - 2X(j;\pm)^2 \\ \pm 2mX(j;\pm) \end{pmatrix} \cos(j\pi x/l) \\ &= \mathrm{Span}\begin{pmatrix} \pm 2mX(j;\pm) \\ -\varepsilon^2\alpha_j^2 - (a + X(j;\pm)^2) \end{pmatrix} \cos(j\pi x/l) = \mathrm{Coker} L_{b-m^2}. \end{aligned}$$

We advance our analysis in the spirit of appealing to a general bifurcation theory as in §4. This time we are just left to check the last part of Theorem 3.2.2 (iv) in [18]. In other words we have to show

$$\left\langle \frac{\partial L_{b-m^2}}{\partial b} \begin{pmatrix} -\alpha_j^2 - 2X(j;\pm)^2 \\ \pm 2mX(j;\pm) \end{pmatrix} \cos(j\pi x/l), \begin{pmatrix} -\alpha_j^2 - 2X(j;\pm)^2 \\ \pm 2mX(j;\pm) \end{pmatrix} \cos(j\pi x/l) \right\rangle \neq 0,$$

which amounts to an algebraic condition

$$4\alpha_j^2 X(j;\pm)^4 - 4(\alpha_j^4 + m^2)X(j;\pm)^2 + \alpha_j^2(\alpha_j^4 - 2m^2(\alpha_j^2 + 1)) \neq 0$$

and we are done.

Now there exists a bifurcating solution to (11) of the form

$$\begin{aligned} u(x;s) &= -(\alpha_j^2 + 2X^2)s\cos\frac{j\pi x}{l} + 2mXr_j(s)\cos\frac{j\pi x}{l} + \sum_{n=1, n\neq j}^{\infty} p_n(s)\cos\frac{n\pi x}{l} \\ v(x;s) &= \pm 2mXs\cos\frac{j\pi x}{l} \pm (\alpha_j^2 + 2X^2)r_j(s)\cos\frac{j\pi x}{l} + \sum_{n=0, n\neq j}^{\infty} q_n(s)\cos\frac{n\pi x}{l} \\ \sqrt{b(s) - m^2} &= X + \lambda_j(s), \end{aligned} \tag{22}$$

where we have put $X = X(j;+)$ and/or $X = X(j;-)$ for simplicity. It is noted that in (22) we avoid troublesome formulation of $b(s) = m^2 + X^2 + \lambda_j(s)$. Smooth functions $r_j(s)$, $\{p_n(s)\}$, $\{q_n(s)\}$ ($|s| \leq \delta$) likewise satisfy

$$r_j(0) = p_n(0) = q_n(0) = r_j'(0) = p_n'(0) = q_n'(0) = 0.$$

We then compute the bifurcation equation, which enables us to handle the $r_j(s)$ terms as higher order terms.

$$\begin{aligned} \int_0^l & -(\alpha_j^2 + 2X^2)\cos\frac{j\pi x}{l}\{\varepsilon^2 u_{xx} - (a + (X + \lambda_j(s))^2 + v^2 \pm 2(X + \lambda_j(s))v)(u + m)\} \\ & \pm 2mX\cos\frac{j\pi x}{l}\{v_{xx} - (u^2 + 2mu + v^2 \pm 2(X + \lambda_j(s))v)(v \pm (X + \lambda_j(s)))\}dx = 0. \end{aligned}$$

After a little tedious calculation it follows that

(23)

$$\begin{aligned} \lambda_j(s) = \frac{1}{4m^2\alpha_j^2 - (\alpha_j^2 + 2X^2)^2}\Big[& -3m^2X\{2m^2X^2 - (\alpha_j^2 + 2X^2)^2\}s^2 \\ & + 2mX\{m^2 - (\alpha_j^2 + 2X^2)\}p_{2j}(s) \mp \frac{1}{2}\{4m^2(\alpha_j^2 - X^2) - (\alpha_j^2 + 2X^2)^2\}q_{2j}(s)\Big] + O(s^3) \end{aligned}$$

as $s \to 0$. We need to bring out the behavior of $p_{2j}(s)$ and $q_{2j}(s)$ as $s \to 0$.

To approach this we observe that

$$\int_0^l \cos\frac{2j\pi x}{l}\{\varepsilon^2 u_{xx} - (a + (X+\lambda_j(s))^2 + v^2 \pm 2(X+\lambda_j(s))v)(u+m)\}dx = 0$$
$$\int_0^l \cos\frac{2j\pi x}{l}\{v_{xx} - (u^2 + 2mu + v^2 \pm 2(X+\lambda_j(s))v)(v \pm (X+\lambda_j(s)))\}dx = 0,$$

which lead respectively to

$$(4\varepsilon^2\alpha_j^2 + a + X^2)p_{2j}(s) \pm 2mXq_{2j}(s) = -2mX^2(m^2 - (\alpha_j^2 + 2X^2))s^2 + O(s^3)$$
$$2mXp_{2j}(s) \pm 2(2\alpha_j^2 + X^2)q_{2j}(s) = \frac{X}{2}\{4m^2(\alpha_j^2 - X^2) - (\alpha_j^2 + 2X^2)^2\}s^2 + O(s^3).$$

As a result we have, after a calculation with employing (17),

$$p_{2j}(s) = \frac{mX^2\{4X^4 + 8\alpha_j^2X^2 - \alpha_j^2(4m^2 - 3\alpha_j^2)\}}{\alpha_j^2\{(1+2\varepsilon^2)X^2 + a + 5\varepsilon^2\alpha_j^2\}}s^2 + O(s^3)$$
$$q_{2j}(s) = \frac{\pm XD(m,\varepsilon,a,j)}{6\alpha_j^2\{(1+2\varepsilon^2)X^2 + a + 5\varepsilon^2\alpha_j^2\}}s^2 + O(s^3),$$

where $D(m,\varepsilon,a,j)$ is defined in (6). Inserting these expressions back in (23) and arranging them, we finish the proof of the second assertion of the theorem. We may safely omit the details.

We are left with the verification of (21). We rearrange the equation (17) as the one for α_j^2.

$$\varepsilon^2\alpha_j^4 + (a + (1+2\varepsilon^2)X^2)\alpha_j^2 + 2X^4 - 2(2m^2 - a)X^2 = 0,$$

which implies that

$$\alpha_j^2 = \frac{-\{a + (1+2\varepsilon^2)X^2\} + \sqrt{\{a + (1+2\varepsilon^2)X^2\}^2 + 8\varepsilon^2X^2(2m^2 - a - X^2)}}{2\varepsilon^2}$$
$$=: s(B) \qquad \text{with } B := X^2.$$

It is easy to check that

$$s(0) = s(2m^2 - a) = 0, \quad s'(0) > 0, \quad s'(2m^2 - a) < 0.$$

Moreover by a computation we have $s''(B) < 0$. Consequently we infer that

$$0 < X(1;-)^2 < X(2;-)^2 < \cdots < X(J;-)^2 < X(J;+)^2 < \cdots < X(1;+)^2 < 2m^2 - a,$$

where J denotes the largest integer j satisfying (20). This proves the condition (21) and the proof of the theorem is finally completed.

5. Discussions

The bifurcation problem of steady state solutions for a model system of phase separation, which is introduced by Eguchi-Oki-Matsumura (EOM), is analyzed. In particular, bifurcations with respect to the principal parameter b, which characterize the system in this model, is examined. One of striking features of the EOM system is that the dynamics is heavily influenced by the single principal parameter; therefore the present bifurcation analysis is indispensable to understanding the model properly.

As to the steady state solution $(u,v) = (m,0)$, bifurcation takes place at $b = m^2 + (j\pi/l)^2$ for every $j = 1, 2, \cdots$. There exists a one-parameter family of nontrivial steady state solutions of the form

$$(u, v; b) = (m + O(s^2), s\cos\frac{j\pi x}{l} + O(s^2); m^2 + \Big(\frac{j\pi}{l}\Big)^2 + \Big(\frac{3}{4} - \frac{m^2}{4}\frac{1}{(2j\pi/l)^2\varepsilon^2 + a}\Big)s^2 + O(s^3))$$

as $s \to 0$. The corresponding bifurcation point is pitch-fork type and if $j^2 < (m^2 - 3a)/12\varepsilon^2$ it is sub-critical, while if $j^2 > (m^2 - 3a)/12\varepsilon^2$ it is super-critical.

On the other hand the situation is much involved for the steady state solution $(u, v) = (m, \pm\sqrt{b - m^2})$. The mode j where bifurcation occurs is bounded above by (5); the bound for j is $O(m^2)$ as $m^2 \to \infty$. Although there also exists a one-parameter family of nontrivial solutions, their asymptotics as $s \to 0$ are rather complicated. We may conclude that the EOM system has a rich enough set of solutions.

There remain several questions to be considered. One of these issues includes the possible interface motion. Because of no degeneracy in the mobility of the EOM model, surface diffusion may not be expected; we are anxious to know answers how the interface really develops. To be a bit precise, what should be the relation between the two different dynamics; interphase boundaries and antiphase boundaries [4][19]. The dependence on the principal parameter, if it is dominant, would be worth investigating forward. We will revisit these topics in the near future.

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