Optimal Portfolio Choice for Unobservable and Regime-Switching Mean Returns

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Abstract

We study dynamic optimal consumption and portfolio choice for a setting in which the mean returns of a risky asset depend on an unobservable regime variable of the economy, which is defined as a continuous-time Markov chain. The investor estimates the current regime by observing past and present asset prices. We compute the optimal consumption and portfolio policies of an investor with power utility. The optimal consumption/portfolio rule of a long-time-horizon investor could be substantially different from that of a short-time-horizon investor. The difference is caused by an investor's hedging demand of assets against fluctuations in the estimated mean returns.

Key Words: Regime-switching, optimal consumption and portfolio, incomplete information, degenerate partial differential equation, stochastic flows

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1 Introduction

We study dynamic optimal consumption and portfolio choice for a setting in which the mean returns of a risky asset depend on an unobservable regime variable of the economy, which is defined as a continuous-time Markov chain. The investor estimates the current regime by observing past and current asset prices. We compute the optimal consumption and portfolio policies of an investor with power utility. A risky asset price and interest rates are exogenously given. The investor is assumed to be a "small investor," in the sense that his or her actions do not influence market prices. The model is a basic building block in an equilibrium problem of agents whose joint actions determine market prices. In addition to its role in theoretical studies of financial markets, the model can form the basis of portfolio management.

Optimal portfolio and consumption choice in a continuous-time setting was investigated by Merton [39] [40]. By assuming a model with constant coefficients and solving the relevant Hamilton-Jacobi-Bellman equation, Merton [39] produces solutions when the utility function is a member of the Hyperbolic Absolute Risk Aversion (HARA) family of utility functions. If coefficients are not assumed to be constant, in other words, the investment opportunity set is time varying, explicit solutions for portfolio weights are available only in special cases, for example, where the investor has log utility.

The difficulty of solving the optimal portfolio problem is particularly unfortunate because there is now considerable evidence that the investment opportunity set is not constant. For example, the evidence for predictable variation in the equity premium is particularly strong (see, for example, Campbell [10], Campbell and Shiller [11] [12], Fama and French [21] [22], Campbell, Lo, and MacKinlay [9], Chapter 7.) Changes in return volatility tend to be persistent, giving rise to the well documented "GARCH" behavior of returns (see, for example, Bollerslev, Chou and Kroner [4].)

A number of recent papers address the issue of portfolio choice by a multi-period investor facing time-varying investment opportunity sets. Brennan, Schwartz, and Lagnando [8], Barberis [1], Campbell and Viceira [13], Lynch [38], and Xia [49] consider portfolio allocations by a multi-period investor confronted with return predictability. Kim and Omberg [32] and Liu [36] obtain exact analytical solutions for a range of continuous-time problems with predictability. All these papers point out the importance of a "hedging portfolio" that is a demand for the asset as a vehicle to hedge against “unfavorable” shifts in the investment opportunity set.

In the above studies, the parameters of the stochastic processes that determine changes in prices and the investment opportunity set are typically supposed to be known. From a practical point of view, the parameters could be estimated from past data. Even if the investment opportunity is constant, this could be problematic, in particular for the drift coefficient of the price process. (See Merton [43] for example.) The precision of the estimator may not be constant because an investor will learn more about the parameters as time passes. In general, the estimates of the unknown parameter values are “state variables” in an investor’s dynamic optimization problem, and it is necessary to hedge against anticipated changes in these state variables. Thus, it is reasonable to expect that an investor’s behavior may differ when taking estimation risk into account. The question is how an investor’s behavior actually changes.

Bawa and Klein [2] examine the effect of estimation risk on optimal portfolio choice in a single period context. The topic is studied in a continuous-time setting by, for example, Detemple [18], Detemple and Murthy [17], Dothan and Feldman [19], Feldman [23], Gennnote [27], Karatzas and Xue [31], and Kuwana [34]. In general, estimation risk could increase or reduce the optimal allocation of the risky assets depending upon models and parameter values. Brennan [7] demonstrates the practical importance of the parameter uncertainty in a simple continuous time setting, in which the stock return is independently and identically distributed. In Brennan’s example, it is shown that for reasonable parameter values the parameter uncertainty could reduce the investor’s optimal allocation to the risky asset by as much as 50%. Xia [49] examines the effects of the parameter uncertainty when the asset prices are possibly predictable and the predictive variables themselves are stochastic. It is shown that the optimal allocation can be quite different from the case where there is no parameter uncertainty or predictability.

In these articles, the processes of the coefficients and the signals have continuous sample paths. Some important sources of uncertainty may be discontinuous, recurrent, and fluctuating. Such significant events include innovations in technique, introduction of new products, natural disasters, and changes in laws or government policies. Although these events may affect the profitability of risky assets, the relationship among these events and the profitability of risky assets can be very complicated. Furthermore, there are numerous events and economic variables that are potentially related to the profitability of risky assets. It is in general difficult to conclude which one is really relevant. Thus investors may not try to count up all potentially relevant events and variables, and simply assume that there are a few possible states, for example, good and bad states, of an asset’s profitability. Investors attempt to learn about the current state of mean returns by observing the market price of assets and to update their expectation with the arrival of new information. Even if an investor is relatively confident about the current profitability of assets, the regime can shift at the next instant. Therefore the precision of the estimator will fluctuate even when investors have so many historical data.

In order to describe this situation, it is assumed in this paper that the unknown drift coefficient of the risky asset price process is an unobserved, two-state, continuous-time Markov chain. The estimated (conditional mean) drift process $\hat{\mu}$ has outcomes in the interval $[\mu_0, \mu_1]$, whose endpoints are the mean-rate-of-return ("drift") parameters of the two regimes. Thus, the investor's optimal control problem has, in effect, the estimated drift process $\hat{\mu}$ as well as the wealth process as its state variables.

The model presented below is similar to David [15], who studies an investor's portfolio and excess returns in equilibrium of a Cox-Ingersoll-Ross production economy with the (unobservable) Markov-switching state variable. Assuming the existence of the value function and symmetric switches between good and bad states in two industries, it is shown that commonly observed stylized facts in financial markets could be explained. In a similar setting, Veronesi [48] also investigates the effects of uncertainty and risk aversion on the form of the stock price function and how this relates to stock market volatility. Both authors mainly focus on the behavior of equilibrium expected return and volatility.

The model presented below uses the underlying framework of David and Veronesi, but differs from it in its emphasis. While they investigate behavior of equilibrium expected return and volatility, this paper investigates the effects of uncertainty and risk aversion on investor portfolios and consumption. Although David [15] also investigates the optimal consumption and portfolio rule, it is difficult to see how each investor's optimal consumption and portfolio rule is affected by uncertainty and estimation risk, since an analytical solution is not available. This paper investigates how various parameters of the model affect the optimal consumption and portfolio rule, for example, investors' risk aversion and terminal horizon, frequency of regime switching, and volatility of returns.

From the technical point of view, this paper makes the following contributions. As is discussed above, to compute the optimal consumption and portfolio rule is in general difficult when the investment opportunity set is time-varying. Brennan, Schwartz, and Lagnando (1996) and Xia (1999), for example, solve the non-linear partial differential equation (PDE), which is associated with the Hamilton-Jacobi-Bellman (HJB) equation. Campbell and Viceira (1998) use log-linear approximations to solve the investor's multi-period discrete-time problem. Kim and Omberg (1996) and Liu (1999) obtain exact analytical solutions for the value function of the problem that allows a range of continuous-time problems with predictability. Brandt [5] uses the investor's Euler equations to estimate his or her portfolio allocation to stocks.

In order to compute the optimal consumption-portfolio rule, we use the Monte Carlo simulation approach in this paper. Our control problem is converted into the martingale formulation by standard arguments, since markets are complete from the investor's point of view. The technique of stochastic flows is then applied to facilitate the computation. (See, for example, Nualart [44] and Protter [46].) It is possible that the result could be derived as a corollary of the general result of Ocone and Karatzas [45] and Detemple, Garcia, and Rindisbacher [16], in which the Malliavin calculus is applied. However, any such relation is not transparent, and the stochastic flow is more directly related to the derivatives of the value function.[1]

The stochastic flow is also effective to prove the existence of a solution to the Hamilton-Jacobi-Bellman (HJB) equation and to derive the optimal portfolio rule explicitly. In this paper, following standard procedures, we obtain the HJB equation, which is a partial differential equation (PDE) with two boundaries for the state variable $\hat{\mu}$. The existence of a solution to the HJB equation is hardly guaranteed because it is non-linear and degenerate. Standard existence theorems cannot be applied.[2] For a particular value of the risk-aversion coefficient, however, the HJB equation is reduced to a linear PDE by a change of variables. We then overcome the difficulty associated with degenerate PDE by showing that the naturally conjectured solution is in fact sufficiently smooth. For the particular value of the risk-aversion coefficient, interesting properties are found by looking at the explicit optimal portfolio rule without carrying out a numerical computation.

From a careful examination of the solution to the HJB equation and numerical examples, we can see how the optimal consumption rate and the optimal portfolio depend on the estimated drift process $\hat{\mu}$, time horizon and risk-aversion of investors, and other parameters. In particular, the terminal

[1] The Malliavin calculus can also be applied to compute option Greeks. See, for example, Fournié, Lasry, Lebuchoux, Lions, and Touzi [25]. Roughly speaking, the Malliavin calculus is a differentiation of a stochastic process on its domain, that is, on a probability space. The stochastic flows is a differentiation of a stochastic process with respect to its initial value. The relation between the Malliavin calculus and the stochastic flow is also discussed in Colwell, Elliott, and Kopp [14].

[2] The degenerate PDE is an essential technical difficulty that is associated with an incomplete information structure. In some cases, coefficients of PDE could be zero when there is no estimation risk. See Kuwana [33] for example.

horizon effect on the optimal portfolio is important. The hedging portfolio cannot be negligible when investors have a long time horizon. Depending on an investor's degree of risk aversion, the sign as well as the size of the hedging portfolio could vary among investors.

This has interesting implications regarding the terminal horizon of the investor. Financial planners often advise older people to invest less in stocks than younger people. (See for example Siegel [47] and Jagannathan and Kocherlakota [29].) Is this conventional advice true in our model? Our results show that longer-time-horizon investors can invest more or less in the risky asset than shorter-time-horizon investors, depending upon the investor's risk aversion. Furthermore, the hedging portfolio could depend on the estimated drift in a very complicated way, if the expected return of an asset in a worse regime is lower than the risk-free rate. Optimal consumption and portfolio should be decided carefully by taking all related parameters into account, and we cannot simply recommend younger people to invest more in stocks.

The remainder of this paper is structured as follows. Section 2 formally describes the model. Some typical properties of the model are discussed in Section 3. Section 4 proves the existence of the value function for a particular risk-aversion coefficient. Section 5 explains the numerical procedure and shows the numerical examples for various parameter values. Section 6 concludes. All proofs appear in the Appendix.

2 Model

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which a standard Brownian motion B and a two-state, continuous-time Markov chain Y are defined. The process Y is right-continuous with values in $\{0, 1\}$ and represents the regime of the economy. We suppose that, at the initial time $t = 0$, Y_0 has outcome 1 with probability p and outcome 0 with probability $1 - p$. The process Y, starting at i, remains there for an exponentially distributed length of time, and then jumps to state $j (\neq i)$. The exponential density has a parameter λ_{ij}. For notational simplicity, we consider only the symmetric case, in which for some $\lambda > 0$,

$$\lambda_{01} = \lambda_{10} = \lambda.$$

The exponentially distributed inter-regime times are independent, and independent of B. The underlying information filtration is $\mathbf{F} = \{\mathcal{F}_t^{B,Y}\}$, where $\mathcal{F}_t^{B,Y} = \sigma(B_s, Y_s, s \leq t)^{\tilde{}}$. That is, $\mathcal{F}_t^{B,Y}$ is the augmented σ-algebra on Ω generated by observation of B_s and Y_s up to t.

One risky asset and one riskless asset are available for investment. The riskless asset price process β satisfies

$$d\beta_t = \beta_t r \, dt, \tag{1}$$

for a positive constant r. The risky asset price process S satisfies

$$dS_t = S_t \mu(Y_t) \, dt + S_t \sigma_S \, dB_t, \qquad S_0 > 0, \tag{2}$$

where σ_S is a constant and $\mu : \{0,1\} \to I\!R$. Let $\mu_0 \equiv \mu(0)$ and $\mu_1 \equiv \mu(1)$. We assume that $\mu_1 > \mu_0$, which means that state 1 is the high-expected-return state. It is assumed (realistically) that the investor can observe neither Y nor $\mu(Y)$ directly. The investor observes only the asset price process S. Thus, the investment environment is described by the riskless asset's price process (1), the risky asset's price process (2), and the investor's information, defined by the filtration $\mathbf{F}^S = \{\mathcal{F}_t^S\}$, where $\mathcal{F}_t^S = \sigma(S_u, u \le t)^{\tilde{}}$. The parameters σ_S, p, λ, μ_0, and μ_1 are supposed to be known constants.

As in previous works, see for example Gennnote [27] and Feldman [23], we can identify a σ-algebra equivalent economy with the filtered probability

$$\pi_t \equiv P\left(Y_t = 1 \;\middle|\; \mathcal{F}_t^S\right) \qquad \text{and} \qquad \pi_0 = p.$$

That is, π_t is the probability that the current regime is the high-expected-return state, given the observations S_s, $0 \le s \le t$. It follows from Theorem 9.1 of Liptser and Shiryayev [35] that π satisfies the stochastic differential equation

$$\begin{aligned} d\pi_t &= \lambda(1 - 2\pi_t) \, dt + \pi_t(1 - \pi_t) \frac{(\mu_1 - \mu_0)}{\sigma_S} \, d\overline{B}_t \\ &\equiv \mu_\pi(\pi_t) \, dt + \sigma_\pi(\pi_t) \, d\overline{B}_t, \end{aligned} \tag{3}$$

where $\overline{B}$ is the standard Brownian motion with respect to $\{\mathcal{F}_t^S\}$ defined by

$$\overline{B}_t = \int_0^t \frac{dS_s - S_s \hat{\mu}(\pi_s) \, ds}{S_s \sigma_S}, \quad \text{with} \quad \hat{\mu}(\pi_t) = \pi_t \mu_1 + (1 - \pi_t)\mu_0.$$

A σ-algebra equivalent economy is then described by the riskless price process (1), the filtered probability (3), the risky price process S satisfying

$$dS_t = S_t \hat{\mu}(\pi_t) \, dt + S_t \sigma_S \, d\overline{B}_t, \tag{4}$$

and the filtration $\mathbf{F}^S$ generated by S. From now on, we work in this Markovian equivalent economy.

The investor's utility is defined on a terminal wealth w_T (a non-negative random variable) and a consumption process. A consumption process is an adapted non-negative jointly measurable process c with $\int_0^T c_t^2\, dt < \infty$ almost surely. Specifically, we suppose that the investor's preferences are given by the utility function U defined by

$$U(c, w_T) = \mathrm{E}\left(\int_0^T e^{-\rho t} u(c_t)\, dt + u(w_T)\right) = \int \left[\int_0^T e^{-\rho t} u(c_t)\, dt + u(w_T)\right] dP,$$

where $\rho > 0$, $u(c) = c^\alpha/\alpha$, and $\alpha < 1$, $\alpha \neq 0$. For the case with $\alpha = 0$, we let $u(c) = \log c$.

A trading strategy is an adapted process $\theta = (\theta^0, \theta^1)$ satisfying technical integrability conditions.[3] This means that the investor holds θ_t^0 units of the risk-free asset and θ_t^1 units of the risky asset at time t, based on information available at that time. For a trading strategy θ, the investor's wealth at time t is $w_t = \theta_t \cdot \mathcal{S}_t$, where $\mathcal{S}_t = (\beta_t, S_t)$. Given an initial wealth $w_0 > 0$, we say that (c, θ) is budget feasible, denoted $(c, \theta) \in \Lambda(w_0)$, if θ satisfies

$$\theta_t \cdot \mathcal{S}_t = w_0 + \int_0^t \theta_s\, d\mathcal{S}_s - \int_0^t c_s\, ds \geq 0, \quad t \in [0, T].$$

Given θ, let φ be the fraction of wealth w invested in the risky assets. That is,

$$\varphi_t = \frac{\theta_t^1 S_t}{w_t} \quad \text{if} \quad w_t \neq 0, \tag{5}$$

and $\varphi_t = 0$ if $w_t = 0$. The wealth process w generated by (c, φ) satisfies

$$\begin{aligned} dw_t &= [w_t \varphi_t(\hat{\mu}(\pi_t) - r) + r w_t - c_t]\, dt + (w_t \varphi_t \sigma_S)\, d\overline{B}_t \\ &\equiv \mu_w(w_t, \pi_t, c_t, \varphi_t)\, dt + \sigma_w(w_t, \varphi_t)\, d\overline{B}_t. \end{aligned}$$

We say that a control (c, φ) is budget feasible if (c, θ) determined by (5) is budget feasible. We write $(c, \varphi) \in \Lambda(w_0)$ if the corresponding (c, θ) is in $\Lambda(w_0)$. The utility of a control (c, φ) is

$$V^{(c,\varphi)}(w, \pi) = \mathrm{E}\left(\int_0^T e^{-\rho s} u(c_s)\, ds + u(w_T) \,\middle|\, \mathcal{F}_0^S\right).$$

The value of a state (w, π) is then defined by

$$V(w, \pi) = \sup_{(c,\varphi) \in \Lambda(w_0)} V^{(c,\varphi)}(w, \pi). \tag{6}$$

If $V^{(c,\varphi)}(w, \pi) = V(w, \pi)$, then (c, φ) is an optimal control at (w, π).

[3] Let $\mathcal{L}^1$ be the space of $\{\mathcal{F}_t^S\}$-adapted progressively measurable processes a satisfying $\int_0^T |a_t|\, dt < \infty$ almost surely for each T. Let $\mathcal{L}^2$ be the space of $\{\mathcal{F}_t^S\}$-adapted progressively measurable processes a satisfying $\int_0^T a_t^2\, dt < \infty$ almost surely for each T. Given an Ito process X with $dX_t = \mu_t\, dt + \sigma_t\, d\overline{B}_t$, we write that a process a is in $\mathcal{L}(X)$ if $\{a_t \mu_t : t \geq 0\}$ is in $\mathcal{L}^1$ and $\{a_t \sigma_t : t \geq 0\}$ is in $\mathcal{L}^2$. A trading strategy $\theta = (\theta^0, \theta^1)$ satisfies the required technical condition if $\theta^0 \in \mathcal{L}(\beta)$ and $\theta^1 \in \mathcal{L}(S)$.

3 Properties of the Model

In this section, we first discuss some properties of the filtering equation. The properties of our filtering model are well explained in David [15], where the stationary distribution of the filtered probability π is effectively used. We repeat some of the properties that are important for the optimal consumption and portfolio decision. It is interesting to see the differences between the filtering equation (3) for the unobserved Markov chain and the filtering equation for an unobserved Ornstein-Uhlenbeck process, which is typically assumed in previous works, for example, Dothan and Feldman [19] and Gennote [27]. In the latter case, the filter is derived by using its Gaussian structure, and consists of the conditional mean process and the conditional variance process of the unobservable process. The unconditionally Gaussian structure leads to a deterministic conditional variance, and the precision of the mean estimate goes from an initial condition to a steady state.[4]

In our case, the precision of the mean estimate fluctuates randomly rather than being deterministic. Both the mean estimate and its precision are summarized in one variable, $\pi(t)$. If $\pi(t)$ takes values near the boundaries of the interval $[0,1]$, then the investor may be fairly confident about the current regime of the economy. On the other hand, when $\pi(t)$ is near $1/2$, the investor is not confident about the current regime. Thus studying dynamic behavior of π in (3) allows us to investigate the fluctuations of the mean estimate and its precision.

It is easily seen that the larger the difference between two states, μ_1 and μ_0, the larger the volatility of the filtered probability. On the other hand, the larger the volatility of asset returns, the smaller the volatility of the filtered probability, because asset returns provide little information ("low-signal-to-noise").

The density λ of regime shift is important in our model. If we suppose $\lambda > 0$, π has a tendency to drift inside the interval, as can be seen in (3). The regime may switch before the investor completely learns about the current regime. Therefore the investor's filtered probability has a mechanism that avoids "too much" confidence. In other words, the possibility of a regime switch makes the speed of learning slower. If λ is so large that the drift term dominates the diffusion term in (3), then the speed of learning is very slow and π may take values near $1/2$ almost always.

If we suppose that $\lambda = 0$, our model is similar to Brennan [7], where the investment opportunity set is constant but uncertain about the mean return

[4] If the initial condition is higher than the steady state, the precision of the mean estimate increases as more returns are observed. See, for example, Proposition 2 in Feldman [23].

on the risky asset. As a special case of Gennote [27] and other previous literature, the conditional variance of the unobserved mean is deterministic. In our model with $\lambda = 0$, (3) is simplified to

$$d\pi_t = \pi_t(1 - \pi_t)\frac{(\mu_1 - \mu_0)}{\sigma_S}\, d\overline{B}_t.$$

The conditional variance of the unobserved mean is stochastic even if $\lambda = 0$. Notwithstanding, it would be reasonable to conclude that our model with $\lambda = 0$ is quite similar to Brennan [7], because there is no mean-reverting effect and the precision of the mean estimate would increase as more returns were observed.

We now study properties of the optimal solution to our control problem (6). One of the standard procedures is to find a function $J : I\!R_+ \times [0,1] \times [0,T] \to I\!R$ that is suitably differentiable and solves the HJB equation

$$\sup_{(c,\varphi)\in I\!R_+ \times I\!R} \mathcal{D}^{(c,\varphi)} J(w,\pi,t) + e^{-\rho t} u(c) = 0, \tag{7}$$

where

$$\begin{aligned} \mathcal{D}^{(c,\varphi)} J(w,\pi,t) &= J_w(w,\pi,t)\mu_w(w,\pi,\varphi) + \frac{1}{2}J_{ww}(w,\pi,t)\sigma_w^2(w,\varphi) + J_\pi(w,\pi,t)\mu_\pi(\pi) \\ &\quad + \frac{1}{2}J_{\pi\pi}(w,\pi,t)\sigma_\pi^2(\pi) + J_{w\pi}(w,\pi,t)\sigma_w(\pi,\varphi)\sigma_\pi(\pi,\varphi) + J_t(w,\pi,t), \end{aligned}$$

with the boundary condition

$$J(w,\pi,T) = u(w).$$

If such a function J exists, and if other technical conditions are satisfied, then we can find the optimal control by a well-known verification argument.

For the moment, let us assume that there exists such a function, J. By standard homogeneity arguments, a natural conjecture is that

$$J(w,\pi,t) = f(\pi,t)\frac{w^\alpha}{\alpha}, \tag{8}$$

where f is in $C^{2,1}([0,1] \times [0,T])$.[5] It follows from the first-order conditions of the HJB equation (7) that the optimal consumption and portfolio rule (c^*, φ^*) implied by this conjectured value function is given by

$$c_t^* = e^{\rho t/(\alpha-1)} f(\pi_t,t)^{1/(\alpha-1)} w_t, \tag{9}$$

[5]That is, f is twice continuously differentiable with respect to π and once continuously differentiable with respect to t.

$$\varphi_t^* = \frac{\hat{\mu}(\pi_t) - r}{(1-\alpha)\sigma_S^2} + \frac{\pi(1-\pi)(\mu_1 - \mu_0)}{(1-\alpha)\sigma_S^2} \frac{f_\pi(\pi_t, t)}{f(\pi_t, t)}. \tag{10}$$

The arguments of f and its partial derivatives are suppressed below for notational convenience. By substituting this policy back into the HJB equation, we obtain the non-linear partial differential equation on $[0,1] \times [0,T]$:

$$\begin{aligned} 0 \;=\; & \left(\frac{(\hat{\mu}(\pi) - r)^2}{2\sigma_S^2(1-\alpha)} + r \right) f^2 + \left(\frac{\pi(1-\pi)(\mu_1 - \mu_0)}{(1-\alpha)\sigma_S^2} (\hat{\mu}(\pi) - r) + \frac{1}{\alpha}\lambda(1 - 2\pi) \right) f f_\pi \\ & + \left(\frac{\pi^2(1-\pi)^2(\mu_1 - \mu_0)^2}{2(1-\alpha)\sigma_S^2} \right) (f_\pi)^2 + \left(\frac{\pi^2(1-\pi)^2(\mu_1 - \mu_0)^2}{2\alpha\sigma_S^2} \right) f f_{\pi\pi} + \frac{1}{\alpha} f f_t \\ & + \left(\frac{1-\alpha}{\alpha} \right) e^{\rho t/(\alpha-1)} f^{(2\alpha-1)/(\alpha-1)}, \qquad (\pi, t) \in [0,1] \times [0,T], \end{aligned} \tag{11}$$

with the boundary condition

$$f(\pi, T) = 1, \qquad \pi \in [0,1]. \tag{12}$$

Thus, questions about the existence of a solution J to the HJB equation are essentially reduced to the existence of the solution f to the PDE (11).

Since (11) is non-linear, it seems quite hard to find a solution in general. Furthermore, to show the existence of a solution to (11) is not straightforward, because it is degenerate, in the sense that some coefficients are zero at the boundaries 0 and 1 for π.

In section 4, it is shown for the case $\alpha = 1/2$ that (11) can be reduced to a linear PDE. The existence of a solution to (11) is proved by using the stochastic flow. Interesting properties are found without using numerical estimation. For a general value of α, we apply the martingale approach in section 5.[6] Since markets are complete from the investor's point of view, our control problem is converted into a martingale formulation by standard arguments. The optimal control to (6) is numerically computed by Monte Carlo simulation. Stochastic flow also plays an important role to simplify numerical computation.

Before closing this section, we briefly review the case of log utility. The natural conjecture of the value function is

$$J(w, \pi, t) = f(w, t) + g(\pi, t),$$

for some differentiable $f : I\!R_+ \times [0,T] \to I\!R$ and $g : [0,1] \times [0,T] \to I\!R$. The PDE, derived as above, is degenerate, and the existence of a solution is not trivial. It has been already shown, however, that certainty equivalence

[6] See, for example, Duffie [20].

holds if and only if the investor has log utility. (See Kuwana [34].)[7] In the current context, certainty equivalence means that the investor behaves as if the estimator $\hat{\mu}$ is the true drift. Thus, the optimal consumption portfolio rule of the log-utility investor is conjectured to be

$$c_t = e^{-\rho t} f_w(w,t) \qquad \text{and} \qquad \varphi_t = \frac{\hat{\mu}(\pi_t) - r}{\sigma_S^2}.$$

Verification of this policy in this setting is relatively straightforward. Thus, for log-utility investors, it turns out that the time horizon and the incomplete observation of mean rates of return do not play a role in determining the optimal portfolio. The optimal consumption rule is irrelevant to the estimator $\hat{\mu}$.

4 Optimal Consumption and Portfolio for $\alpha = 1/2$

In this section, we show the optimal consumption and portfolio rule and the value function for $\alpha = 1/2$. As discussed above, it seems quite hard to show the existence of a solution to (11) in general, since it is non-linear and degenerate. Although the PDE (11) is non-linear, it can be transformed into a linear PDE by the change of variables for the case $\alpha = 1/2$:

$$h(\pi,t) \equiv (f(\pi,t))^2, \tag{13}$$

leaving

$$\begin{aligned} 0 \;=\; & \left(\frac{(\hat{\mu}(\pi) - r)^2}{\sigma_S^2} + r\right) h + \left(\frac{\pi(1-\pi)(\mu_1 - \mu_0)}{\sigma_S^2}(\hat{\mu}(\pi) - r) + \lambda(1 - 2\pi)\right) h_\pi \\ & + \left(\frac{\pi^2(1-\pi)^2(\mu_1 - \mu_0)^2}{\sigma_S^2}\right)\frac{1}{2} h_{\pi\pi} + h_t + e^{\rho t/(\alpha - 1)}, \end{aligned} \tag{14}$$

with the boundary condition $h(\pi, T) = 1$.

To show the existence of a solution to the PDE (14) is still not straightforward, because standard existence theorems cannot be applied.[8] We overcome this difficulty by using stochastic-flow techniques.

[7] See also the argument about the "generalized myopia" and the log utility investor for the Gaussian uncertainty case in Feldman [24].

[8] The standard references usually assume the non-degeneracy of PDE in order to guarantee the existence of its solution. See, for example, Karatzas and Shreve [30], Remark 7.8.

Proposition 4.1. *Suppose that $u(c) = c^\alpha/\alpha$, with the risk-aversion coefficient $\alpha = 1/2$. Then there is a solution f to (11)-(12) and the value function J solving (7) is given by*

$$J(w, \pi, t) = 2f(\pi, t)w^{1/2}.$$

Moreover, $V(w, \pi) = J(w, \pi, 0)$ for all $w \geq 0$ and $\pi \in [0, 1]$. The unique optimal control (c^*, φ^*) of problem (6) is given by

$$c_t^* = e^{-2\rho t} f(\pi_t, t)^{-2} w_t^* \quad \text{and} \quad \varphi_t^* = \frac{2(\hat{\mu}(\pi_t) - r)}{\sigma_S^2} + \frac{2\pi_t(1 - \pi_t)(\mu_1 - \mu_0)}{\sigma_S^2} \frac{f_\pi(\pi_t, t)}{f(\pi_t, t)}, \tag{15}$$

where w^* is the unique solution of

$$dw_t^* = \mu_w(w_t^*, \pi_t, c_t^*, \varphi_t^*)\, dt + \sigma_w(w_t^*, \varphi_t^*)\, d\overline{B}_t.$$

Our proof strategy is to show that a naturally conjectured solution h of the PDE (14) is in fact sufficiently smooth and solves (14). We then confirm that the obtained properties of $f = \sqrt{h}$ are sufficient for the verification argument. From "Feynman-Kac" reasoning, the naturally conjectured solution of the PDE (14) is[9]

$$h(x, t) = \mathrm{E}\left(\int_t^T e^{-2\rho s} \phi_{t,s}\, ds + \phi_{t,T} \,\middle|\, \pi_t = x \right), \tag{16}$$

where

$$\phi_{t,s} = \exp\left(\int_t^s \left(\frac{(\hat{\mu}(\pi_\tau) - r)^2}{\sigma_S^2} + r \right) d\tau \right).$$

The next proposition shows, using the theory of stochastic flows, that the function h defined by (16) is sufficiently smooth for our purpose.

Proposition 4.2. *The function $h(\,\cdot\,, \cdot\,)$ given by (16) is in $C^{2,1}([0, 1] \times [0, T])$. The derivatives $\partial h/\partial x$ and $\partial f/\partial x$ are bounded.*

Proof. See Appendix. ▮

[9] This conjecture comes from the Feynman-Kac Formula. We can also think of it as the "generalized" solution of a PDE, whose existence and uniqueness can be shown. See Freidlin [26], III. 3.5 for details.

In the proof of proposition 4.2, it is shown that $\partial h/\partial x$ is given by

$$\begin{aligned}\frac{\partial h(x,t)}{\partial x} &= \mathrm{E}\left(\int_t^T e^{-2\rho s}\phi_{t,s}\left(\int_t^s k'(\pi_t^x)W_\tau\,d\tau\right)ds\,\middle|\,\pi_t = x\right)\\ &\quad + \mathrm{E}\left(\phi_{t,T}\int_t^T k'(\pi_\tau^x)W_\tau\,d\tau\,\middle|\,\pi_t = x\right),\end{aligned} \tag{17}$$

where π^x is the process that solves (3) with the initial condition $\pi_t = x$. The function $k(\cdot)$ is given by

$$k(\pi) \equiv \frac{(\hat{\mu}(\pi) - r)^2}{\sigma_S^2} + r.$$

The process W is the stochastic flow of π and is defined by

$$W_t \equiv \frac{\partial}{\partial x}\pi_t^x, \tag{18}$$

satisfying

$$dW_t = -2\lambda W_t\,dt + (1 - 2\pi_t)\frac{\mu_1 - \mu_0}{\sigma_S}W_t\,d\overline{B}_t,$$

with $W_0 = 1$.

We now verify that h given by (16) satisfies the PDE (14).

Proposition 4.3. *The function h given by (16) satisfies the PDE (14).*

Proof. See Appendix. ▮

We can then prove proposition 4.1 by a relatively standard verification argument. We define a function $J : I\!R_+ \times [0,1] \times [0,T] \to I\!R$ by

$$J(w,\pi,t) = 2f(\pi,t)w^{1/2}, \tag{19}$$

where $f(\cdot\,,\cdot)$ is defined by (13) and (16).

Proof of Proposition 4.1. See Appendix.

We can observe some properties of f directly from (13) and (17). When $\mu_0 > r$, it turns out that f is a relatively simple function of π.

Proposition 4.4. *Suppose $\mu_0 > r$. For all $t \in [0,T]$, $f(\pi,t)$ is increasing with respect to π.*

Proof of Proposition 4.4. See Appendix.

The investor thus expects higher remaining utility if it is more likely that the market is in the high-expected-return regime ($Y_t = 1$). The optimal consumption rate is thus decreasing with respect to π. The investor with $\alpha = 1/2$ thus consumes less when it is more likely that the market is in the high-expected-return state ($Y_t = 1$). In other words, the investor with $\alpha = 1/2$ invests more in the assets to increase the future wealth level, when the market is more likely to be in the high-expected-return state. We will compare the optimal consumption rule with other values of α and consider this property again in the next section.

The optimal portfolio φ^* in (15) consists of two parts. The first term,

$$\frac{2(\hat{\mu}(\pi_t) - r)}{\sigma_S^2}, \tag{20}$$

is the usual demand for a risky asset by a single-period mean-variance maximizer. (See Merton [41], equation (36), for example.) This part depends linearly on the filtered probability π_t. The terminal horizon does not play a role.

The second component of φ^*,

$$\frac{2\pi_t(1-\pi_t)(\mu_1 - \mu_0)}{\sigma_S^2} \frac{f_\pi(\pi_t, t)}{f(\pi_t, t)}, \tag{21}$$

is the so-called "hedging demand," a term coined by Merton [42]. This is a demand for the asset as a vehicle to hedge against "unfavorable" shifts in the investment opportunity set. The hedging demand is always positive if $\mu_0 > r$, since f is increasing and all other terms are positive. Since f is increasing with respect to π, the marginal utility of wealth, V_w, is larger in the better state. Thus, the investor with risk-aversion coefficient $\alpha = 1/2$ "tries" to have more wealth in the better state. Ignoring the effect of the drift in (3), the filtered probability π rises when the asset price S rises. Therefore, by holding more of the risky asset, the investor will have more wealth when investment opportunities will be better. We can conjecture that risk-averse investors with $\alpha < 0$ may show the opposite behavior, because the log investor ($\alpha = 0$) does not have a hedging demand. This conjecture is partially confirmed by numerical examples in the following section.

The part of the hedging demand defined by

$$\frac{2\pi(1-\pi)(\mu_1 - \mu_0)}{\sigma_S^2} \tag{22}$$

is easy to analyze. This term is large if $\mu_1 - \mu_0$ is large. In other words, the hedging demand is large if there is a large difference in expected returns between the two regimes. The term (22) is also large if the price volatility σ_S is small. Since small σ_S means small noise in observation, the conditional probability π reacts to price changes more sensitively. In other words, the volatility of π is larger, against which the investor needs to hedge more. Furthermore, if π is near its boundaries and estimation risks are small, the hedging demand is small because of the term $\pi(1-\pi)$. Thus the hedge against changes in π may be large if (i) the difference in the two states is large (large $\mu_1 - \mu_0$), (ii) the noise in the price process is small and inference of the expected return is easy (small σ_S), and (iii) the investor is not confident about the current state (π is near 1/2).

The actual size of the hedging demand is the product of f_π/f and (22). Because of the boundary condition, $f(\pi, T) = 1$ for all π, for a short-horizon investor, f_π is almost zero for all π, which makes the hedging demand small. For long-horizon investors, f_π/f can achieve a notable size. Furthermore, the effect of parameters through the term (22) could be strengthened or weakened by f_π/f, since this term also depends on parameters. We will show this through the numerical example in the following section.

5 Optimal consumption and portfolio for general risk-aversion coefficients

In this section, we study the value function and the optimal consumption/portfolio rule for various values of risk-aversion coefficient α. Since an analytical solution is not available, we first study general properties of optimal rule and then compute the optimal solution numerically by Monte Carlo simulation.

It follows from the first-order condition for (7) that

$$u_c(c^*) = J_w(w, \pi, t), \tag{23}$$

$$\varphi^* = -\frac{J_w(w, \pi, t)}{wJ_{ww}(w, \pi, t)} \left(\frac{\hat{\mu}(\pi) - r}{\sigma_S^2} \right) - \frac{J_{w\pi}(w, \pi, t)}{wJ_{ww}(w, \pi, t)} \left(\frac{\sigma_\pi(\pi)}{\sigma_S} \right). \tag{24}$$

Following previous literature, such as Merton [42] and Breeden [6], we study properties of the optimal consumption/portfolio rule (23) and (24). As discussed above, the optimal portfolio rule (24) consists of two parts: the mean-variance portfolio and the hedging portfolio. How much an investor should invest in the mean-variance portfolio depends on the investor's relative risk tolerance $-J_w/wJ_{ww}$, which is constant $1/(1-\alpha)$ in our case. On the other hand, how much an investor should invest in the hedging portfolio depends on

$$-\frac{J_{w\pi}(w,\pi,t)}{wJ_{ww}(w,\pi,t)} = \frac{1}{1-\alpha}\frac{f_\pi(\pi,t)}{f(\pi,t)}, \tag{25}$$

which is not trivial. Applying implicit function theorem to the function for marginal utility of wealth J_w, it follows from (23) and Breeden [6] (equation (18), p.286) that

$$-\frac{J_{w\pi}}{wJ_{ww}} = \frac{1}{1-\alpha}\frac{f_\pi}{f} = \left.\frac{\partial w}{\partial \pi}\right|_{J_w} = \left.\frac{\partial w}{\partial \pi}\right|_{u}.$$

That is, (25) is the compensating variation in wealth for a change in π that is required to maintain the current level of marginal utility of wealth. Thus we can see that (25) will provide state-contingent wealth that combines with investment opportunity changes to maintain the utility of current consumption.

To indicate the implications, consider the effect on an individual of an increase in π. Assume that an increase in π has a real wealth effect that is positive, thereby tending to increase current consumption expenditure. However, the change has a negative effect on current consumption, in that the price of current consumption has increased relative to the price of future consumption. The net result on current consumption is ambiguous; thus, an individual's demand component may be either positive or negative, depending upon the sign of f_π/f. We can see from (9) and (10) that those who consume more with an increase in π would tend to be short in the hedging portfolio, and those who consume less would tend to be long in the hedging portfolio. We cannot however confirm who consumes more with an increase in π, because the value of f_π/f is not known.

Since an individual chooses consumption and asset portfolio to maximize his expected utility of lifetime consumption, we then consider the use of a risky asset in stabilizing an individual's expected utility of lifetime consumption. A perfect hedge, if possible, is a portfolio of assets whose return in the various states of the world is such that the individual's utility of lifetime consumption $J(w,\pi,t)$ is the same in all states of the world. Although a perfect hedge may not be possible in our case, an investor may try to have a hedging portfolio that realizes the compensating variations in wealth required to maintain expected lifetime utility. In order to see how expected lifetime utility J depends on the state variable π, we have to study function f. In the following, we compute f and f_π numerically.

Our control problem (6) is converted into a martingale formulation by standard arguments, since markets are complete from the investor's point of view. It follows from the first-order condition of the HJB equation (7) that the optimal portfolio rule can be computed by evaluating a function f in (8) and its first derivative f_π. As shown below, a function f can be easily estimated by Monte Carlo simulation. We could estimate f for many values of π and approximate f_π by taking differences between two grid points. However, to evaluate function f for many points is time consuming. Furthermore it may not be easy to find a good grid size that approximates f_π with sufficient accuracy. We can resolve these difficulties by using the stochastic flow technique. The value of f and f_π are estimated in one simulation procedure.

The risky asset price S satisfies

$$dS_t = \hat{\mu}_t S_t\, dt + \sigma_S S_t\, d\overline{B}_t,$$

and the estimator $\hat{\mu}$ satisfies

$$d\hat{\mu}_t = \lambda(\mu_1 + \mu_0 - 2\hat{\mu}_t)\, dt + \frac{1}{\sigma_S}(\hat{\mu}_t - \mu_0)(\mu_1 - \hat{\mu}_t)\, d\overline{B}_t.$$

Since markets are complete from the investor's point of view, the equivalent martingale measure is uniquely determined. We define processes γ and ξ by

$$\gamma_t = \frac{\hat{\mu}_t - r}{\sigma_S} \quad \text{and} \quad \xi_t = \exp\left(-\int_0^t \gamma_s\, d\overline{B}_s - \frac{1}{2}\int_0^t \gamma_s^2\, ds\right).$$

It is easy to show that ξ satisfies $d\xi_t = -\xi_t \gamma_t\, d\overline{B}_t$ and $\xi_0 = 1$. Problem (6) is converted into a martingale formulation by standard arguments:[10]

$$\begin{aligned} \sup \qquad & \mathrm{E}\left[\int_0^T e^{-\rho t} \tfrac{c_t^\alpha}{\alpha}\, dt + e^{-\rho T} \tfrac{w_T^\alpha}{\alpha}\right] \\ \text{s.t.} \quad w_0 = {} & \mathrm{E}\left[\int_0^T e^{-rt} \xi_t c_t\, dt + e^{-rT} \xi_T w_T\right]. \end{aligned} \tag{26}$$

The Lagrangean of problem (26) is given by

$$\mathrm{E}\left[\int_0^T e^{-\rho t} \frac{c_t^\alpha}{\alpha}\, dt + e^{-\rho T} \frac{w_T^\alpha}{\alpha}\right] + \eta\left[w_0 - \int_0^T e^{-rt} \xi_t c_t\, dt - e^{-rT} \xi_T w_T\right],$$

where $\eta > 0$ is a scalar Lagrange multiplier. The first-order conditions for optimality are given, state by state, by $(c_t^*)^{\alpha-1} = \eta e^{(\rho - r)t} \xi_t$ and $(w_T^*)^{\alpha-1} = \eta e^{(\rho-r)T} \xi_T$, where c^* and w_T^* are optimal consumption rate and terminal wealth. We define X by

$$X = \mathrm{E}\left[\int_0^T \exp\left(\frac{\rho - \alpha r}{\alpha - 1} t\right) \xi_t^{\frac{\alpha}{\alpha-1}}\, dt + \exp\left(\frac{\rho - \alpha r}{\alpha - 1} T\right) \xi_T^{\frac{\alpha}{\alpha-1}}\right]. \tag{27}$$

[10] See, for example, Duffie [20].

It is then easy to show that

$$\mathrm{E}\left[\int_0^T e^{-\rho t}\frac{(c_t^*)^\alpha}{\alpha}\,dt + e^{-\rho T}\frac{(w_T^*)^\alpha}{\alpha}\right] = \frac{w_0^\alpha}{\alpha}X^{1-\alpha}.$$

Since the left-hand side of the above equation is the maximized value of expected utility, $X^{1-\alpha}$ coincides with $f(\pi, 0)$ where f is defined in (8).

Let π^x be the process that solves (3) with the initial condition $\pi_0 = x$. The stochastic flow W of π is defined by (18). Since we can think of X as a function of the initial value x of π, $f_\pi(x, 0)$ is given by

$$f_\pi(x,0) = (1-\alpha)X^{-\alpha}\frac{\partial X}{\partial x},$$

where

$$\frac{\partial X}{\partial x} = \frac{\alpha}{\alpha-1}E\left[\int_0^T \exp\left(\frac{\rho-\alpha r}{\alpha-1}t\right)\xi_t^{\frac{1}{\alpha-1}}\frac{\partial \xi_t}{\partial x}\,dt + \exp\left(\frac{\rho-\alpha r}{\alpha-1}T\right)\xi_T^{\frac{1}{\alpha-1}}\frac{\partial \xi_T}{\partial x}\right],$$

$$\frac{\partial \xi_t}{\partial x} = \xi_t\left(-\int_0^t\left(\frac{\mu_1-\mu_0}{\sigma_S}\right)W_s\,d\overline{B}_t - \int_0^t \gamma_s\left(\frac{\mu_1-\mu_0}{\sigma_S}\right)W_s\,ds\right),$$

$$W_s = -2\lambda W_s\,dt + \left(\frac{\mu_1-\mu_0}{\sigma_S}\right)(1-2\pi_t)W_s\,d\overline{B}_t,$$

with $W_0 = 1$. We can thus estimate f and f_π by Monte Carlo simulation:

$$X \approx \frac{1}{N}\sum_{n=1}^N\left[\int_0^T \exp\left(\frac{\rho-\alpha r}{\alpha-1}t\right)(\xi_t^{(n)})^{\frac{\alpha}{\alpha-1}}\,dt + \exp\left(\frac{\rho-\alpha r}{\alpha-1}T\right)(\xi_T^{(n)})^{\frac{\alpha}{\alpha-1}}\right]$$

$$\frac{\partial X}{\partial x} \approx \frac{1}{N}\frac{\alpha}{\alpha-1}\sum_{n=1}^N\left[\int_0^T \exp\left(\frac{\rho-\alpha r}{\alpha-1}t\right)(\xi_t^{(n)})^{\frac{1}{\alpha-1}}\frac{\partial \xi_t^{(n)}}{\partial x}\,dt + \exp\left(\frac{\rho-\alpha r}{\alpha-1}T\right)(\xi_T^{(n)})^{\frac{1}{\alpha-1}}\frac{\partial \xi_T^{(n)}}{\partial x}\right],$$

where $\xi^{(n)}$ and $\partial\xi^{(n)}/\partial x$ are n-th sample path of ξ and $\partial\xi/\partial x$.

In Panel A and B of Table 1, we show $f(\,\cdot\,,\,\cdot\,)$ and $f_\pi(\cdot,\cdot)$ for investors with $\alpha = 0.5$, 0, -0.5, -5, -10, and -15 for the parameters:

$$\mu_1 = 0.20,\quad \mu_0 = 0.07,\quad \lambda = 0.25,\quad \sigma_S = 0.25,\quad r = 0.05,\quad \rho = 0.07,$$

$$w_0 = 100,\quad N = 20000,\quad T = 10.$$

These parameter values are used in the following examples unless otherwise stated. As shown in Proposition 4.4, the function f for an investor with $\alpha = 0.5$ is increasing with respect to π. The log-investor ($\alpha = 0$) is the knife-edge case and the function f does not depend on π and t. It is seen that the function f is increasing with respect to π for $\alpha = 0.5$ and decreasing for $\alpha = -0.5$ and -5. The investors with $\alpha = -0.5$ and -5 thus expect lower remaining utility if it is more likely that the market is in the high-expected-return regime ($Y_t = 1$). It is interesting that f is decreasing near $\pi = 0$ and increasing near $\pi = 1$ for $\alpha = -10$, and that f is increasing with respect to π for $\alpha = -15$. In this example, the investor with very low risk tolerance may expect higher remaining utility if the market is more likely in the high-expected-return regime.

It may be rather counter-intuitive that the slope of f is positive for investors with $\alpha = -15$. In Panel C of Table 1, we compute

$$\overline{X} = E\left[\int_0^T \exp(-rt)\xi_t \, dt + \exp(-rT)\xi_T\right],$$

which would be a good approximation of X in (27) for investors with very low risk tolerance. It turns out that $\overline{X}$ is increasing with respect to π for our parameter combination. As we can see easily, $\overline{X}$ is the price of a bond maturing at T whose face value is one unit of account and dividend is one unit of consumption for $t \in [0, T]$. In other words, $\overline{X}$ is the price of the stable future consumption stream. This bond is expensive in our example when the market is more likely in a higher-expected-return regime. Thus it may not be curious that investors with very low risk tolerance have lower expected lifetime utility for higher π.

Table 2A shows the optimal consumption rate for investors with $\alpha = 0.5$, 0, -0.5, -5, -10, and -15. As shown in section 4, the optimal consumption rate for the investor with $\alpha = 0.5$ is decreasing with respect to π. On the other hand, the optimal consumption rate for the investor with $\alpha = -0.5$ and -5 is increasing with respect to π. Since f is increasing with respect to π for $\alpha = -15$, the optimal consumption rate is decreasing with respect to π. For each value of π, the optimal consumption rate of the investor with $\alpha = -0.5$ is highest in our example. The optimal consumption rate decreases as the investor becomes more risk averse or more risk tolerant than the investor with $\alpha = -0.5$.

As is well known, the log-investor maximizes the growth rate of his wealth, and the utility of the investor with $\alpha > 0$ increases with respect to the variance of the log of return. The investor with $\alpha < 0$ is risk averse in a dynamic context in the sense that the utility decreases with respect to the variance of the logarithm of return. (See for example Luenberger [37], p.427.) When the market is more likely in the high-expected-return regime, investors with $\alpha > 0$, who are very aggressive in the above sense, increase investment in the risky asset and try to increase their future consumption. Investors with

$\alpha = -0.5$ and -5, on the other hand, increase their current consumption rate when the expected return of the risky asset is higher. They appreciate a gain in their purchasing power, and the income effect of higher expected return dominates the substitution effect. In our example, this tendency is reversed for more risk-averse investors. Given that $\overline{X}$ is increasing with respect to π, the price of stable future consumption stream is expensive when the market is more likely in the high-expected-return regime. Since stable future consumption is strongly preferred by investors with very low risk tolerance, the current consumption is decreasing with respect to π.

The level of consumption rate is also informative. The optimal consumption rate for investors with $\alpha = 0.5$ is low and very sensitive to π, because these investors are eager to increase their future wealth level. Investors with $\alpha = -0.5$ have a higher consumption rate, which is not very sensitive to π. More risk-averse investors, such as $\alpha = -10$ and -15, try to maintain a similar consumption level in the various states of the world, and the optimal consumption rate is not sensitive to π.

Table 2C shows the hedging portfolio. The main determinant of the hedging demand is estimation risk, which is large when the investor is not confident about the current regime. On the other hand, as π approaches the boundaries, the hedging demand is small because of the term $\pi(1-\pi)$ in the hedging portfolio.

Since f is increasing with respect to π for $\alpha = 0.5$, the hedging portfolio of the investor with $\alpha = 0.5$ is positive. On the other hand, the hedging portfolio of the investor with $\alpha = -0.5$ and -5 is negative. Ignoring the effect of the drift in (3), the filtered probability π is positively correlated to the asset price S. Thus, by short selling the risky asset, the investor can transfer the wealth from the higher $\hat{\mu}$ state to the lower $\hat{\mu}$ state. On the other hand, if investors long the risky asset, they can transfer their wealth from the lower $\hat{\mu}$ state to the higher $\hat{\mu}$ state. In any case, investors stabilize their lifetime utility using the hedging portfolio.

It is interesting to see that the hedging portfolios of investors with very low risk tolerance and $\alpha = -15$ is positive. They transfer their wealth from the lower $\hat{\mu}$ state to the higher $\hat{\mu}$ state. Since the price $\overline{X}$ of stable future consumption and income is expensive in the higher $\hat{\mu}$ state, they may want to have more wealth to stabilize the future consumption stream.

Although the size of the hedging portfolio is smaller for more risk-averse investors, Table 2D shows that the ratio of the hedging portfolio to the optimal portfolio is larger for them. For example, when $\alpha = -15$ and $\pi = 0.4$, 16.1% of risky-asset-holding is motivated for hedging reasons. As in other literature, such as Lynch [38] and Xia [49], the hedging demand could be important in our example.

Table 3 compares the optimal consumption and portfolio of investors with $\alpha = -0.5$ who have different time horizons $T = 1$, 5, 10, 30, and 50. In general, the hedging portfolio is larger for longer-horizon investors, because they pay more attention to changes in investment opportunity sets. For example, the hedging portfolio is -0.012 for $T = 1$ but -0.035 for $T = 50$, when $\pi = 0.5$. We can thus see that the hedging portfolio is more important for the longer-time-horizon investors.

Table 4 shows the optimal consumption and portfolio for price volatility (σ_S) parameters of 0.15, 0.25, or 0.35, when the risk aversion is $\alpha = -0.5$. The other parameters are as above. The parameter σ_S affects the optimal consumption and portfolio rule through two channels. First of all, the risky asset is more attractive when σ_S is smaller, for given parameters μ_0 and μ_1. The optimal consumption rate increases as σ_S decreases, since the same wealth level in the future could be realized by a smaller quantity of investment in the risky asset. When $\pi = 0.5$, for example, the optimal consumption rate is 12.995 if $\sigma_S = 0.35$ and 15.093 if $\sigma_S = 0.15$. Secondly, a small σ_S makes estimation of $\mu(Y_t)$ easier, because the volatility of observations is smaller. When σ_S is small, the investor then can actively hedge against changes in the investment opportunity set, and the hedging portfolio is larger. Not only the size of hedging demands, but also the ratio of the hedging portfolio to the optimal portfolio, increases as σ_S becomes smaller. When $\pi = 0.5$, for example, the ratio of the hedging portfolio to the optimal portfolio is -0.018 if $\sigma_S = 0.35$ and -0.1 if $\sigma_S = 0.15$. We can thus see that smaller estimation risk increases the importance of the hedging portfolio in our model.

Table 5 compares the optimal consumption and portfolio for different low-expected-return parameters (μ_0) of 0.07, 0, or -0.05. The risk-aversion coefficient is $\alpha = -0.5$. The other parameters are as for the base case. In general, the optimal consumption rate decreases as μ_0 decreases since the risky asset becomes less attractive. Since the difference $\mu_1 - \mu_0$ is one of a component in the hedging portfolio (21), the size of the hedging portfolio is large if $|\mu_1 - \mu_0|$ is large.

When the expected return of the risky asset is negative and $\mu_0 = -0.05$, the optimal consumption and the hedging portfolio depend on π in a complicated way. The optimal consumption rate is high when the market is more likely in a high-expected-return regime as in the case of positive μ_0. The optimal consumption rate is decreasing with respect to π near $\pi = 0$ and increasing near $\pi = 1$. As opposed to the case where μ_0 is positive, the hedging portfolio is positive for small π and is negative for large π. The investor with a longer time horizon can invest more in the risky asset than the shorter-time-horizon investor.

Table 6 considers the case where jump intensities are $\lambda = 0.05$, 0.25, 0.5, and 5. The risk-aversion coefficient is $\alpha = -0.5$. We can see that the consumption rate is more sensitive to π and the size of hedging demand is larger when regime switches are infrequent. When jump intensities are large, the value of π_t is less important because the filtered probability π_t is likely to take values near $1/2$. On the other hand, when regime switches do not happen very frequently, information about a current regime is more important and hedging against a current regime is large. When $\pi = 0.5$, for example, the ratio of the hedging portfolio to the optimal portfolio is -0.083 if $\lambda = 0.05$ and -0.002 if $\lambda = 5$.

In summary, our examples show that the optimal consumption rate and the hedging portfolio have the following properties:

- When $\alpha = 0.5$, investors are aggressive. They consume less and invest more in assets if the market is more likely in the high-expected-return regime. The hedging portfolio is positive so that they can transfer their wealth from a lower $\hat{\mu}$ state to a higher $\hat{\mu}$ state.

- When $\alpha = -0.5$ and -5, investors consume more and invest less in assets if the market is more likely in the high-expected-return regime. The income effect of higher expected returns dominates the substitution effect. The hedging portfolio is negative so that they can maintain similar wealth levels among different $\hat{\mu}$ states.

- Investors who are less risk tolerant ($\alpha = -15$) consume less and invest more in assets if the market is more likely in the high-expected-return regime. The hedging portfolio is positive so that they can transfer their wealth from a lower $\hat{\mu}$ state to a higher $\hat{\mu}$ state. In our numerical example, the price of the security that delivers a stable future consumption stream is higher if the market is more likely in the high-expected-return regime. Investors with $\alpha = -15$ want to have more wealth in a higher $\hat{\mu}$ state so that they can spend more money to stabilize the future consumption stream.

- The hedging portfolio is more important when the time horizon is longer. In general, the longer the time horizon, the larger the size of the hedging portfolio.

- Smaller noise in the risky asset price makes estimation easier and the investment in the risky asset more attractive. Both the hedging portfolio and the consumption rate are more sensitive to π when σ_S is smaller.

- The difference $|\mu_1 - \mu_0|$ between the expected returns in the two states makes the hedging portfolio large. When the expected return μ_0 in a worse state is lower than the short rate, both the optimal consumption rate and the hedging portfolio depend on the estimation π in a very complicated way.

- If switching between the two regimes is not frequent, the regime switching is more important from the investor's point of view. The hedging portfolio is larger, and the optimal consumption rate is more sensitive to π.

In our example, the recommendation that younger people invest more in stocks could be right or wrong depending upon the parameters. There is no simple rule between the hedging portfolio and the time horizon of investors. For example, the hedging portfolio can be positive or negative depending upon the risk-aversion coefficient α for our example parameter set. When the drift of the risky asset could be smaller than the riskless rate, the hedging portfolio takes both positive and negative value depending upon the value of π. The time horizon effect is so complicated that we cannot simply recommend younger people to invest more in the risky asset as a rule of thumb. The complexity of the terminal horizon effects is consistent with findings in other recent papers, such as Lynch [38] and Xia [49].

6 Conclusion

In this paper, we study dynamic optimal consumption and portfolio choice for a setting in which the mean returns of a risky asset depend on an unobservable regime variable of the economy. The underlying optimization problem was solved both analytically and numerically with the stochastic flow. The optimal portfolio of a long-time-horizon investor can be substantially different from the optimal portfolio of a short-time-horizon investor. The difference is caused by an investor's hedging demand of assets against fluctuations in the estimated mean returns. The optimal consumption rate of a long-time-horizon investor is also sensitive to the estimated mean return. The optimal rule depends on the estimated mean return in a very complicated way. An investor's degree of risk aversion and time horizon, parameters of asset returns, and frequency of regime switching are key factors in his or her decision.

7 Appendix: Proof of Propositions

Proof of Proposition 4.1. Let

$$\begin{aligned}
\mathcal{D}^{(c,\varphi)}J(w,\pi,t) &= fw^{-1/2}(w\varphi(\hat{\mu}(\pi)-r)+rw-c)-\frac{1}{4}fw^{-3/2}(w\varphi\sigma_S)^2 \\
&\quad + 2w^{1/2}f_\pi\lambda(1-2\pi)+w^{1/2}f_{\pi\pi}\pi^2(1-\pi)^2\left(\frac{\mu_1-\mu_0}{\sigma_S}\right)^2 \\
&\quad + f_\pi w^{1/2}\pi(1-\pi)(\mu_1-\mu_0)\varphi+2f_t w^{1/2}.
\end{aligned}$$

From the first-order conditions, the maximum of $\mathcal{D}^{(c,\varphi)}J(w,\pi,t)+e^{-\rho t}u(c)$, as a function of (c,φ), is attained at

$$c^* = e^{-2\rho t}f(\pi,t)^{-2}w \quad \text{and} \quad \varphi^* = \frac{2(\hat{\mu}(\pi)-r)}{\sigma_S^2}+\frac{2\pi(1-\pi)(\mu_1-\mu_0)}{\sigma_S^2}\,\frac{f_\pi(\pi,t)}{f(\pi,t)}.$$

Since we know that f solves the PDE (11),

$$\mathcal{D}^{(c^*,\varphi^*)}J(w,\pi,t)+e^{-\rho t}u(c^*)=0.$$

Thus, for any (c,φ),

$$\mathcal{D}^{(c,\varphi)}J(w,\pi,t)+e^{-\rho t}u(c)\leq 0. \tag{28}$$

We now consider an arbitrarily chosen $(c,\varphi)\in\Lambda(w_0)$. Because f is sufficiently differentiable, it follows from Ito's Lemma that

$$\begin{aligned}
J(w_T,\pi_T,T) &= J(w_0,\pi_0,0)+\int_0^T \mathcal{D}^{(c_t,\varphi_t)}J(w_t,\pi_t,t)\,dt+\int_0^T f(\pi_t,t)w_t^{1/2}\varphi_t\sigma_S\,d\overline{B}_t \\
&\quad +\int_0^T 2f_\pi(\pi_t,t)w_t^{1/2}\pi_t(1-\pi_t)\frac{\mu_1-\mu_0}{\sigma_S}\,d\overline{B}_t.
\end{aligned} \tag{29}$$

It follows from the HJB equation that

$$\begin{aligned}
J(w_T,\pi_T,T)+\int_0^T e^{-\rho t}u(c_t)\,dt &\leq J(w_0,\pi_0,0)+\int_0^T f(\pi_t,t)w_t^{1/2}\varphi_t\sigma_S\,d\overline{B}_t \\
&\quad +\int_0^T f_\pi(\pi_t,t)2w_t^{1/2}\pi_t(1-\pi_t)\frac{\mu_1-\mu_0}{\sigma_S}\,d\overline{B}_t.
\end{aligned} \tag{30}$$

Since the left-hand side of (30) is non-negative, the right-hand side is non-negative.

It is then easy to show that

$$f(\pi_t,t)w_t^{1/2}\varphi_t\sigma_S \quad \text{and} \quad f_\pi(\pi_t,t)(2w_t^{1/2})\pi_t(1-\pi_t)((\mu_1-\mu_0)/\sigma_S)$$

are in $\mathcal{L}^2$, and both

$$\int_0^T f(\pi_t,t)w_t^{1/2}\varphi_t\sigma_S\,d\overline{B}_t \quad \text{and} \quad \int_0^T f_\pi(\pi_t,t)(2w_t^{1/2})\pi_t(1-\pi_t)((\mu_1-\mu_0)/\sigma_S)\,d\overline{B}_t$$

are local martingales. Since a non-negative local martingale is a supermartingale, the right-hand side of (30) is supermartingale.

We can now take the expectation of each side of (30) and obtain

$$\mathrm{E}\left(u(w_T)+\int_0^T e^{-\rho t}u(c_t)\,dt\right)\le J(w_0,\pi_0,0),$$

where the boundary condition of J is used on the left-hand side and the supermartingale property is used on the right-hand side.

It remains for us to show that, for the candidate optimal control (c^*,φ^*), we have $J(w_0,\pi_0,0)=\mathrm{E}[u(w_T)+\int_0^T e^{-\rho t}u(c_t)\,dt]$. It follows from the HJB equation that the inequality in (30) is replaced with equality if the candidate optimal control (c^*,φ^*) is used. That is,

$$\begin{aligned} J(w_T^*,\pi_T,T)+\int_0^T e^{-\rho t}u(c_t^*)\,dt &= J(w_0,\pi_0,0)+\int_0^T f(\pi_t,t)(w_t^*)^{1/2}\varphi_t^*\sigma_S\,d\overline{B}_t \\ &\quad +\int_0^T 2f_\pi(\pi_t,t)(w_t^*)^{1/2}\pi_t(1-\pi_t)\frac{\mu_1-\mu_0}{\sigma_S}\,d\overline{B}_t. \quad (31)\end{aligned}$$

We want to show that the last two terms on the right-hand side are martingales. In order to show this, it is enough to show that $E\left[\int_0^T w_t^*\,dt\right]<\infty$. By direct computation,

$$\begin{aligned} dw_t^* &= \left[\frac{2(\hat\mu(\pi_t)-r)^2}{\sigma_S^2}+\frac{2\pi(1-\pi)(\mu_1-\mu_0)}{\sigma_S^2}\frac{f_\pi(\pi_t,t)}{f(\pi_t,t)}(\hat\mu(\pi_t)-r)\right]w_t^*\,dt \\ &\quad +[r-e^{\rho t/(\alpha-1)}f(\pi_t,t)^{1/(\alpha-1)}]w_t^*\,dt+\left[\frac{2(\hat\mu(\pi_t)-r)^2}{\sigma_S}+\frac{2\pi(1-\pi)(\mu_1-\mu_0)}{\sigma_S}\right]w_t^*\,d\overline{B}_t.\end{aligned}$$

By Chapter 5, Lemma 2 of Protter [46], we can see $\|\sup_t|w_t^*|\|_{L^p}<\infty$ for $1\le p<\infty$, which implies that $\mathrm{E}\left[\int_0^T w_t^*\,dt\right]<\infty$. Since f is bounded, we can see that

$$\mathrm{E}\left[\int_0^T\left(f(\pi_t,t)(w_t^*)^{1/2}\varphi_t^*\sigma_S\right)^2\,dt\right]<\infty.$$

Similarly, we can see that

$$\mathrm{E}\left[\int_0^T\left(f_\pi(\pi_t,t)\left(2(w_t^*)^{1/2}\right)\pi_t(1-\pi_t)\left(\frac{\mu_1-\mu_0}{\sigma_S}\right)\right)^2\,dt\right]<\infty.$$

Then, by Proposition 5B of Duffie [20], the last two terms of the right-hand side of (31) are martingales. Therefore, taking expectations on both sides of (31) leaves $J(w_0, \pi_0, 0) = \mathrm{E}[u(w_T) + \int_0^T e^{-\rho t} u(c_t)\, dt]$, verifying that a candidate optimal policy is actually given by (c^*, φ^*). Uniqueness follows from the strict concavity of u. ∎

Proof of Proposition 4.2. Let π^x be the process that solves (3) with the initial condition $\pi_0 = x$. It is easy to see that the diffusion and drift coefficients of (3) are globally Lipschitz and have linear growth. That is, the conditions (h1) and (h2) of Nualart [44], (p.99, Section 2.2) are satisfied. It follows from Corollary 2.2.1 and Theorem 2.2.1 of Nualart [44] or Theorem 39 in Chapter 5 of Protter [46] that the random variable $\pi_t^x(\omega)$ is differentiable with respect to x, ω by ω, and that the process W defined by

$$W_t \equiv \frac{\partial}{\partial x} \pi_t^x$$

satisfies

$$dW_t = -2\lambda W_t\, dt + (1 - 2\pi_t) \frac{\mu_1 - \mu_0}{\sigma_S} W_t\, d\overline{B}_t,$$

with $W_0 = 1$. Furthermore, by Lemma 2.2.2 or Theorem 2.2.2 of Nualart [44] or Theorem 40 in Chapter 5 of Protter [46], the process Z defined by

$$Z_t \equiv \frac{\partial}{\partial x} W_t$$

satisfies

$$dZ_t = -2\lambda Z_t\, dt + \frac{\mu_1 - \mu_0}{\sigma_S} (-2W_t^2 + (1 - 2\pi_t) Z_t)\, d\overline{B}_t,$$

with $Z_0 = 0$. Since the coefficients of the processes W and Z are random Lipschitz, all conditions in Lemma 2 in Chapter 5 of Protter [46] are satisfied. Thus we can find constants C_W and C_Z such that, for $1 \leq p < \infty$,

$$\| \sup_t |W_t| \|_{L^p} \leq C_W \quad \text{and} \quad \| \sup_t |Z_t| \|_{L^p} \leq C_Z.$$

(See Protter [46], Chapter 5, Section 2.)

The first derivative of h with respect to x, if it exists, is given by

$$\frac{\partial h(x,t)}{\partial x} = \frac{\partial}{\partial x} \mathrm{E} \left(\int_t^T e^{-2\rho s} \phi_{t,s}\, ds + \phi_{t,T} \,\middle|\, \pi_t \right).$$

If we can exchange the order of differentiation and integration, then the derivative of h is given by

$$\frac{\partial h(x,t)}{\partial x} = \mathrm{E} \left(\int_t^T e^{-2\rho s} \frac{\partial}{\partial x} \phi_{t,s}\, ds + \frac{\partial}{\partial x} \phi_{t,T} \,\middle|\, \pi_t \right). \tag{32}$$

By the Dominated Convergence Theorem, the exchange is possible if there are measurable functions $A : \Omega \times [0,T] \to I\!R$ and $C : \Omega \to I\!R$ such that

$$\left| e^{-2\rho s} \frac{\partial}{\partial x} \phi_{t,s} \right| < |A_{t,s}|, \qquad \left| \frac{\partial}{\partial x} \phi_{t,T} \right| \leq |C_{t,T}|,$$

$$\mathrm{E}\left[\int_t^T |A_{t,s}| \, ds \,\middle|\, \pi_t \right] < \infty, \text{ and } \mathrm{E}\left[\, |C_{t,T}| \,|\, \pi_t \right] < \infty.$$

(See Billingsley [3], Theorem 16.8, for example.)

We define for convenience

$$k(\pi) \equiv \frac{(\hat{\mu}(\pi) - r)^2}{\sigma_S^2} + r.$$

Because

$$\left| \frac{\partial}{\partial x} \left(\int_t^T k(\pi_\tau^x) \, d\tau \right) \right| = \left| \int_t^s k'(\pi_\tau^x) W_\tau \, d\tau \right| \leq \frac{2(\mu_1 - \mu_0)}{\sigma_S^2} \max\{|\mu_1 - r|, |\mu_0 - r|\} \int_t^s W_\tau \, d\tau$$

and

$$|\phi_{t,s}| \leq \left| \exp\left(\int_t^s \frac{(\max\{|\mu_1 - r|, |\mu_0 - r|\})^2}{\sigma_S^2} + r \, d\tau \right) \right|,$$

we can find constants K_0 and K_1 such that

$$\left| \frac{\partial}{\partial x} \phi_{t,s} \right| \leq K_0 \exp(K_1(s-t)) \left(\int_t^s k'(\pi_\tau^x) W_\tau \, d\tau \right).$$

Thus, by the Cauchy-Schwartz inequality,

$$\begin{aligned} \left| \mathrm{E}\left(\int_t^T \frac{\partial}{\partial x} \phi_{t,s} \, ds \,\middle|\, \pi_t \right) \right| \quad \leq \quad & K_0 \left[\mathrm{E}\left(\int_t^T \exp(2K_1(s-t)) \, ds \,\middle|\, \pi_t \right) \right]^{1/2} \\ & \left[\mathrm{E}\left(\int_t^T \left(\int_t^s W_\tau \, d\tau \right)^2 ds \,\middle|\, \pi_t \right) \right]^{1/2} . \end{aligned} \tag{33}$$

The right-hand side is finite because $\| \sup_t |W_t| \|_{L^p}$ is finite. By similar arguments, we can also find an integrable function that dominates $(\partial/\partial x)\phi_{t,T}$. Therefore h is differentiable with respect to x and

$$\begin{aligned} \frac{\partial h(x,t)}{\partial x} \quad = \quad & \mathrm{E}\left(\int_t^T e^{-2\rho s} \phi_{t,s} \left(\int_t^s k'(\pi_t^x) W_\tau \, d\tau \right) ds \,\middle|\, \pi_t \right) \\ & + \mathrm{E}\left(\phi_{t,T} \int_t^T k'(\pi_\tau^x) W_\tau \, d\tau \,\middle|\, \pi_t \right). \end{aligned}$$

For the second derivative, we remark that

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\phi_{t,s} &= \left(\frac{\partial}{\partial x}\phi_{t,s}\right)\left(\int_t^s k'(\pi_\tau^x)W_\tau\,d\tau\right) + \phi_{t,s}\left(\frac{\partial}{\partial x}\left(\int_t^s k'(\pi_\tau^x)W_\tau\,d\tau\right)\right) \\
&\le \left(\frac{\partial}{\partial x}\phi_{t,s}\right)\left(\int_t^s \frac{2(\mu_1-\mu_0)}{\sigma_S^2}\max\{|\mu_1-r|,|\mu_0-r|\}W_\tau\,d\tau\right) \\
&\quad + \phi_{t,s}\left(\int_t^s \frac{2(\mu_1-\mu_0)}{\sigma_S^2}W_\tau\,d\tau + \int_t^s \frac{2(\mu_1-\mu_0)}{\sigma_S^2}\max\{|\mu_1-r|,|\mu_0-r|\}|Z_\tau|\,d\tau\right).
\end{aligned}$$

By similar arguments and because $\|\sup_t|W_t|\|_{L^p}$ and $\|\sup_t|Z_t|\|_{L^p}$ are finite, the expectation of the right-hand side is finite. Thus

$$\frac{\partial^2}{\partial x^2}\phi_{t,s}$$

is dominated by an integrable function, and the second derivative of $h(x,t)$ with respect to x is given by

$$\frac{\partial^2 h(x,t)}{\partial x^2} = \mathrm{E}\left(\int_t^T e^{-2\rho s}\frac{\partial^2}{\partial x^2}\phi_{t,s}\,ds + \frac{\partial^2}{\partial x^2}\phi_{t,T}\,\middle|\,\pi_t\right),$$

where

$$\frac{\partial^2}{\partial x^2}\phi_{t,s} = \left(\frac{\partial}{\partial x}\phi_{t,s}\right)\left(\int_t^s k'(\pi_\tau^x)W_\tau\,d\tau\right) + \phi_{t,s}\left(\int_t^s k''(\pi_\tau^x)\,d\tau + \int_t^s k'(\pi_\tau^x)Z_\tau\,d\tau\right).$$

It follows from direct computation that the partial derivative with respect to t exists and is given by

$$\frac{\partial h(\pi,t)}{\partial t} = e^{-2\rho t} + \left(\frac{(\hat\mu(\pi_t)-r)^2}{\sigma_S^2} + r\right)\mathrm{E}\left[e^{-2\rho s}\phi_{t,s}\,ds + \phi_{t,T}\,\middle|\,\pi_t\right].$$

For the boundedness, notice that

$$\phi_{t,s} \le \exp\left(\int_t^s \left(\frac{(\mu_1-r)^2}{\sigma_S^2} + r\right)d\tau\right)$$

and that

$$\frac{\partial}{\partial x}\phi_{t,s} \le \phi_{t,s}\int_t^s \frac{2(\mu_1-r)(\mu_1-\mu_0)}{\sigma_S^2}W_\tau\,d\tau.$$

Thus h and $\partial h/\partial x$ are bounded for $\mathrm{E}[|W_t|] < \infty$ by Theorem 2.2.1 of Nualart [44]. Since $\partial f/\partial x = (1/2)h^{-1/2}(\partial h/\partial x)$, we have that $\partial f/\partial x$ is bounded. ∎

Proof of Proposition 4.3. Let H be the Ito process defined by $H_s = h(\pi_t, t)$, $s < t$, and

$$H_s = h(\pi_s, s)\phi_{t,s}, \qquad s \geq t.$$

By Ito's Lemma,

$$\begin{aligned} H_T &= h(\pi_t, t) + \int_t^T \phi_{t,s} \left[\mathcal{D}h(\pi_s, s) + \left(\frac{(\hat{\mu}(\pi_s) - r)^2}{\sigma_S^2} + r \right) h(\pi_s, s) \right] ds \\ &\quad + \int_t^T \phi_{t,s} h_\pi(\pi_s, s) \frac{\mu_1 - \mu_0}{\sigma_S} \pi_s (1 - \pi_s)\, d\overline{B}_s, \end{aligned} \tag{34}$$

where

$$\begin{aligned} \mathcal{D}h(\pi_s, s) &= h_\pi(\pi_s, s) \left(\lambda(1 - 2\pi_s) + \frac{\pi(1-\pi)(\mu_1 - \mu_0)}{\sigma_S^2} (\hat{\mu}(\pi_s) - r) \right) \\ &\quad + \frac{1}{2} h_{\pi\pi}(\pi_s, s) \pi_s^2 (1 - \pi_s)^2 \left(\frac{\mu_1 - \mu_0}{\sigma_S} \right)^2 + h_s(\pi_s, s). \end{aligned}$$

If follows from Proposition 4.2 that both $\phi_{t,s}$ and h_π are bounded. Thus the last term on the right-hand side of (34) is martingale. (See, for example, Duffie [20], Section 5B.) By taking expectations through each side,

$$h(\pi_t, t) = \mathrm{E} \left[\int_t^T \phi_{t,s} e^{-2\rho t}\, ds + \phi_{t,T} \,\middle|\, \pi_t \right].$$

This completes the proof. ∎

Proof of Proposition 4.4. Note that h_π is given by (17). Since $\mu_0 > r$, $\phi_{t,s}$ and $k'(\pi_s)$ are positive. We know that $W \equiv \frac{\partial}{\partial x} \pi_t^x$ is a positive process because $W_0 = 1$. Thus $h_\pi > 0$ for all π, which implies that f is increasing in π. ∎

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Table 1: Utility Multiplier and its First Derivative for Various Risk-Aversion Coefficients

This table reports utility multiplier $f(\pi, t)$ and its first derivative $f_\pi(\pi, t)$ for different values of investor's risk-aversion coefficient $\alpha = 0.5$, 0, -0.5, -5, -10, and -15. Other parameters are given by $\mu_1 = 0.20$, $\mu_0 = 0.07$, $\lambda = 0.25$, $\sigma_S = 0.25$, $r = 0.05$, $\rho = 0.07$, $w_0 = 100$, $N = 20000$, and $T = 10$. We also report $\overline{X}$, which may be a good approximation of X in (27) for investors with very low risk tolerance.

Panel A: Utility multiplier $f(\pi, t)$

π	$\alpha = 0.5$	$\alpha = 0$	$\alpha = -0.5$	$\alpha = -5$	$\alpha = -10$	$\alpha = -15$
0	3.247E+00	7.699E+00	2.117E+01	2.910E+05	1.319E+10	7.122E+14
0.1	3.313E+00	7.699E+00	2.107E+01	2.875E+05	1.314E+10	7.286E+14
0.2	3.389E+00	7.699E+00	2.094E+01	2.835E+05	1.308E+10	7.499E+14
0.3	3.476E+00	7.699E+00	2.079E+01	2.792E+05	1.303E+10	7.765E+14
0.4	3.572E+00	7.699E+00	2.063E+01	2.746E+05	1.298E+10	8.089E+14
0.5	3.708E+00	7.699E+00	2.040E+01	2.684E+05	1.293E+10	8.577E+14
0.6	3.828E+00	7.699E+00	2.021E+01	2.632E+05	1.290E+10	9.034E+14
0.7	3.960E+00	7.699E+00	2.000E+01	2.578E+05	1.288E+10	9.548E+14
0.8	4.104E+00	7.699E+00	1.977E+01	2.522E+05	1.289E+10	1.012E+15
0.9	4.263E+00	7.699E+00	1.953E+01	2.464E+05	1.292E+10	1.074E+15
1	4.440E+00	7.699E+00	1.928E+01	2.405E+05	1.300E+10	1.143E+15

Panel B: First derivative $f_\pi(\pi, t)$

π	$\alpha = 0.5$	$\alpha = 0$	$\alpha = -0.5$	$\alpha = -5$	$\alpha = -10$	$\alpha = -15$
0	6.132E-01	0.000E+00	-1.033E+00	-3.398E+04	-1.119E+09	1.396E+14
0.1	7.200E-01	0.000E+00	-1.221E+00	-3.837E+04	-1.143E+09	1.846E+14
0.2	8.233E-01	0.000E+00	-1.397E+00	-4.219E+04	-1.109E+09	2.370E+14
0.3	9.245E-01	0.000E+00	-1.564E+00	-4.553E+04	-1.030E+09	2.946E+14
0.4	1.025E+00	0.000E+00	-1.722E+00	-4.843E+04	-9.113E+08	3.554E+14
0.5	1.151E+00	0.000E+00	-1.908E+00	-5.153E+04	-7.066E+08	4.335E+14
0.6	1.257E+00	0.000E+00	-2.049E+00	-5.363E+04	-4.823E+08	4.957E+14
0.7	1.369E+00	0.000E+00	-2.181E+00	-5.542E+04	-1.742E+08	5.564E+14
0.8	1.495E+00	0.000E+00	-2.306E+00	-5.689E+04	2.740E+08	6.159E+14
0.9	1.645E+00	0.000E+00	-2.423E+00	-5.796E+04	9.865E+08	6.787E+14
1	1.839E+00	0.000E+00	-2.534E+00	-5.838E+04	2.338E+09	7.623E+14

Panel C: $\overline{X} = \mathrm{E}\left[\int_0^T \exp(-rt)\xi_t\,dt + \exp(-rT)\xi_T\right]$

π	
0	8.4872
0.1	8.4883
0.2	8.4894
0.3	8.4907
0.4	8.4915
0.5	8.4928
0.6	8.4937
0.7	8.4946
0.8	8.4954
0.9	8.4960
1	8.4967

Table 2: Optimal Consumption and Portfolio Rule for Various Risk-Aversion Coefficients

This table reports, for different values of investor's risk-aversion coefficient $\alpha = 0.5$, 0, -0.5, -5, -10, and -15, the optimal consumption rate $c_t^* = e^{\rho t/(\alpha-1)} f(\pi_t, t)^{1/(\alpha-1)} w_t$, the mean-variance portfolio

$$\varphi_{mvp}^* \equiv \frac{\hat{\mu}(\pi_t) - r}{(1-\alpha)\sigma_S^2},$$

the hedging portfolio

$$\varphi_{hp}^* \equiv \frac{\pi(1-\pi)(\mu_1 - \mu_0)}{(1-\alpha)\sigma_S^2} \frac{f_\pi(\pi_t, t)}{f(\pi_t, t)},$$

and the ratio of the hedging portfolio to the optimal portfolio, that is, $\varphi_{hp}^*/(\varphi_{mvp}^* + \varphi_{hp}^*)$. Other parameters are given by $\mu_1 = 0.20$, $\mu_0 = 0.07$, $\lambda = 0.25$, $\sigma_S = 0.25$, $r = 0.05$, $\rho = 0.07$, $w_0 = 100$, $N = 20000$, and $T = 10$.

Panel A: Optimal consumption rate

π	$\alpha = 0.5$	$\alpha = 0$	$\alpha = -0.5$	$\alpha = -5$	$\alpha = -10$	$\alpha = -15$
0	9.487	12.989	13.066	12.284	12.022	11.795
0.1	9.113	12.989	13.111	12.309	12.027	11.779
0.2	8.707	12.989	13.164	12.338	12.031	11.757
0.3	8.278	12.989	13.225	12.369	12.036	11.732
0.4	7.836	12.989	13.294	12.404	12.040	11.702
0.5	7.273	12.989	13.392	12.451	12.044	11.659
0.6	6.823	12.989	13.479	12.491	12.046	11.621
0.7	6.377	12.989	13.574	12.535	12.048	11.581
0.8	5.936	12.989	13.677	12.581	12.047	11.539
0.9	5.502	12.989	13.788	12.630	12.045	11.496
1	5.073	12.989	13.907	12.681	12.038	11.452

Panel B: Mean-variance portfolio

π	$\alpha = 0.5$	$\alpha = 0$	$\alpha = -0.5$	$\alpha = -5$	$\alpha = -10$	$\alpha = -15$
0	0.640	0.320	0.213	0.053	0.029	0.020
0.1	1.046	0.523	0.349	0.087	0.048	0.033
0.2	1.452	0.726	0.484	0.121	0.066	0.045
0.3	1.858	0.929	0.619	0.155	0.084	0.058
0.4	2.263	1.132	0.754	0.189	0.103	0.071
0.5	2.771	1.385	0.924	0.231	0.126	0.087
0.6	3.177	1.588	1.059	0.265	0.144	0.099
0.7	3.582	1.791	1.194	0.299	0.163	0.112
0.8	3.988	1.994	1.329	0.332	0.181	0.125
0.9	4.394	2.197	1.465	0.366	0.200	0.137
1	4.800	2.400	1.600	0.400	0.218	0.150

Panel C: Hedging portfolio

π	$\alpha = 0.5$	$\alpha = 0$	$\alpha = -0.5$	$\alpha = -5$	$\alpha = -10$	$\alpha = -15$
0	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.080	0.000	-0.007	-0.004	-0.001	0.003
0.2	0.159	0.000	-0.015	-0.008	-0.003	0.006
0.3	0.229	0.000	-0.022	-0.012	-0.003	0.010
0.4	0.284	0.000	-0.028	-0.015	-0.003	0.014
0.5	0.323	0.000	-0.032	-0.017	-0.003	0.016
0.6	0.325	0.000	-0.033	-0.017	-0.002	0.017
0.7	0.298	0.000	-0.031	-0.015	-0.001	0.016
0.8	0.238	0.000	-0.025	-0.012	0.001	0.012
0.9	0.141	0.000	-0.015	-0.007	0.001	0.007
1	0.000	0.000	0.000	0.000	0.000	0.000

Panel D: Ratio of hedging portfolio to optimal portfolio

π	$\alpha = 0.5$	$\alpha = 0$	$\alpha = -0.5$	$\alpha = -5$	$\alpha = -10$	$\alpha = -15$
0	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.071	0.000	-0.021	-0.049	-0.031	0.081
0.2	0.099	0.000	-0.031	-0.072	-0.040	0.125
0.3	0.110	0.000	-0.036	-0.082	-0.038	0.150
0.4	0.111	0.000	-0.038	-0.084	-0.032	0.161
0.5	0.104	0.000	-0.036	-0.078	-0.021	0.159
0.6	0.093	0.000	-0.033	-0.068	-0.012	0.146
0.7	0.077	0.000	-0.027	-0.054	-0.003	0.123
0.8	0.056	0.000	-0.019	-0.038	0.003	0.091
0.9	0.031	0.000	-0.010	-0.020	0.006	0.050
1	0.000	0.000	0.000	0.000	0.000	0.000

Table 3: Optimal Consumption and Portfolio Rule for Various Terminal horizons

This table reports, for different values of investor's time horizon $T = 1$, 5, 10, 30, and 50, the optimal consumption rate c^*, the hedging portfolio φ^*_{hp}, and the ratio of the hedging portfolio to the optimal portfolio $\varphi^*_{hp}/(\varphi^*_{mvp} + \varphi^*_{hp})$. Other parameters are given by $\alpha = -0.5$, $\mu_1 = 0.20$, $\mu_0 = 0.07$, $\lambda = 0.25$, $\sigma_S = 0.25$, $r = 0.05$, $\rho = 0.07$, $w_0 = 100$, and $N = 20000$.

Panel A: Optimal consumption rate

π	$T = 1$	$T = 5$	$T = 10$	$T = 30$	$T = 50$
0	52.459	20.240	13.066	8.213	7.602
0.1	52.512	20.300	13.111	8.248	7.636
0.2	52.581	20.369	13.164	8.288	7.673
0.3	52.665	20.448	13.225	8.332	7.715
0.4	52.764	20.536	13.294	8.380	7.760
0.5	52.909	20.660	13.392	8.447	7.823
0.6	53.042	20.769	13.479	8.505	7.878
0.7	53.191	20.888	13.574	8.567	7.936
0.8	53.356	21.017	13.677	8.635	8.000
0.9	53.538	21.157	13.788	8.708	8.068
1	53.737	21.307	13.907	8.786	8.141

Panel B: Hedging portfolio

π	$T = 1$	$T = 5$	$T = 10$	$T = 30$	$T = 50$
0	0.000	0.000	0.000	0.000	0.000
0.1	-0.002	-0.006	-0.007	-0.008	-0.009
0.2	-0.005	-0.012	-0.015	-0.017	-0.017
0.3	-0.008	-0.018	-0.022	-0.024	-0.025
0.4	-0.010	-0.023	-0.028	-0.030	-0.031
0.5	-0.012	-0.027	-0.032	-0.035	-0.035
0.6	-0.013	-0.028	-0.033	-0.035	-0.036
0.7	-0.013	-0.026	-0.031	-0.033	-0.033
0.8	-0.011	-0.021	-0.025	-0.026	-0.027
0.9	-0.006	-0.013	-0.015	-0.016	-0.016
1	0.000	0.000	0.000	0.000	0.000

Panel C: Ratio of hedging portfolio to optimal portfolio

π	$T = 1$	$T = 5$	$T = 10$	$T = 30$	$T = 50$
0	0.000	0.000	0.000	0.000	0.000
0.1	-0.006	-0.017	-0.021	-0.025	-0.025
0.2	-0.010	-0.026	-0.031	-0.036	-0.037
0.3	-0.012	-0.030	-0.036	-0.041	-0.042
0.4	-0.014	-0.031	-0.038	-0.042	-0.043
0.5	-0.014	-0.030	-0.036	-0.039	-0.040
0.6	-0.013	-0.027	-0.033	-0.035	-0.035
0.7	-0.011	-0.022	-0.027	-0.028	-0.029
0.8	-0.008	-0.016	-0.019	-0.020	-0.021
0.9	-0.004	-0.009	-0.010	-0.011	-0.011
1	0.000	0.000	0.000	0.000	0.000

Table 4: Optimal Consumption and Portfolio Rule for Various Price Volatilities

This table reports, for different values of risky asset price volatilities $\sigma_S = 0.35$, 0.25, and 0.15, the optimal consumption rate c^*, the hedging portfolio φ^*_{hp}, and the ratio of the hedging portfolio to the optimal portfolio $\varphi^*_{hp}/(\varphi^*_{mvp} + \varphi^*_{hp})$. Other parameters are given by $\alpha = -0.5$, $\mu_1 = 0.20$, $\mu_0 = 0.07$, $\lambda = 0.25$, $r = 0.05$, $\rho = 0.07$, $w_0 = 100$, $N = 20000$, and $T = 10$.

Panel A: Optimal consumption rate

π	$\sigma_S = 0.35$	$\sigma_S = 0.25$	$\sigma_S = 0.15$
0	12.832	13.077	14.072
0.1	12.856	13.125	14.222
0.2	12.883	13.180	14.394
0.3	12.914	13.241	14.586
0.4	12.948	13.309	14.799
0.5	12.995	13.404	15.093
0.6	13.036	13.487	15.354
0.7	13.081	13.577	15.639
0.8	13.130	13.675	15.950
0.9	13.182	13.781	16.292
1	13.238	13.894	16.670

Panel B: Hedging portfolio

π	$\sigma_S = 0.35$	$\sigma_S = 0.25$	$\sigma_S = 0.15$
0	0.000	0.000	0.000
0.1	-0.002	-0.007	-0.057
0.2	-0.004	-0.015	-0.114
0.3	-0.006	-0.021	-0.164
0.4	-0.007	-0.027	-0.204
0.5	-0.008	-0.031	-0.233
0.6	-0.009	-0.032	-0.237
0.7	-0.008	-0.030	-0.219
0.8	-0.006	-0.024	-0.177
0.9	-0.004	-0.015	-0.106
1	0.000	0.000	0.000

Panel C: Ratio of hedging portfolio to optimal portfolio

π	$\sigma_S = 0.35$	$\sigma_S = 0.25$	$\sigma_S = 0.15$
0	0.000	0.000	0.000
0.1	-0.011	-0.021	-0.063
0.2	-0.016	-0.031	-0.093
0.3	-0.018	-0.036	-0.105
0.4	-0.019	-0.037	-0.108
0.5	-0.018	-0.035	-0.100
0.6	-0.016	-0.031	-0.088
0.7	-0.013	-0.026	-0.071
0.8	-0.010	-0.019	-0.050
0.9	-0.005	-0.010	-0.027
1	0.000	0.000	0.000

Table 5: Optimal Consumption and Portfolio Rule for Various Price Drifts

This table reports, for different values of risky asset price volatilities $\mu_0 = 0.07, -0.05$, and 0, the optimal consumption rate c^*, the hedging portfolio φ^*_{hp}, and the ratio of the hedging portfolio to the optimal portfolio $\varphi^*_{hp}/(\varphi^*_{mvp} + \varphi^*_{hp})$. Other parameters are given by $\alpha = -0.5$, $\mu_1 = 0.20$, $\lambda = 0.25$, $\sigma_S = 9.25$, $r = 0.05$, $\rho = 0.07$, $w_0 = 100$, $N = 20000$, and $T = 10$.

Panel A: Optimal consumption rate

π	$\mu_0 = 0.07$	$\mu_0 = -0.05$	$\mu_0 = 0$
0	13.105	12.942	12.789
0.1	13.153	12.892	12.795
0.2	13.208	12.866	12.816
0.3	13.270	12.860	12.851
0.4	13.338	12.874	12.899
0.5	13.431	12.916	12.978
0.6	13.513	12.970	13.056
0.7	13.602	13.042	13.148
0.8	13.697	13.135	13.255
0.9	13.801	13.251	13.378
1	13.914	13.393	13.518

Panel B: Hedging portfolio

π	$\mu_0 = 0.07$	$\mu_0 = -0.05$	$\mu_0 = 0$
0	0.000	0.000	0.000
0.1	-0.007	0.010	-0.003
0.2	-0.015	0.008	-0.011
0.3	-0.021	-0.003	-0.022
0.4	-0.026	-0.017	-0.033
0.5	-0.030	-0.035	-0.045
0.6	-0.031	-0.047	-0.050
0.7	-0.029	-0.053	-0.050
0.8	-0.023	-0.050	-0.043
0.9	-0.014	-0.034	-0.028
1	0.000	0.000	0.000

Panel C: Ratio of hedging portfolio to optimal portfolio

π	$\mu_0 = 0.07$	$\mu_0 = -0.05$	$\mu_0 = 0$
0	0.000	0.000	0.000
0.1	-0.021	0.029	-0.009
0.2	-0.031	0.015	-0.024
0.3	-0.035	-0.005	-0.037
0.4	-0.036	-0.023	-0.046
0.5	-0.034	-0.040	-0.051
0.6	-0.030	-0.046	-0.050
0.7	-0.025	-0.046	-0.044
0.8	-0.018	-0.039	-0.034
0.9	-0.010	-0.024	-0.019
1	0.000	0.000	0.000

Table 6: Optimal Consumption and Portfolio Rule for Various Jump Intensities

This table reports, for different values of regime-switch-jump-intensities $\lambda = 0.05$, 0.25, 0.5, and 5, the optimal consumption rate c^*, the hedging portfolio φ^*_{hp}, and the ratio of the hedging portfolio to the optimal portfolio $\varphi^*_{hp}/(\varphi^*_{mvp} + \varphi^*_{hp})$. Other parameters are given by $\alpha = -0.5$, $\mu_1 = 0.20$, $\mu_0 = 0.07$, $\sigma_S = 9.25$, $r = 0.05$, $\rho = 0.07$, $w_0 = 100$, $N = 20000$, and $T = 10$.

Panel A: Optimal consumption rate

π	$\lambda = 0.05$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 5$
0	12.759	13.066	13.202	13.363
0.1	12.850	13.111	13.230	13.366
0.2	12.963	13.164	13.262	13.369
0.3	13.094	13.225	13.297	13.373
0.4	13.243	13.294	13.336	13.377
0.5	13.455	13.392	13.390	13.383
0.6	13.645	13.479	13.438	13.388
0.7	13.856	13.574	13.489	13.394
0.8	14.087	13.677	13.544	13.400
0.9	14.344	13.788	13.604	13.407
1	14.631	13.907	13.667	13.414

Panel B: Hedging portfolio

π	$\lambda = 0.05$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 5$
0	0.000	0.000	0.000	0.000
0.1	-0.015	-0.007	-0.004	0.000
0.2	-0.031	-0.015	-0.008	-0.001
0.3	-0.047	-0.022	-0.012	-0.001
0.4	-0.060	-0.028	-0.015	-0.002
0.5	-0.071	-0.032	-0.018	-0.002
0.6	-0.073	-0.033	-0.018	-0.002
0.7	-0.069	-0.031	-0.017	-0.002
0.8	-0.057	-0.025	-0.014	-0.002
0.9	-0.034	-0.015	-0.008	-0.001
1	0.000	0.000	0.000	0.000

Panel C: Ratio of hedging portfolio to optimal portfolio

π	$\lambda = 0.05$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 5$
0	0.000	0.000	0.000	0.000
0.1	-0.045	-0.021	-0.012	-0.001
0.2	-0.069	-0.031	-0.018	-0.002
0.3	-0.082	-0.036	-0.020	-0.002
0.4	-0.086	-0.038	-0.021	-0.002
0.5	-0.083	-0.036	-0.020	-0.002
0.6	-0.074	-0.033	-0.018	-0.002
0.7	-0.061	-0.027	-0.015	-0.002
0.8	-0.044	-0.019	-0.011	-0.001
0.9	-0.024	-0.010	-0.006	-0.001
1	0.000	0.000	0.000	0.000