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# ORDER STATISTIC GAMES BY PRICE TAKERS*

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## *Abstract*

This paper investigates the consequence of price-taking behavior in quantal response equilibrium model of order-statistic games. In contrast to the result of Yi (2003, *Journal of Economic Behavior and Organization*) that QRE selects the efficient equilibrium, if players ignore the influences of their own choices on the game outcome, or behave as price takers, the selection depends entirely on the prespecified order-statistic and the number of players, and an inefficient outcome could result.

*Keywords:* Quantal Response Equilibrium, Coordination, Price-Taking Behavior
*JEL Classification number:* C79, C92

## I. *Introduction*

Anderson, Goeree and Holt (2001) and Yi (2003) applied "quantal response equilibrium" (QRE) model to a class of coordination games, so-called order-statistic games, and showed that QRE produced a unique selection and had a potential to explain the associated experimental results in Van Huyck et al.'s (1990, 1991, 2001). The present paper studies a consequence of price-taking behavior as a complement to the standard QRE model for a fuller explanation of Van Huyck et al.'s (2001) experimental result.

In the order statistic games in Van Huyck et al. (2001) (VHBR), either five or seven

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subjects simultaneously chose among 101 efforts, the order-statistic being either the second or the fourth from the lowest effort among five or seven effort choices, and each player's payoff is increasing in the prespecified order statistic of his own and others' efforts and quadratically decreasing in the distance between the resulting order statistic and his own effort. In these games, any configuration in which all players choose the same effort is a strict, symmetric, pure-strategy equilibrium, and these equilibria are Pareto-ranked. Other things being equal, the closer the subjects' efforts are to the order statistic, the higher their payoffs are, with all players preferring equilibria with higher efforts to those with lower efforts. However, there is a tension between the higher payoffs of the Pareto-efficient equilibrium and its greater fragility, which makes it riskier to play the efficient equilibrium strategy when others' responses are not perfectly predictable. These games capture important aspects of coordination problems in economic environments and resemble a number of economic models, including the models of Keynesian coordination failure in Bryant (1983) and Cooper and John (1988).[1]

In VHBR's experiment, the results were quite heterogeneous across treatments as well as sessions, sometimes converging to the efficient equilibrium and sometimes not. The numbers of sessions in each treatment converging to the efficient outcome are: six out of eight sessions (6/8) in 5-person fourth order statistic game, 6/10 in 7-person fourth order statistic game, 4/8 in 5-person second order statistic game, 2/10 in 7-person second order statistic game. Otherwise, the play exhibited fairly strong history dependence although the dynamic features are quite different across sessions and treatments. The smaller the number of players and the higher the order statistics are, the more likely the play converges to the efficient outcome.

Yi (2003) applied QRE model to the order statistic games and showed that, for all order-statistic games with payoff decreasing quadratically in the difference between the order statistic and a player's choice, no matter what the order statistic and the number of players are, QRE selects the Pareto-efficient equilibrium when players have a bounded continuum set of strategies. Since the expected benefit of raising effort depends linearly on the change while the associated expected penalty quadratically, with fine strategy spaces the expected benefit from raising effort level by a sufficiently small amount always dominates the associated cost. Together with the discreteness of choice space, the standard QRE model provides sensible explanations for the efficient outcome.[2] However, it fails to explain some aspects of the experimental results in VHBR as the play failed to achieve the efficient outcome in many sessions. In particular, in some sessions of 7-person second order statistic game, the realized order statistic was decreasing over periods and such behavior is difficult to explain with the standard QRE model. The present paper proposes a plausible behavioral assumption as a possible explanation of the result.

When an order statistic game is played, it is conceivable that players response myopically so that they stick to the order statistic that resulted in the previous period. Such a behavior can be incorporated in QRE framework assuming price-taking behavior. The present analysis shows that if individual players ignore their own influences on the resulting order-statistic as

[1] For example, in a conveyer system, the output level is determined by the worker's performance whose productivity is the lowest, and the situation can be modeled as a minimum game.

[2] The coarse space tends to increase opportunity cost of choosing higher effort than the resulting order statistic in the previous period. In Van Huyck et al.'s (1991) 9-person median game experiment with 7 effort levels to choose from, the play always locked in on the equilibrium determined by the initial median of their efforts even though it varied across sessions and was usually inefficient.

a rule of thumb, the efficient outcome is not guaranteed. In this case, since each individual player would try to place his choice as close as possible to the expected resulting order-statistic, each has no incentive at all to raise his effort over the expected value of the order-statistic. If players make stochastic choices as in QRE model, the outcome entirely depends on the number of players and the prespecified order-statistic. That is, only the smallest or largest effort level can be supported as a unique prediction of QRE except in median games.

The rest of the paper is organized as follows. Section 2 introduces a logit equilibrium model of a class of order statistic games with bounded, continuous strategy spaces with a brief summary of Yi's (2003) main results. Section 3 characterizes a variant of logit equilibrium with players ignoring their own influences on the order statistic. Section 4 concludes.

## II. *Order Statistic Games and Logit Equilibrium*

In an $n$-person order-statistic game, a player's payoff is determined by his own effort and a prespecified order statistic of his own and the other players' efforts. Each player chooses an effort level $x_i \in [0, \bar{x}]$, $i=1, \cdots, n$, and $\bar{x}$ is a finite maximum effort level. Each player is assumed to have a risk-neutral preference. Let $\pi_i(x_i, m)$ be the player $i$'s payoff when he plays $x_i$ and the prespecified order statistic is $m$. In this paper, we use a specific functional form which has been used in Van Huyck et al. (1991, 2001),[3]

$$\pi_i(x_i, m_{j:n}) = am_{j:n} - b(m_{j:n} - x_i)^2 + c, \qquad a, b \geq 0 \tag{1}$$

where $m_{j:n}$ is the $j^{th}$ inclusive order statistic which is defined by $m_{1:n} \leq m_{2:n} \leq \cdots \leq m_{n:n}$, where the $m_{j:n}$ is the $j^{th}$ element of choice combinations $\{x_1, \cdots, x_n\}$ arranged in increasing order. When no confusion arises, $\pi_i(x_i, m)$ is denoted by $\pi_i(x_i)$. The main results can be directly extended to games where the penalty function, the second term in Eq.(1), is symmetric around $m_{j:n}$.

In a QRE, a player does not always choose the strategy with the highest expected payoff, as in standard analyses. Instead their strategy choices are noisy, and strategies with higher payoffs are chosen with higher probabilities. As in standard equilibrium analysis, players take the noise in each other's strategy choices into account rationally, correctly predicting the distributions of other's strategies in evaluating the expected payoffs of their own strategies.

In this paper, we focus on a specialized version of QRE where the choice probability is an analogue of the standard multinomial logit distribution. The probability density of player $i$'s choosing $x_i$ is a function of the expected payoff $\pi_i^e(x_i)$ and the density of each choice is an increasing function of the expected payoff for that choice:

$$f_i(x_i) = \frac{\exp(\lambda \pi_i^e(x_i))}{\int_0^{\bar{x}} \exp(\lambda \pi_i^e(y))dy} \tag{2}$$

where $0 \leq \lambda \leq \infty$ measures the amount of noise, or equivalently, the degree of rationality. This functional form is called a logit function where the odds are determined by the exponential transformation of the utility times a given non-negative constant $\lambda$. The ratio of probabilities

[3] In Van Huyck et al. (1990), the payoff function is

$$u_i(x_i, m_{j:n}) = am_{1:n} - b(x_i - m_{1:n}) + \mathrm{c}, \qquad a, b \geq 0$$

so that the penalty for the difference from the minimum is linear.

of two different effort choices is $f_i(x_i)/f_i(x_i')=\exp\,[\lambda(\pi_i^e(x_i)-\pi_i^e(x_i'))]$ and the logit function is invariant to the transformation of expected payoffs by changing the origin. As $\lambda\rightarrow\infty$, the probability of the choice having the highest expected payoff becomes one, if it is unique, so that the choice behavior becomes best response; when $\lambda=0$ all choices have equal probability.[4] Logit equilibrium for $\lambda$ is defined by a fixed point in these probability distributions so that in equilibrium $f_i$ are mutually "noisy best responses." Since only the best responses can be played with positive probabilities in the limit of $\lambda$, the limiting logit equilibrium can be viewed as a selection among Nash equilibria.

The limiting logit equilibrium of order statistic games with quadratic payoff function is characterized in the following proposition.

Proposition 1 (Yi, 2003) *There exists a logit equilibrium for every $\lambda\geq 0$. When $a>0$, as $\lambda\rightarrow\infty$ the logit equilibrium converges to the efficient Nash equilibrium, $x_i=\bar{x}$ for all $i$.*

To see the intuition behind this result, consider the first derivative of the expected payoff with respect to $x_i$:

$$\frac{\partial \pi_i^e(x_i)}{\partial x_i}=a\,\frac{\partial E_i(m_{j:n}\,|\,x_i)}{\partial x_i}$$
$$-2bE_i\left[\{(m_{j:n}-x_i)-x_i\}\left\{\frac{\partial(m_{j:n}\,|\,x_i)}{\partial x_i}-1\right\}\right].$$

Since the first term is strictly positive and $\dfrac{\partial(m_{j:n}\,|\,x_i)}{\partial x_i}-1\leq 0$ in the second term, at $x_i^*=E_i(m_{j:n}\,|\,x_i^*)$, each individual player has a strict incentive to raise his effort by a small amount when $a>0$. In other words, at $x_i^*=E_i(m_{j:n}\,|\,x_i^*)$, the benefit from a sufficiently small increase in one's effort always dominates the cost. As long as players are aware of their own influences on the expected order statistic, this small "tilt" in favor of higher efforts tips the balance of the benefit and the cost in favor of the efficient equilibrium.

## III. *Competitive Logit Equilibria*

In general, when a game involves a large number of players, players might use a "competitive" approximation such that players ignore their own influence on "market signal" (here it is the order statistic) as if the market signal is given. Although such an approximation can be justified with an infinite number of players, it would be still plausible that each individual player behaves as if the order statistic is determined independent of his own effort choice even with a finite number of players. In many economic applications, it often provides a good description of a game with a large number of players. However, such behavior could lead the play far from the efficient equilibrium in order-statistic games.

A "competitive logit equilibrium" is defined in a similar way as the original logit equilibrium except that players do not take into account their own influences on the order

[4] In principle, logit equilibrium permits different $\lambda$ across players. All the results in this paper hold with heterogeneous $\lambda$ by defining a limting logit equilibrium as a limit of $\min_i\lambda_i\rightarrow\infty$.

statistic. In the analysis, it is assumed that each player takes other players' strategy choices rationally into account but with the condition of $\frac{\partial E_i(m_{j:n}|x_i)}{\partial x_i}=0$. In this case, however, since it is not clear how to express each player's expectation on $m_{j:n}$ in an explicit form, a set of assumptions on expectations is imposed instead of defining the exact distribution of the order statistic

Let $F_i(x)$ denote the cumulative distribution function associated with $f_i(x)$. Let $G^i_{j:n-1}(x)$ be the cumulative distribution function of $j^{th}$ order statistic regarding $\{x_i, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}$ drawn from distributions, $\{F_i, \cdots, F_{i-1}, F_{i+1}, \cdots, F_n\}$, respectively. Let $g^i_{j:n-1}(x)$ be the associated probability density function. Let $G^i_{c,j:n}(x)$ be the distribution of player $i$'s expectation of $m_{j:n}$.

Assumption 1. $G^i_{c,j:n}(x)$ satisfies basic properties of a probability distribution function and $G^i_{c,j:n}(x)$ is continuous in $F_{-i}(x)$, where $F_{-i}\equiv\{F_i, \cdots, F_{i-1}, F_{i+1}, \cdots, F_n\}$.

Assumption 2. Given $F_{-i}(x)$ and $F_{-k}(x)$, if at least one of the components in $F_{-i}(x)$ first-order stochastically dominates one of the component in $F_{-k}(x)$ and others are the same, then $G^i_{c,j:n}(x)$ first-order stochastically dominates $G^k_{c,j:n}(x)$.

Assumption 3. Given $F_{-i}(x)$, $G^i_{j-1:n-1}(x)\geq G^i_{c,j:n}(x)\geq G^i_{j:n-1}(x)$ for all $x\in[0,\bar{x}]$.

Assumptions 1 and 2 make $G^i_{c,j:n}(x)$ satisfy basic properties of order statistics. Assumption 2 implies identical $G^i_{c,j:n}(x)$ across players in equilibrium (Lemma 2 in Appendix) and is also crucial in every proof in this section on its own. Assumption 3 links players' perceptions of the order statistic to the objective order statistic, and it is necessary to prove the existence using the same technique as the existence of logit equilibrium.[5] In this environment, it would be natural to consider that $G^i_{c,j:n}(x)$ is the distribution of $m_{j:n}$ regarding $\{x_i, \cdots, x_i, \cdots, x_n\}$ but with players ignoring the influences of their own choices on the resulting order statistic.[6] Clearly, this is a special case of our fomulation and the present approach allows various interpretations of $G^i_{c,j:n}(x)$ as long as the assumptions are met.

Let $E^i_c(m_{j:n})=\int_0^{\bar{x}} xdG^i_{c,j:n}(x)$ be the player $i$'s expectation of $m_{j:n}$ based on his belief when he believes $\frac{\partial E_i(m_{j:n}|x_i)}{\partial x_i}=0$. Then, the expected payoff is

$$\pi^e_{c,i}(x_i)=aE^i_c(m_{j:n})-b(E^i_c(m_{j:n})-x_i)^2+c$$

Notice that $a$ matters in determining expected payoffs but it does not change the relative expected payoffs. $\pi^e_{c,i}(x_i)$ can therefore be rescaled as

$$\pi_{c,i}(x_i)=2bE^i_c(m_{j:n})x_i-bx_i^2 \tag{3}$$

and the effort density is

$$f^i_c(x_i)=\frac{\exp[\lambda b(2E^i_c(m_{j:n})x_i-x_i^2)]}{\int_0^{\bar{x}}\exp[\lambda b(2E^i_c(m_{j:n})y-y^2)]dy}. \tag{4}$$

Because each player thinks that the resulting order statistic is independent of his own choice,

[5] The existence proof is identical to that in Yi (2003) with a slight modification.

[6] We are grateful to the anonymous referee who suggested this interpretation.

his only concern is how close his choice is to $E_c^i(m_{j:n})$. Therefore, the problem each individual player faces is similar to a game with $a=0$, but the competitive logit equilibrium outcome is still Pareto-ranked as long as $a>0$.

For the intuition behind the main result, consider a finite-person order statistic game with $j<\frac{n+1}{2}$ and suppose that $E_c^i(m_{j:n})=E_i(m_{j-1:n-1})$ for all $i$. If $E_i(m_{j-1:n-1})>0$, the best response in the order statistic game is $E_i(m_{j-1:n-1})$ at which the effort density should attain its maximum. Since the expected payoff depends only on the distance to $E_i(m_{j-1:n-1})$, the effort density is symmetric around $E_i(m_{j-1:n-1})$. Moreover, for a sufficiently large $\lambda$, only the expected payoffs of $\varepsilon$-neighbor of $E_i(m_{j-1:n-1})$ are relevant to determining $E_i(m_{j-1:n-1})$ such that $G_{c,j:n}^i(E_i(m_{j-1:n-1}))\approx 1/2$. Since $j<\frac{n+1}{2}$, by the nature of the order statistic (Lemma 5 in Appendix), there is a force which pushes $E_i(m_{j-1:n-1})$ toward 0. Even though there is no incentive for individual players to change their effort levels, the equilibrium effort density is determined entirely by $j$ and $n$, depending on whether $j$ is larger or less than $\frac{n+1}{2}$.

For the main result, minimum and maximum games are not considered. For instance, if players believe that they cannot influence the minimum, the analysis in this section remains valid. However, this is implausible in a minimum game: if a player believes the expected minimum is $0<m<\bar{x}$, the price-taking assumption requires that he should believe that the expected minimum is $m$ even when he chooses 0. For the reason, the main analysis considers only games with $2\leq j\leq n-1$.[7]

Proposition 2 *If $G_{c,j:n}^i(x)$ satisfies Assumption 1-3, as $\lambda$ goes to infinity, the competitive logit equilibrium converges to the point-mass at 0 when $1<j\leq \frac{n+1}{2}-1$ and it converges to $\bar{x}$ when $\frac{n+1}{2}+1\leq j<n$.*

The proof is in Appendix and the proof is valid for the games with symmetric penalty functions in the sense that $\pi_{c,i}^e(x_i)=\pi_{c,i}^e(x_i')$ for all plausible $x_i$ and $x_i'$ such that $|E_c^i(m_{j:n})-x_i|=|E_c^i(m_{j:n})-x_i'|$. The three assumptions on players' beliefs are mainly for an analytical convenience. As long as players ignore their own influences on the resulting order statistic, one can expect the same result. For instance, in games with continuum of players, where each individual player cannot influence the resulting 'order statistic' or the quantile, the limiting

[7] If player $i$ takes the effect of his own choice into account holding $E_i(m_{1:n-1})$ constant, his expected payoff is:

$$\pi_i(x_i)=a\min[x_i, E_i(m_{1:n-1})]-b(\min[x_i, E_i(m_{1:n-1})]-x_i)^2$$

which is not differentiable and that makes the analysis difficult. However, an informal analysis is still possible. Since effort choice depends on the expected payoffs of $\varepsilon$-neighbour of $E_i(m_{1:n-1})$ for a sufficiently large $\lambda$ ($E_i(m_{1:n-1})$ is a unique best response), and

$$\pi_i(E_i(m_{1:n-1})-\varepsilon)=aE_i(m_{1:n-1})-a\varepsilon$$

and

$$\pi_i(E_i(m_{1:n-1})+\varepsilon)=aE_i(m_{1:n-1})-b\varepsilon^2$$

a conjecture can made that a strictly positive $a$ is enough to lead the play to the efficient Nash equilibrium outcome. As in standard logit equilibrium, once players recognize their own influence on the order statistic, the quadratic payoff fucntion would work in favor of the efficient outcome.

logit equilibrium is determined by the relative values of $j$ and $n$.[8]

The prediction of the competitive logit equilibrium is similar to that of Kandori, Mailath and Rob's (1993) model. Robles (1997) showed that the model selects a unique pure-strategy equilibrium with the effort choice of 0 ($\bar{x}$) for $j < \frac{n+1}{2}$ $\left(j > \frac{n+1}{2}\right)$; when $j = \frac{n+1}{2}$, every strict Nash equilibrium has the same size of basin of attraction (or resistance) and it does not produce a unique selection.[9] Because of players' ignorance of their own influences on the resulting order statistic, both the competitive logit equilibrium and the Kandori, Mailath and Rob's notion of long-run equilibrium depend on the statistical property of order statistics, which determines the sizes of basins of attraction.

When $\frac{n}{2} \leq j \leq \frac{n+2}{2}$, the limiting competitive logit equilibrium depends on the players' perceptions of the order statistic. The result is summarized in following corollary.

Corollary 1 *Under Assumptions 1-3, as $\lambda$ goes to infinity, if $E_c^i(m_{j:n}) < E_i(m_{\frac{n}{2}:n-1})$, the competitive logit equilibrium converges to the worst Nash equilibrium; if $E_c^i(m_{j:n}) > E_i(m_{\frac{n}{2}:n-1})$, it converges to the efficient one; if $E_c^i(m_{j:n}) = E_i(m_{\frac{n}{2}:n-1})$ or $E_c^i(m_{j:n}) = E_i(m_{\frac{n-1}{2}:n-2})$,[10] it converges to a point-mass at $\frac{\bar{x}}{2}$.*[11]

## IV. *Concluding Remark*

In many games, a Nash equilibrium of a game with a large but finite number of players can be approximated by the analogous notion in which players ignore their own influences as the outcome is barely affected by the choice of each individual player. In coordination games, however, the ignorance of interaction among players could lead the play far away from a standard logit equilibrium, even in the nonpathological class of order statistic games studied here.

## Appendix: Proof of Proposition 2

Let's define some notation for the proof. Let $f_{c,j:n}^i(x)$ and $F_{c,j:n}^i(x)$ be player $i$'s competitive equilibrium effort density and distribution function in the game with $j$ and $n$, respectively. $E_f(m_{j:n})$ and $E_f(x)$ denote unconditional expectations of order statistic $m_{j:n}$ and $x$ with given

[8] With a continuum of players, an order statistic is not defined. However, one can define a quantile game as a limit of a $n$-person $j^{th}$ order statistic game for $n \to \infty$ and the quantile $q = \frac{j}{n+1}$. Yi (2004) characterized QRE in quantile game.

[9] Robles (1997) considered a more general payoff structure such that $\pi_i(m, m) \geq \pi_i(m', m')$ for all $m > m'$, and $\pi_i(x_i, m) \geq \pi_i(x_i', m)$ for all $|x_i - m| < |x_i' - m|$.

[10] This is the case of $E_c^i(m_{j:n}) = \frac{1}{2}\left[E_i(m_{\frac{n-1}{2}:n-1}) + E_i(m_{\frac{n+1}{2}:n-1})\right]$ when $n$ is odd.

[11] The first part rephrases Proposition 2, and the second part follows from Lemma 4 ($f_i^c(x_i)$ is symmetric around $m_{j:n}$) and Lemma 5 ($F_i^c(m_{j:n}) = \frac{1}{2}$ in this case).

$f$. Let $f_{j-1:n-1}(x)$ and $f_{j:n-1}(x)$ denote the equilibrium effort densities associated with $E_c(m_{j:n}) = E(m_{j-1:n-1})$ and $E_c(m_{j:n}) = E(m_{j:n-1})$, respectively. $F_{j:n-1}(x)$ and $F_{j-1:n-1}(x)$ denote corresponding cumulative distribution functions.

Lemma 1 *Given* $\lambda$, *for any* $i_1$, $i_2$, $j_1$, $j_2$, *and* $n_1$, $n_2$, *if* $E_c^{i_1}(m_{j_1:n_1}) > E_c^{i_2}(m_{j_2:n_2})$, *then* $F^{i_1}_{c,j_1:n_1}$ *first-order stochastically dominates* $F^{i_2}_{c,j_2:n_2}$, *and if* $E_c^{i_1}(m_{j_1:n_1}) = E_c^{i_2}(m_{j_2:n_2})$, *then* $F^{i_1}_{c,j_1:n_1}(x) = F^{i_2}_{c,j_2:n_2}(x)$ *for all* $x$.

Proof. From Eq.(4), $D_x f^i_{c,j:n}(x_i) = 2\lambda b f^i_{c,j:n}(x_i)(E^i_c(m_{j:n}) - x_i)$,

$$\frac{D_x f^{i_1}_{c,j_1:n_1}(x)}{f^{i_1}_{c,j_1:n_1}(x)} - \frac{D_x f^{i_2}_{c,j_2:n_2}(x)}{f^{i_2}_{c,j_2:n_2}(x)} = 2\lambda b[E_c^{i_1}(m_{j_1:n_1}) - E_c^{i_2}(m_{j_2:n_2})] > 0$$

and $f^i_{c,j:n}(0)$ is decreasing in $E^i_c(m_{j:n})$. Thus they cannot cross more than once and the first result follows. The second result is direct from Eq.(4). ■

Lemma 2 *Every competitive logit equilibrium is symmetric across players.*

Proof. By Lemma 1, if two players have different $E^i_c(m_{j:n})$'s, then the effort densities should be different. Suppose $E^1_c(m_{j:n}) > E^2_c(m_{j:n})$. $f^1_{c,j:n}(x)$ depends on $f^2_{c,j:n}(x)$ as well as the other $f^i_{c,j:n}(x)$'s but not on itself, and so does $f^2_{c,j:n}(x)$. By Lemma 1, $F^1_{c,j:n}(x)$ first-order stochastically dominates $F^2_{c,j:n}(x)$ and, by Assumption 2, $G^2_{c,j:n}(x)$ first-order stochastically dominates $G^1_{c,j:n}(x)$. Therefore, $E^1_c(m_{j:n}) < E^2_c(m_{j:n})$, which is a contradiction. ■

By Lemma 2, now on the superscript $i$ will be suppressed.

Lemma 3 *Given* $\lambda$, *for any* $j_1 < j_2$, *if* $E_c(m_{j_1:n})$ *and* $E_c(m_{j_2:n})$ *are expected values in competitive logit equilibria, then* $E_c(m_{j_1:n}) < E_c(m_{j_2:n})$.

Proof. For a given $\lambda$, in order to reach a contradiction, suppose $E_c(m_{j_1:n}) \geq E_c(m_{j_2:n})$ in a equilibrium. When $\lambda = 0$, by Assumption 3, $f_{c,j:n}(x)$ is uniformly distributed and $E_c(m_{j_1:n}) < E_c(m_{j_2:n})$. Since $E_c(m_{j:n})$ is continuous in $\lambda$, there must exist a $\lambda^*$ such that $E_c(m_{j_1:n}) = E_c(m_{j_2:n})$. Then $f_{c,j_1:n}(x) = f_{c,j_2:n}(x)$ by Lemma 1. However, if $f_{c,j_1:n}(x) = f_{c,j_2:n}(x)$, $E_c(m_{j_1:n}) < E_c(m_{j_2:n})$ by Assumption 3, which is a contradiction. ■

Lemma 3 enables to compare competitive logit equilibrium effort densities based only on the $E_c(m_{j:n})$ given $\lambda$. From Eq.(3), since there is a unique "best response", $E_c(m_{j:n})$, $f_{c,j:n}(x)$ converges to a point-mass at $E_c(m_{j:n})$. Together with Assumption 3, by comparing "unconditional expectations" $E(m_{j:n})$'s, one can "order" $E_c(m_{j:n})$s by Lemma 3. Therefore, the proof of Proposition 2 requires only that $E(m_{\frac{n+1}{2}+1:n-1})$ converges to $\bar{x}$ and $E(m_{\frac{n+1}{2}-1:n-1})$ goes to 0 as $\lambda$ goes to infinity.

Lemma 4 *Given* $\lambda$, *for any* $2 \leq j \leq n-1$ *and* $n$, $f_{c,j:n}(x)$ *is symmetric around* $E_c(m_{j:n})$. *That is, if* $(2E_c(m_{j:n}) - x) \in [0, \bar{x}]$, *then* $f_{c,j:n}(x) = f_{c,j:n}(2E_c(m_{j:n}) - x)$.

Proof. If $(2E_c(m_{j:n}) - x) \in [0, \bar{x}]$, it follows directly from $\pi(x) = \pi(2E_c(m_{j:n}) - x)$. ■

Lemma 5 (Ali and Chen, 1965) *If a distribution function*, $F(x)$, *is symmetric, continuous, strictly positive and unimodal, for* $j > \frac{n+1}{2}$, $E(m_{j:n}) \geq F^{-1}\left(\frac{j}{n+1}\right)$ *and for* $j = \frac{n+1}{2}$, $E(m_{j:n})$

$=F^{-1}\left(\frac{j}{n+1}\right)$, *where F is unimodal if there exists at least one real c such that* $F^{-1}$ *is concave for* $x<c$ *and convex for* $x>c$.

Lemma 6 *In a competitive logit equilibrium, for all* $\lambda$, *if* $j\leq\frac{n+1}{2}-1$, *then* $E_c(m_{j:n})<\frac{\bar{x}}{2}$; *if* $j\geq\frac{n+1}{2}+1$, *then* $E_c(m_{j:n})>\frac{\bar{x}}{2}$.

Proof. By Lemma 3 and Assumption 2, it is sufficient to show that $E_{f_c}(m_{j-1:n-1})>\frac{\bar{x}}{2}$ when $j=\frac{n+1}{2}+1$ or $j=\frac{n}{2}+2$, and $E_{f_c}(m_{j-1:n-1})<\frac{\bar{x}}{2}$ when $j=\frac{n+1}{2}-1$ or $j=\frac{n}{2}-1$. When $n$ is even, consider the case of $j=\frac{n}{2}+1$. Suppose $f_{c,j:n}(x)=f_{j-1:n-1}(x)$ and $E_{f_c}(m_{j-1:n-1})<\frac{\bar{x}}{2}$. (This is a median game in terms of players' perceptions.) Define a density $f^*(x)$ as:

$$f^*(x)=f_{j-1:n-1}(x)+\frac{\int_{2E_f(m_{j-1:n-1})}^{\bar{x}}f_{j-1:n-1}(y)dy}{2E_f(m_{j-1:n-1})}\quad\text{if } x\leq 2E_f(m_{j-1:n-1}),$$
$$=0\qquad\text{if } x>2E_f(m_{j-1:n-1})$$

so that $f^*(x)$ is the density that distributes the probability mass of $1-F_{j-1:n-1}(2E_f(m_{j-1:n-1}))$ equally over $[0,\ 2E_f(m_{j-1:n-1})]$ to $f_{j-1:n-1}(x)$. By Lemma 4, $f^*(x)$ satisfies the conditions in Lemma 5 and $E_f(m_{j-1:n-1})=E_{f^*}(m_{j-1:n-1})$. If $E_f(m_{j-1:n-1})<\frac{\bar{x}}{2}$, $f_c(x)$ first-order stochastically dominates $f^*(x)$ and it contradicts the inequality. Therefore $E_f(m_{j-1:n-1})\geq\frac{\bar{x}}{2}$. Since $j=\frac{n}{2}+1<\frac{n+1}{2}+1$, $\frac{n}{2}+2$ is the smallest integer which is greater than $\frac{n+1}{2}+1$. By Assumption 3,

$$\frac{\bar{x}}{2}\leq E(m_{\frac{n}{2}:n-1})<E(m_{\frac{n}{2}+1:n-1})\leq E_c(m_{\frac{n}{2}+2:n})$$

and the result follows. The same arguments hold for $j=\frac{n}{2}$ with

$$f^{**}(x)=f_{j:n-1}(x)+\frac{\int_0^{\bar{x}-2E_f(m_{j:n-1})}f_{j:n-1}(y)dy}{2(\bar{x}-E_f(m_{j:n-1}))}\quad\text{if } x\geq 2E_f(m_{j:n-1})\tag{5}$$
$$=0\qquad\text{if } x<2E_f(m_{j:n-1})-\bar{x}$$

That is, if $j=\frac{n}{2}-1$ and $f_{c,j:n}(x)=f_{j:n-1}(x)$, then $E_{f_c}(m_{j-1:n-1})<\frac{\bar{x}}{2}$.

When $n$ is odd, consider the case of $j=\frac{n+1}{2}+1$. Suppose $E_f(m_{j-1:n-1})\leq\frac{\bar{x}}{2}$, then by Lemma 4 and 5, $E_{f^*}(m_{j:n-1})=E_f(m_{j-1:n-1})$. If $E_f(m_{j-1:n-1})=\frac{\bar{x}}{2}$, $f_{j-1:n-1}(x)$ and $f^*(x)$ are identical, but by the property of order statistic it is not possible. Therefore $F_{j-1:n-1}(x)$ should first-order stochastically dominates $F^*(x)$. Since $E_{f^*}(m_{j:n-1})=E_{f^*}(m_{j-1:n-1})$, $E_{f^*}(m_{j-1:n-1})>E_f(m_{j-1:n-1})$. However, $F_{j-1:n-1}(x)$ first-order stochastically dominates $F^*(x)$, which is a contradiction. The proof for $j=\frac{n+1}{2}-1$ is identical with $f^{**}(x)$. ■

Lemma 7 *In a competitive logit equilibrium, given $\lambda$, for any $j_1$ and $j_2$, if $E_c(m_{j_1:n})=E(m_{j-1:n-1})$ and $E_c(m_{j_2:n})=E(m_{n-j+1:n-1})$, then $E_c(m_{j_1:n})=\bar{x}-E_c(m_{j_2:n})$.*

Proof. Suppose that $F_{c,j-1:n-1}(x)=F_{j-1:n-1}(x)$ in a game with $j$ and $n$ and let

$$F^{\dagger}(x)=\frac{\int_0^x \exp[\lambda b\{2(\bar{x}-E(m_{j-1:n-1}))y-y^2\}]dy}{\int_0^{\bar{x}} \exp[\lambda b\{2(\bar{x}-E(m_{j-1:n-1}))y-y^2\}]dy}$$

so that $F^{\dagger}(x)=1-F_{j-1:n-1}(x)$.[12] Next step is to show that $F^*(x)$ is an equilibrium density identical to $F_{c,\,n-j+1:n-1}(x)$ or $E_{f^{\dagger}}(m_{n-j+1:n-1})=\bar{x}-E(m_{j-1:n-1})$.

$$\begin{aligned}E_{f^{\dagger}}(m_{n-j+1:n-1})&=\bar{x}-\int_0^{\bar{x}} G^{\dagger}_{c,\,n-j+1:n-1}(y)dy\\&=\bar{x}-\int_0^{\bar{x}}\sum_{k=n-j+1}^{n-1}\binom{n-1}{k}[F^{\dagger}(y)]^k[1-F^{\dagger}(y)]^{n-k-1}dy\\&=\bar{x}-\int_0^{\bar{x}}\sum_{k=n-j+1}^{n-1}\binom{n-1}{k}[1-F_{j-1:n-1}(x)]^k[F_{j-1:n-1}(x)]^{n-k-1}dy\end{aligned}$$

Since $\binom{n-1}{k}=\binom{n-1}{n-k-1}$, by substituting $r=n-k-1$,

$$\begin{aligned}E_{f^{\dagger}}(m_{n-j+1:n-1})&=\bar{x}-\int_0^{\bar{x}}\sum_{r=0}^{j-2}\binom{n-1}{r}[F_{j-1:n-1}(y)]^r[1-F_{j-1:n-1}(y)]^{n-r-1}dy\\&=\int_0^{\bar{x}}\left[1-\sum_{r=0}^{j-2}\binom{n-1}{r}[F_{j-1:n-1}(y)]^r[1-F_{j-1:n-1}(y)]^{n-r-1}\right]dy\end{aligned}$$

Since $\sum_{r=0}^{n-1}\binom{n-1}{r}[F_{j-1:n-1}(y)]^{n-r-1}[1-F_{j-1:n-1}(y)]^r=1$,

$$\begin{aligned}E_{f^{\dagger}}(m_{n-j+1:n-1})&=\int_0^{\bar{x}}\sum_{r=j-1}^{n-1}\binom{n-1}{r}[F_{j-1:n-1}(y)]^{n-r-1}[1-F_{j-1:n-1}(y)]^r dy\\&=\int_0^{\bar{x}} G_{j-1:n-1}(y)dy=\bar{x}-E(m_{j-1:n-1}).\end{aligned}$$

■

Proof of Proposition 2. When $j\geq\frac{n+1}{2}+1$, by Assumption 2 and Lemma 3, it is sufficient to show that $E(m_{j-1:n-1})=E(m_{\frac{n+1}{2}:n-1})$ converges to $\bar{x}$. By Lemma 6, $E(m_{j-1:n-1})$ should be greater than $\frac{\bar{x}}{2}$. By Lemma 4,

[12] This relationship holds because the competitive effort density depends only on the distance between the expected order statistic and the choice.

$$2E_f(m_{j-1:n-1}-\bar{x})f_{j-1:n-1}(\bar{x})=(2E_f(m_{j-1:n-1})-\bar{x})f_{j-1:n-1}(2E_f(m_{j-1:n-1})-\bar{x}) \quad (6)$$

and

$$\begin{aligned}\int_0^{2E_f(m_{j-1:n-1})-\bar{x}} f_{j-1:n-1}(y)dy &= 1-\int_{2E_f(m_{j-1:n-1})-\bar{x}}^{\bar{x}} f_{j-1:n-1}(y)dy \quad (7)\\ &= 1-2\int_{E_f(m_{j-1:n-1})}^{\bar{x}} f_{j-1:n-1}(y)dy\\ &= 1-2[1-F_{j-1:n-1}(E_f(m_{j-1:n-1}))]\\ &= 2F_{j-1:n-1}(E_f(m_{j-1:n-1}))-1\end{aligned}$$

Suppose $f_{j-1:n-1}(x)$ converges to a mass-point at $m<\bar{x}$, then $f_{j-1:n-1}(\bar{x})$ converges to 0 and $f_{j-1:n-1}(2E_f(m_{j-1n-1})-\bar{x})$ converges to 0. That implies that for every $\varepsilon>0$, there exists a $\lambda^*$ such that $|f_{j-1n-1}(x)-f^{**}(x)|<\varepsilon$ for all $\lambda>\lambda^*$ where $f^{**}(x)$ is defined as in Eq.(5).

Let $|F_{j-1n-1}(E_f(m_{j-1:n-1}))-F^{**}(E_{f^{**}}(m_{j-1:n-1}))|=\delta_\lambda$, then $\delta_\lambda<\bar{x}f_c(\bar{x})$. Since $f_{j-1:n-1}(\bar{x}) \geq f_{j-1:n-1}(x)$ for every $x\in[0, 2E_f(m_{j-1:n-1})-\bar{x}]$, from Eq.(6) and Eq.(7),

$$\begin{aligned}(2E_f(m_{j-1:n-1})-\bar{x})f_{j-1:n-1}(\bar{x}) &\geq \int_0^{2E_f(m_{j-1:n-1})-\bar{x}} f_{j-1:n-1}(y)dy\\ &= 2F_{j-1:n-1}(E_f(m_{j-1:n-1}))-1\\ &= 2F^{**}_{j-1:n-1}(E_{f^{**}}(m_{j-1:n-1}))-1-2\delta_\lambda\end{aligned}$$

By Lemma 5, $F^{**}_{j-1:n-1}(E_{f_c^{**}}(m_{j-1:n-1}))\geq \dfrac{j-1}{n}$, and

$$(2E_f(m_{j-1:n-1})-\bar{x})f_{j-1:n-1}(\bar{x})+2\delta_\lambda > \frac{2(j-1)}{n}-1=\frac{1}{n}$$

By the hypothesis, $f_{j-1:n-1}(\bar{x})$ and $\delta_\lambda$ vanish as $\lambda$ goes to infinity. This is a contradiction.

When $j\leq \dfrac{n+1}{2}-1$, by Assumption 3 and Lemma 3, it is sufficient to show that $E_f(m_{j:n-1})=E_f(m_{\frac{n+1}{2}-1:n-1})$ converges to 0. By Lemma 7, $E_f(m_{\frac{n+1}{2}-1:n-1})=\bar{x}-E_f(m_{\frac{n+1}{2}-1:n-1})$. By taking smallest $E_f(m_{\frac{n+1}{2}-1:n-1})$, if not unique, $E_f(m_{\frac{n+1}{2}-1:n-1})$ becomes the greatest equilibrium value. The result follows from that $E_f(m_{\frac{n+1}{2}-1:n-1})$ converges to $\bar{x}$ as $\lambda$ increases. ■

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