Remarks on third-order ODEs relevant to
the Kuramoto-Sivashinsky equation

Naoyuki ISHIMURA

Department of Mathematics, Faculty of Economics, Hitotsubashi University,
Kunitachi, Tokyo 186-8601, Japan
E-mail: ishimura@math.hit-u.ac.jp

We are interested in the third-order ordinary differential equations (ODEs)
which are related to the Kuramoto-Sivashinsky equation. So-called steady
solutions of the Kuramoto-Sivashinsky equation are known to admit several
types; for example, bounded global solutions or periodic solutions. We show
that, in addition to these, there exist solutions which blow up on bounded
intervals. Moreover, for certain class of these ODEs, the nonexistence of non-
trivial bounded entire solutions is exhibited.

Key Words: Kuramoto-Sivashinsky equation, third-order ordinary differential
equation, blowing up solutions

1. INTRODUCTION AND RESULTS

We are concerned with the third-order ordinary differential equations
(ODEs) of the form
$$\lambda y''' + y' = f(y), \quad y = y(x) \text{ for } x \in \mathbb{R},$$
(1)
where $\lambda > 0$ is a parameter. $f$ stands for a given smooth function assumed
to satisfy the following hypotheses:
(H1) $f(y) = f(-y)$ and $f' < 0$ on $\{y > 0\}$;
(H2) there exists $\alpha \geq 0$ such that $f(\alpha) = 0$ and $f < 0$ on $\{y > \alpha\}$;
(H3) there exist $p > 1$ and positive constants $\gamma_1 \geq \gamma_2$ such that
$$-\gamma_1 \leq \liminf_{y \to \infty} \frac{f(y)}{y^p} \quad \text{and} \quad \limsup_{y \to \infty} \frac{f(y)}{y^p} \leq -\gamma_2.$$ 

Typical examples of $f$ we have in mind include
\[ f(y) = 1 - y^2 \quad \text{for which } \alpha = 1 \text{ and } p = 2; \quad (2) \]
\[ f(y) = -y^2 \quad \text{for which } \alpha = 0 \text{ and } p = 2. \quad (3) \]

As easily seen, both nonlinearities (2)(3) are derived from the Kuramoto-Sivashinsky equation

\[ u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u_x^2 = 0, \quad u = u(x, t), \quad (4) \]

which arises in a wide variety of fascinating physical phenomena. For instance, we recall that the Kuramoto-Sivashinsky equation is introduced to describe pattern formation in reaction-diffusion systems [10][11], and to model the instability of flame front propagation [14][16]. If we postulate \( u(x, t) = -c^2 t + v(x) \) in (4), whose form is numerically suggested by [13] with the velocity \( c \approx \sqrt{1.2} \), and define \( y(x) = 2^{-1/2}c^{-1}v(x)(2^{1/2}c^{-1}x) \), then we obtain (1) with (2) and \( \lambda = c^2/2 \). Solutions in this case are customary referred to as steady solutions of the Kuramoto-Sivashinsky equation. We remark that (1) with (2) can be recovered from (4) by a usual traveling wave setting \( u(x, t) = v(x - ct) \) and suitable computation. While we just put \( u(x, t) = v(x) \) in (4) and define \( y(x) = 2^{-1/2}c^{-1}v(x)(2^{1/2}c^{-1}x) \), then we find (1) with (3) and \( \lambda = c^2/2 \).

The structure of steady solutions of the Kuramoto-Sivashinsky equation is rather complicated; the equation (1) with (2) gives rise to the existence and/or nonexistence of solutions of various kinds. We will quickly review on several results. Troy [18] made a thorough research in the case \( \lambda = 1/2 \) (here and hereafter we make a scaling to adjust to our formulation); he imposes the initial condition \( y(0) = 0, \ y'(0) = \beta > 0, \ y''(0) = 0 \), and pursues the trajectory in detail regarding \( \beta \) as a parameter. The main results of [18] state that (1)(2) with \( \lambda = 1/2 \) has at least two odd periodic solutions and that there in addition exist at least two bounded global solutions obeying \( y(x) \to \mp 1 \) as \( x \to \pm \infty \), respectively. In [19], the same method is applied to conclude the nonexistence of monotonic solutions; that is, there is no solution to (1)(2) with \( \lambda \approx 1/2 \) such that \( y'(x) > 0 \) for all \( x \in \mathbb{R} \) and \( y(x) \to \pm 1 \) as \( x \to \pm \infty \), respectively. The nonexistence of monotonic global solutions is later extended in [8] to hold for all \( \lambda > 0 \). See also [15]. The method of proof in [8] should be compared with the one [1] given for a similar third-order ODE

\[ \varepsilon y''' + y' = \cos y, \quad y = y(x), \quad \varepsilon > 0, \quad (5) \]
which is presented as a simple model of dendritic crystal growth [12]. After a pioneering observation by [9], it is now established that there is no solution of (5) which fulfills \( y'(x) > 0 \) for all \( x \in \mathbb{R} \) and \( y(x) \to \pm \pi/2 \) as \( x \to \pm \infty \), respectively. Our previous work [7], on the other hand, gives a simple unified treatment toward the nonexistence of monotonic global solutions for certain class of ODEs including (2) and (5); however, [7] restricts the range of parameters \( \lambda \) and \( \varepsilon \) depending on each nonlinearity. We remark that Toland [17] already made an elementary but ingenious approach to proving the nonexistence of monotonic solutions of (1)(2) if \( \lambda \geq 2/9 \).

More on the existence of solutions to (1)(2), we recall that Jones, Troy, and MacGillivray [8] exhibit that there is an odd periodic solution for all small \( \lambda > 0 \). We again refer to [15], where is shown the existence of a unique monotonic solution to (1)(2) with \( \lambda = 1/2 \) on the half interval \( \{0 \leq x < \infty \} \) which fulfills \( y(0) = 0 \) and \( y(x) \to 1 \) as \( x \to \infty \).

The purpose of this paper is twofold: First, we provide another type of solution from those mentioned above, which seems missing in the foregoing literature and applies to the nonlinearity involving both (2) and (3). These solutions blow up on finite intervals in the sense that \( y(x) \to \mp \infty \) as \( x \to \pm x_\beta \), respectively, where \( x_\beta > 0 \) is finite. To be stated more precisely, we consider (1) supplemented with the initial condition

\[
y(0) = 0, \quad y'(0) = -\beta < 0, \quad y''(0) = 0. \tag{6}
\]

Note that under (6) the solutions of (1) turn out to be odd. Our first results now read as follows.

**Theorem 1.1.** Suppose that \( f \) satisfies (H1)(H2)(H3). Then there exists a large \( \beta_f > 0 \) such that for all \( \beta > \beta_f \), there is a finite \( x_\beta > 0 \) for which the solution \( y \) of (1)(6) defined on the interval \( (-x_\beta, x_\beta) \) satisfies \( y' < 0 \) and

\[
y(x) \to \mp \infty \text{ as } x \to \pm x_\beta, \text{ respectively.}
\]

Moreover there hold

\[
\limsup_{x \uparrow x_\beta}(x_\beta - x)^{3/(p-1)}(y(x)) \leq \left\{ \frac{3\sqrt{2}(2p + 1)^{3/2}\sqrt{p + 2\lambda \sqrt{\gamma_1}}}{\gamma_2^{3/2}(p - 1)^3} \right\}^{1/(p-1)},
\]

\[
\liminf_{x \uparrow x_\beta}(x_\beta - x)^{3/(p-1)}(y(x)) \geq \left\{ \frac{3(p + 2)^2\lambda}{2(p - 1)^3\gamma_1} \right\}^{1/(p-1)}. \tag{7}
\]

\[
\limsup_{x \uparrow x_\beta}(x_\beta - x)^{3/(p-1)}(y(x)) \leq \left\{ \frac{3\sqrt{2}(2p + 1)^{3/2}\sqrt{p + 2\lambda \sqrt{\gamma_1}}}{\gamma_2^{3/2}(p - 1)^3} \right\}^{1/(p-1)},
\]

\[
\liminf_{x \uparrow x_\beta}(x_\beta - x)^{3/(p-1)}(y(x)) \geq \left\{ \frac{3(p + 2)^2\lambda}{2(p - 1)^3\gamma_1} \right\}^{1/(p-1)}. \tag{8}
\]
The proof of Theorem 1.1 relies on the monotonicity of solutions, which enables us to reduce the third-order equation to the second-order one. See §2 below for details. The asymptotic profile at $x = x_0$ is calculated by method in the same spirit.

Our next results, which is the second aim of this paper, deal with the nonexistence of global solutions concerning the nonlinearities involving (3). This is a Liouville-type property for (1) if $\alpha = 0$ and $f(0) = 0$ in (H2), whose precise statement is formulated as follows.

**Theorem 1.2.** Suppose that $f$ satisfies (H1)/(H2) with $\alpha = 0$ and $f(0) = 0$ in (H2). Then every bounded entire solution to (1) is trivial; in other words, no bounded nontrivial solution exists on the whole interval $\mathbb{R}$.

We remark that (H3) is not required for Theorem 1.2, whose proof will be developed in §3.

We conclude this introduction with comments on the study of (1) by ODE communities. Most researchers seem to examine the case of odd $f$; that is, $f(y) = -f(-y)$. We refer to a book by Greguš [5], where extensive perspective on this issue is given. In many physically relevant problems, however, it seems that the nonlinearity is an even function; i.e., $f(y) = f(-y)$, on which our current interest is placed.

**2. PROOF OF THEOREM 1.1**

We begin with transforming the equation (1) and constructing solutions. By symmetry it suffices to implement the construction on the half interval $\{x < 0\}$. Since $y'(0) = -\beta < 0$, there is a small $x_0 > 0$ such that $y$ is monotone decreasing if $-x_0 \leq x \leq 0$, where we can regard $y$ as an independent variable. Let $x(y)$ denote the inverse function of $y(x)$ defined on $0 \leq y \leq t_0 := y(x_0)$. We put

$$
t = y \quad \text{and} \quad v(t) = \{y'(x(t))\}^2,
$$

and we compute

$$
v'(t) = 2y'(x(t))x'(t)y''(x(t)) = 2y''(x(t)),
\lambda v''(t) = 2\lambda y'''(x(t))x'(t) = -2 \frac{f(t)}{\sqrt{v(t)}} - 2.
$$

Here we recall that $f(t) = 1 - t^2$ for the example (2) and $f(t) = -t^2$ for (3).
The reduction (10) of the third-order equation into the second-order one, in this context, is introduced by Toland [17] and independently by Bernis, Peletier, and Williams [4], which is exploited in [2][3] to analyze other third-order ODEs different from (1). The application of this technique to the Kuramoto-Sivashinsky equation is undertaken in our previous work [7]. See also a book of Hartman [6].

The problem we want to solve now becomes the following.

\begin{align}
\lambda v''(t) &= -2 \frac{f(t)}{\sqrt{v(t)}} - 2, \\
v(0) &= \beta^2 > 0, \quad v'(0) = 0,
\end{align}

where the differential equation is temporarily understood to hold on the interval \(0 < t < t_0\). We proceed with a series of lemmas.

**Lemma 2.1.** For all sufficiently large \(\beta > 0\), there exists a first \(t_1 > 0\) such that

\[ v(t_1) > 0, \quad v'(t_1) = 0, \quad v''(t_1) > 0. \]

**Proof.** By virtue of \(v'(0) = 0\) and \(v''(0) = -2f(0)/\beta - 2\), it follows that \(v'(t) < 0\) at least for small \(t > 0\) if \(\beta > f(0)\). We distinguish two cases.

1. There is a first \(t_1 > 0\) with \(v(t_1) = 0\) and \(v' < 0\) on \(\{0 < t < t_1\}\);
2. There is a first \(t_1 > 0\) with \(v'(t_1) = 0\) and \(v > 0\) on \(\{0 \leq t \leq t_1\}\).

Our intention is to eliminate the possibility of Case 1 provided \(\beta > 0\) is sufficiently large. We argue indirectly and suppose Case 1 occurs. Taking into account that \(f > 0\) on \(\{0 \leq t < \alpha\}\), we integrate (11) twice on the interval \(0 \leq t \leq \alpha\) to obtain

\begin{align}
\lambda v'(t) &= -2\left(\frac{f(0)}{\sqrt{v(\alpha)}} + 1\right)t \\
\lambda v(t) &= \lambda \beta^2 - \left(\frac{f(0)}{\sqrt{v(\alpha)}} + 1\right)t^2,
\end{align}

from which we especially infer that \(v(\alpha) \geq \beta^2/4\) if we choose \(\beta\) sufficiently large. The last lower bound still holds if \(\alpha = 0\) in (H3) with redundant division. We continue to integrate (11) twice on \(t \geq \alpha\).
\[ \lambda v'(t) > \lambda v'(\alpha) - \frac{4}{\beta} \int_\alpha^t f(s) \, ds - 2(t - \alpha), \]
\[ \lambda v(t) > \frac{\lambda}{4} \beta^2 + \lambda v'(\alpha)(t - \alpha) - \frac{4}{\beta} \int_\alpha^t (t - s)f(s) \, ds - (t - \alpha)^2. \]

Since (H3) implies that for every small \( \varepsilon > 0 \) there exists a \( t_\varepsilon \) such that 
\(-f(t) \geq (\gamma_2 - \varepsilon)t^p \) if \( t \geq t_\varepsilon \), we deduce that
\[ -\int_\alpha^t f(s) \, ds \geq -C_1 + \frac{\gamma_2 - \varepsilon}{2(p + 1)} t^{p+1}, \]
\[ -\int_\alpha^t (t - s)f(s) \, ds \geq -C_2 + \frac{\gamma_2 - \varepsilon}{2(p + 2)(p + 1)} t^{p+2}, \]

for \( t \geq \alpha \) with appropriate positive constants \( C_1, C_2 \). These considerations show that \( v(t) > 0 \) for all \( t \geq 0 \) and there must exist a first \( t_1 > \alpha \) such that \( v'(t_1) = 0 \), if \( \beta \) is sufficiently large; Case 1 cannot occur.

At \( t_1 \) there holds \( v''(t_1) > 0 \). If \( v'''(t_1) = 0 \), then we discover that \( v''''(t_1) > 0 \) upon differentiating the equation (11), which means that \( v''(t) < 0 \) in the left neighborhood of \( t_1 \); a contradiction with the definition of \( t_1 \). We therefore arrive at \( v''(t_1) > 0 \) as desired; Lemma 2.1 is settled. 

**Lemma 2.2.** There holds that
\[ v(t) \geq v(t_1) \quad \text{for all } t \geq 0. \]

In particular the transformation (9) is legitimate for all \( t \geq 0. \)

**Proof.** Suppose that there exists a first \( t_2 > t_1 \) such that \( v(t_2) = v(t_1) \).

We multiply the equation (11) by \( v'(t) \) and integrate on the interval \( t_1 < t < t_2. \)

\[ \frac{1}{2}(v'(t_2))^2 = -4 \int_{t_1}^{t_2} f(t)(\sqrt{v(t)})' \, dt - 2(v(t_2) - v(t_1)) \]
\[ = -4(f(t_2) - f(t_1))\sqrt{v(t_1)} + 4 \int_{t_1}^{t_2} f'(t)\sqrt{v(t)} \, dt \]
\[ < -4(f(t_2) - f(t_1))\sqrt{v(t_1)} + 4\sqrt{v(t_1)} \int_{t_1}^{t_2} f'(t) \, dt = 0, \]
where we have invoked $v(t) > v(t_1) = v(t_2)$ on $t_1 < t < t_2$. This is a contradiction and we obtain $v(t) \geq v(t_1)$ for all $t \geq 0$.

**Remark 2.1.** There might exist finite sequences $\{t_{2i-1}\}_{i=1}^{n+1}$ of local minima and $\{t_{2i}\}_{i=1}^{n}$ of local maxima such that $t_{2i-1} < t_{2i} < t_{2i+1}$ for $i = 1, 2, \cdots, n$ and

$$v(t_{2i-1}) < v(t_{2i+1}), \quad v''(t_{2i-1}) > 0,$$
$$v(t_{2i}) < v(t_{2i+2}), \quad v''(t_{2i}) < 0.$$  

However, for these circumstances, artificial behaviors of $f$ should be requested; concerning typical nonlinearities $(2)(3)$, such oscillations do not take place as we perceive from the rest of proof.

In view of the lower bound of Lemma 2.2, we conclude that the solution of (11) exists for all $t \geq 0$.

We next turn our attention to determining the asymptotic profile. The behavior of $v(t)$ as $t \to \infty$ is described in the following lemma.

**Lemma 2.3.** For all large $t$, we have

$$\left\{ \frac{3^4 \gamma_2}{2 \lambda^2 (2p + 1)^3 (p + 2) \gamma_1} \right\}^{1/3} t^{2(p+2)/3} \leq v(t) \leq \left\{ \frac{18 \gamma_1}{(p + 2)^2 \lambda} \right\}^{2/3} t^{2(p+2)/3}.$$

**Proof.** First we estimate $v(t)$ from above. We introduce a functional

$$F[v](t) = \frac{\lambda}{2} (v'(t))^2 + 2v(t) + 4f(t) \sqrt{v(t)}.$$

We compute

$$\frac{d}{dt} F[v](t) = 4f'(t) \sqrt{v(t)} < 0 \quad \text{for all } t > 0,$$

$$F[v](t_1) = 2v(t_1) + 4f(t_1) \sqrt{v(t_1)} < 4v(t_1) + 4f(t_1) \sqrt{v(t_1)} = -2\lambda v(t_1) v''(t_1) < 0,$$

where $t_1$ is given in Lemma 2.1. Since for every small $\varepsilon > 0$ there exists a $t_\varepsilon > t_1$ such that $-f(t) \leq (\gamma_1 + \varepsilon)t^p$ if $t \geq t_\varepsilon$, we infer that
\[
\frac{\lambda}{2} (v'(t))^2 < -2v(t) - 4f(t) \sqrt{v(t)} < -4f(t) \sqrt{v(t)} < 4(\gamma_1 + \varepsilon) t^p \sqrt{v(t)} < -\sqrt{\frac{8(\gamma_1 + \varepsilon)}{\lambda}} t^{p/2} v(t)^{1/4} < v'(t) < \sqrt{\frac{8(\gamma_1 + \varepsilon)}{\lambda}} t^{p/2} v(t)^{1/4} \quad \text{if } t \geq t_\varepsilon.
\]

(12)

Integrating the last differential inequality, we find that there is another large \( t_\varepsilon \) (denoted by the same \( t_\varepsilon \)) such that

\[
v(t)^{3/4} < \frac{3}{p+2} \sqrt{\frac{2(\gamma_1 + \varepsilon)}{\lambda}} (t^{(p+2)/2} - v(t_\varepsilon)^{(p+2)/2}) + v(t_\varepsilon)^{3/4}
\]

\[
v(t) \leq \left\{ \frac{18(\gamma_1 + 2\varepsilon)}{(p + 2)^2 \lambda} \right\}^{2/3} t^{2(p+2)/3} \quad \text{if } t \geq t_\varepsilon.
\]

The arbitrariness of \( \varepsilon \) implies an upper bound.

Next we wish to establish a lower bound. Recalling that for any small \( \varepsilon > 0 \) there exists a \( t_\varepsilon \) such that \(-f(t) \geq \gamma_2 - \varepsilon \) \( t^p \) if \( t \geq t_\varepsilon \), we observe that

\[
\lambda v''(t) = -2 \frac{f(t)}{\sqrt{v(t)}} - 2 > 2(\gamma_2 - \varepsilon) \left\{ \frac{2(p+2)^2 \lambda}{3^2 \gamma_1} \right\}^{1/3} t^{2(p-1)/3} - 2.
\]

An integration of the above inequality twice leads us to conclude that there is another large \( t_\varepsilon \) (still denoted by the same) such that

\[
v(t) \geq \left\{ \frac{3^4(\gamma_2 - 2\varepsilon)^3}{2 \lambda^2 (2p + 1)^3 (p + 2) \gamma_1} \right\}^{1/3} t^{2(p+2)/3} \quad \text{if } t \geq t_\varepsilon.
\]

Since \( \varepsilon \) is arbitrary we arrive at a lower bound. This finishes the proof of Lemma 2.3.  

\textit{Remark 2.} 2. If \( \gamma_1 = \gamma_2 =: \gamma \) and \( \lim_{t \to \infty} f(t)/t^p = -\gamma \) instead of (H3), then a similar procedure as in \$3\$ of [3] yields also

\[
\liminf_{t \to \infty} \frac{v(t)}{t^{2(p+2)/3}} \leq \left( \frac{3^2 \gamma}{\lambda(p + 2)(2p + 1)} \right)^{2/3} \leq \limsup_{t \to \infty} \frac{v(t)}{t^{2(p+2)/3}}.
\]
Remark 2. 3. As a byproduct of (12) in the above proof, we deduce that \( t_1 > \alpha \).

By virtue of \( y'(x) = -\sqrt{v(y(x))} \) if \( 0 < x < x_\beta \), we assert that

\[
x_\beta := \int_0^\infty \frac{dt}{\sqrt{v(t)}} < \int_0^{t_+} \frac{dt}{\sqrt{v(t_1)}} + \int_{t_+}^\infty \frac{dt}{\sqrt{v(t)}} < \infty,
\]

where \( t_+ \) is so large that Lemma 2.3 is applicable. We are therefore led to the bounded existence interval of the solution to (11).

Lemma 2.3 is also utilized to prove (7)(8). We show (8); the proof of (7) is handled similarly. Since there holds

\[
x = \int_0^{-y(x)} \frac{dt}{\sqrt{v(t)}},
\]

we have

\[
x_\beta - x = \int_{-y(x)}^\infty \frac{dt}{\sqrt{v(t)}} \geq \int_{-y(x)}^\infty \left\{ \frac{(p + 2)^2}{18 \gamma_1} \right\}^{1/3} (x^{p+2})^{1/3} dt,
\]

for all \( x < x_\beta \) with \( x_\beta - x \) being sufficiently small. Carrying out the computation, we derive (8); the proof of Theorem 1.1 is complete.

Remark 2. 4. It is doubtful that (5) possesses similar blow-up solutions.

3. PROOF OF THEOREM 1.2

Our strategy of proof is to discard the possibilities step by step. We begin with the following lemma.

Lemma 3.1. Under the hypotheses of Theorem 1.2, if there exists a non-trivial solution \( y \) of (1) satisfying \( y(x) \to 0 \) as \( x \to \infty \) (or \( y(x) \to 0 \) as \( x \to -\infty \)), then \( y \) must be unbounded.

Proof. The solution \( y \) of (1) which tends to zero as \( x \to \infty \) can be expressed as
\[ y(x) = - \int_{x}^{\infty} \left(1 - \cos \frac{x - t}{\sqrt{\lambda}} \right) f(y(t)) \, dt, \quad (13) \]

which implies especially that \( y(x) \) is nonnegative on \( \mathbb{R} \). If there is a point \( x_0 \in \mathbb{R} \) such that \( y(x_0) = 0 \) (\( x_0 = -\infty \) may take place), then we have

\[ 0 = y(x_0) = - \int_{x_0}^{\infty} \left(1 - \cos \frac{x - t}{\sqrt{\lambda}} \right) f(y(t)) \, dt. \]

Since the integrand is nonnegative, it follows that \( f(y(t)) \equiv 0 \); namely, \( y(x) \equiv 0 \) on \( x_0 \leq x < \infty \), and therefore \( y \equiv 0 \) on the whole \( \mathbb{R} \), which violates the assumption of nontriviality of \( y \). In particular we infer that

\[ \liminf_{x \to -\infty} y(x) \geq 2\varepsilon > 0 \]

for some positive \( \varepsilon \).

Now we take another point \( x_0 \in \mathbb{R} \) (still denoted by the same) so that \( \inf_{x \in (-\infty, x_0)} y(x) \geq \varepsilon \) and define for \( n = 1, 2, 3, \cdots \),

\[ x_n := x_0 - 2\pi n \sqrt{\lambda}. \]

We compute

\[ y(x_n) = - \int_{x_n}^{\infty} \left(1 - \cos \frac{x_n - t}{\sqrt{\lambda}} \right) f(y(t)) \, dt \]

\[ = - \left( \int_{x_n}^{x_{n-1}} + \int_{x_{n-1}}^{\infty} \right) \left(1 - \cos \frac{x_{n-1} - 2\pi \sqrt{\lambda} - t}{\sqrt{\lambda}} \right) f(y(t)) \, dt \]

\[ = - \int_{x_n}^{x_{n-1}} \left(1 - \cos \frac{x_0 - t}{\sqrt{\lambda}} \right) f(y(t)) \, dt + y(x_{n-1}) \]

\[ \geq 2\pi \sqrt{\lambda} f(\varepsilon) + y(x_{n-1}), \]

from which we conclude that

\[ y(x_n) \geq 2\pi n \sqrt{\lambda} f(\varepsilon) + y(x_0) \to \infty \quad \text{as} \ n \to \infty. \]

This proves the lemma. \[ \square \]
Every solution to (1) turns out to converge to null as \( x \to \infty \) or \( x \to -\infty \) and hence it is given by (13) or by similar expression. The next lemma asserts this fact, which completes the proof of Theorem 1.2 thanks to Lemma 3.1.

**Lemma 3.2.** Under the hypotheses of Theorem 1.2, every solution to (1) has to tend to zero as \( x \to \infty \) or as \( x \to -\infty \).

**Proof.** Assume to the contrary that there is a solution \( y \) of (1) which does not converge to zero as \( x \to \infty \) nor as \( x \to -\infty \). Since \( y \) is supposed to be bounded, we may further assume that, say, \( \liminf_{x \to \infty} y(x) = y_0 < 0 \).

First we claim that \( y \) is never eventually monotone increasing nor monotone decreasing. Indeed, if \( y \) is happen to be eventually monotone increasing and \( y'(x) \geq 0 \) for \( x > x_0 \) with \( x_0 \in \mathbb{R} \). We regard \( y \) as an independent variable and define (9) on \( t_0 - \delta < t \leq t_0 < 0 \), where \( t_0 = y_0 \) and \( \delta \) is a suitably chosen small positive constant. Since \( \lim_{x \to \infty} y'(x) = \lim_{x \to \infty} y''(x) = 0 \) in this case, the corresponding second-order equation results in the form

\[
\lambda y''(t) = 2 \frac{f(t)}{\sqrt{v(t)}} - 2,
\]

\[
v(t) \downarrow 0 \quad \text{and} \quad v'(t) \to 0 \quad \text{as} \quad t \uparrow t_0 \neq 0,
\]

whose boundary condition is desperately ill-posed; the solution of (14) does not exist. The other options are excluded similarly.

Consequently there exist sequences \( \{x_{2i-1}\}_{i=-\infty}^{\infty} \) of local minima and \( \{x_{2i}\}_{i=-\infty}^{\infty} \) of local maxima such that \(-\infty < \cdots < x_{2i-1} < x_{2i} < x_{2i+1} < \cdots < \infty \). We claim that both sequences \( \{y(x_{2i-1})\} \) and \( \{y(x_{2i})\} \) are strictly monotone decreasing. To accomplish this, consider the sequence \( \{y(x_{2i})\} \) for example. The case \( \{y(x_{2i-1})\} \) is handled similarly. If there exists a point \( x_+ > x_{2i} \) such that \( y(x_+) = y(x_{2i}) \), then we observe that

\[
y(x_+) = y(x_{2i}) + \lambda y''(x_{2i})(1 - \cos \frac{x_+ - x_{2i}}{\sqrt{\lambda}})
\]

\[
+ \int_{x_{2i}}^{x_+} (1 - \cos \frac{x_+ - t}{\sqrt{\lambda}}) f(y(t)) \, dt
\]

\[
y(x_{2i}),
\]

taking \( y'(x_{2i}) = 0 \) into account. Invoking \( y''(x_{2i}) \leq 0 \) and \( f(y) \leq 0 \) with strict inequality if \( y \neq 0 \), we find that the above equality is impossible; \( y \) enjoys \( y(x) < y(x_{2i}) \) for all \( x > x_{2i} \).
Now select one local minimum $x_{2i-1}$ such that $y(x_{2i-1}) < y_0/2$ and define for $n = 1, 2, 3, \ldots$,

$$x_n^i := x_{2i-1} + 2\pi n \sqrt{\lambda}.$$  

In view of $y'(x_{2i-1}) = 0$, we obtain

$$y(x_{n+1}^i) = y(x_{2i-1}) + \lambda y''(x_{2i-1})(1 - \cos \frac{x_{n+1}^i - x_{2i-1}}{\sqrt{\lambda}})$$

$$+ \int_{x_{2i-1}}^{x_{n+1}^i} (1 - \cos \frac{x_{n+1}^i - t}{\sqrt{\lambda}}) f(y(t)) \, dt$$

$$= y(x_n^i) + \int_{x_n^i}^{x_{n+1}^i} (1 - \cos \frac{x_{2i+1} - t}{\sqrt{\lambda}}) f(y(t)) \, dt$$

$$\leq y(x_n^i),$$

where the last inequality is strict if $y \neq 0$ somewhere on $(x_n^i, x_{n+1}^i)$. Since there holds \( \liminf_{n \to \infty} y(x_n^i) \geq y_0 \), the sequence \{\(y(x_n^i)\)\}_{n=1}^{\infty} converges to a limit; we deduce that \(\sup_{x \in (x_n^i, x_{n+1}^i)} |f(y(x))| \to 0\) as \(n \to \infty\) and thus we arrive at \(y(x) \to 0\) as \(x \to \infty\), a contradiction. This absurdity establishes the lemma.  

**ACKNOWLEDGMENT**  

We are grateful to Professor MasaAki Nakamura for encouragement and to Shin'ya Matsui for informing us about [6]. Thanks are also due to the referee for carefully reading the manuscript and valuable comments. This work is partially supported by Grants-in-Aids for Scientific Research (Nos.10555023, 11640136), from Japan Ministry of Education, Science, and Culture.

**REFERENCES**