A random walk analogue of Lévy's theorem [1]

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Abstract: In this paper, we will give a simple symmetric random walk analogue of Lévy's Theorem. We will take a new definition of a local time of the simple symmetric random walk. We apply a discrete Itô formula to some absolute value like function to obtain a discrete Tanaka formula. Results in this paper rely upon a discrete Skorokhod reflection argument. This random walk analogue of Lévy's theorem was already obtained by G.Simons([14]) but it is still worth noting because we will use a discrete stochastic analysis to obtain it and this method is applicable to other research. We note some connection with previous results by Csáki, Révész, Csörgő and Szabados. Finally we observe that the discrete Lévy transformation in the present version is not ergodic. Lastly we give a Lévy's theorem for simple nonsymmetric random walk using a discrete bang-bang process.

Keywords: simple symmetric random walk, local time, discrete Itô's formula, Lévy's Theorem, discrete Tanaka-Meyer formula, discrete Skorokhod equation, Lévy transformation, simple nonsymmetric random walk, discrete bang-bang process.

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1 Introduction

The following celebrated Lévy's Theorem([8], [12]) is well known:

$$(M_{\cdot} - W_{\cdot},\, M_{\cdot}) = (|W_{\cdot}|,\, L_{\cdot}) \quad \text{in law},$$

where W_t is a Brownian motion, $M_t = \max_{0 \leqq s \leqq t} W_s$ and L_t=the local time of W_t at 0=$(\lim_{\epsilon \to 0}(1/2\epsilon)\int_0^t 1_{(-\epsilon,\epsilon)}(W_s)ds)$.

First this paper shows that there exists a simple symmetric random walk analogue of this theorem. Second we remark that Discrete Lévy's transformation is not ergodic on path space, while the question (the original Lévy's transformation is ergodic or not) is still open. Last we give the Lévy's theorem for a simple nonsymmetric random walk, using a discrete bang-bang process.

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2 Facts

Let Z_t be a simple symmetric random walk, that is, $Z_t = \xi_1 + \xi_2 + \cdots \xi_t, Z_0 = 0$ where $\xi_1, \ldots, \xi_t$ are i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. We put $M_t = \max_{0 \leqq s \leqq t} Z_s$, while the local time of Z_t at 0 up to the time t is defined as:

$$L_t = \sharp\{i | i = 0, \ldots t-1 (Z_i = 0 \cap Z_{i+1} = 1) \cup (Z_i = 1 \cap Z_{i+1} = 0)\}.$$

Then we obtain the following theorem:

Theorem 1.

$$(M. - Z., M.) = (\lceil Z. \rceil, L.) \quad \text{in law.}$$

where $\lceil x \rceil := \max(x-1, -x)$ (a quasi-absolute value function)

Before proving this theorem, we prepare a discrete Itô formula for the simple symmetric random walk. This formula was obtained in [5] when the author studied a derivative pricing in a discrete time model.

Lemma (Discrete Itô formula [5])

$$f(Z_{t+1}) - f(Z_t) = \frac{f(Z_t+1) - f(Z_t-1)}{2}(Z_{t+1} - Z_t) + \frac{f(Z_t+1) - 2f(Z_t) + f(Z_t-1)}{2}.$$

$$f(Z_t) - f(0) = \sum_{i=0}^{t-1} \frac{f(Z_i+1) - f(Z_i-1)}{2}(Z_{i+1} - Z_i)$$

$$+ \sum_{i=0}^{t-1} \frac{f(Z_i+1) - 2f(Z_i) + f(Z_i-1)}{2}.$$

(Doob-Meyer Decomposition of $f(Z_t)$)

Proof

$$f(Z_{t+1}) - f(Z_t) - \frac{f(Z_t+1) - 2f(Z_t) + f(Z_t-1)}{2}$$

$$= \begin{cases} \frac{f(Z_t+1) - f(Z_t-1)}{2} & \text{if } \xi_{t+1} = 1 \\ \frac{f(Z_t-1) - f(Z_t+1)}{2} & \text{if } \xi_{t+1} = -1 \end{cases}$$

$$= \frac{f(Z_t+1)-f(Z_t-1)}{2}(Z_{t+1}-Z_t).$$

(q.e.d.)

Remark 1.

Let $Z_t^{(p)}$ be a simple nonsymmetric random walk,that is, $Z_t^{(p)} = \xi_1^{(p)} + \xi_2^{(p)} + \cdots \xi_t^{(p)}, X_0 = 0$ where $\xi_1^{(p)}, \ldots, \xi_t$ are i.i.d. with $P(\xi_i^{(p)} = 1) = p, P(\xi_i^{(p)} = -1) = q = 1 - p$. In this case, Itô's formula is the same as in the simple symmetric random walk case, as follows:

$$f(Z_{t+1}^{(p)}) - f(Z_t^{(p)}) = \frac{f(Z_i^{(p)}+1) - f(Z_i^{(p)}-1)}{2}(Z_{t+1}^{(p)} - Z_t^{(p)})$$

$$+\frac{f(Z_i^{(p)}+1) - 2f(Z_i^{(p)}) + f(Z_i^{(p)}-1)}{2}$$

$$f(Z_t^{(p)}) - f(0) = \sum_{i=0}^{t-1} \frac{f(Z_i^{(p)}+1) - f(Z_i^{(p)}-1)}{2}(Z_{i+1}^{(p)} - Z_i^{(p)} - (2p-1))$$

$$+\sum_{i=0}^{t-1}(\frac{f(Z_i^{(p)}+1) - 2f(Z_i^{(p)}) + f(Z_i^{(p)}-1))}{2} + (2p-1)\frac{f(Z_i^{(p)}+1) - f(Z_i^{(p)}-1)}{2}$$

(Doob Meyer Decomposition of $f(Z_t^{(p)})$.)

Remark 2.

Szabados ([15]) obtained the following discrete version of Itô formula.
Let $g(k) = \epsilon_k \left\{(1/2)f(0) + \sum_{j=1}^{|k|-1} f(\epsilon_k j) + (1/2)f(k)\right\}$ where

$$\epsilon_k = \begin{cases} 1 & \cdots (k > 0) \\ 0 & \cdots (k = 0) \\ -1 & \cdots (k < 0) \end{cases}$$

He called g a *trapezoidal sum* of f.

The following is Szabados's version of the discrete Itô formula.

$$g(Z_t) = \sum_{i=0}^{t-1} f(Z_i)\xi_{i+1} + (1/2)\sum_{i=0}^{t-1} \frac{f(Z_{i+1}) - f(Z_i)}{\xi_{i+1}}.$$

Using this formula and limiting procedure, Szabados proved Itô's formula for Brownian motion in the following form: $\int_0^{W_t} f(s)ds = \int_0^t f(W_s)dW_s + (1/2)\int_0^t f'(W_s)ds$ and investigated some consequences. We note that this paper's version of discrete Itô formula also yields Itô formula $f(W_t) - f(0) = \int_0^t f'(W_s)dW_s + (1/2)\int_0^t f''(W_s)ds$ by using limiting procedure by an appropriate scale change taking $Z_t^{\delta t} = \sqrt{\delta t} Z_{[t/\delta t]}$. Actually Fujita and Kawanishi([6]) prooved the Itô formula using this paper version of discerte Itô's fromula.

So if we consider the limit case, both discrete versions of Itô's formula give the same result. But within the discrete case there exist some differences because this paper's version gives the Doob Meyer decomposition of $f(Z_t)$, while Szabados's version does not give it in general.

Proof of Theorem 1.

Applying $f(x) = \lceil x \rceil$ to discrete Itô formula, we have that

$$\lceil Z_t \rceil = \sum_{i=0}^{t-1} \frac{\lceil Z_i + 1 \rceil - \lceil Z_i - 1 \rceil}{2}(Z_{i+1} - Z_i)$$

$$+(1/2)\sum_{i=0}^{t-1}(\lceil Z_i + 1 \rceil - 2\lceil Z_i \rceil + \lceil Z_i - 1 \rceil).$$

We note that

$$\frac{\lceil x+1 \rceil - \lceil x-1 \rceil}{2} = \begin{cases} 1 & \cdots x \geqq 2 \\ 1/2 & \cdots x = 1 \\ -1/2 & \cdots x = 0 \\ -1 & \cdots x \leqq -1 \end{cases}.$$

So putting

$$\mathrm{sgn}(x) = \begin{cases} 1 & \cdots x \geqq 1 \\ -1 & \cdots x \leqq 0 \end{cases},$$

$$h_1(x) = \begin{cases} 1/2 & \cdots x = 1 \\ -1/2 & \cdots x = 0 \\ 0 & \cdots otherwise \end{cases},$$

$$h_2(x) = \frac{\lceil x+1 \rceil - 2\lceil x \rceil + \lceil x-1 \rceil}{2} = \begin{cases} 1/2 & \cdots x = 0, 1 \\ 0 & \cdots otherwise \end{cases},$$

$$\lceil Z_t \rceil = \sum_{i=0}^{t-1} \mathrm{sgn}(Z_i)(Z_{i+1} - Z_i) + \sum_{i=0}^{t-1} h_2(Z_i) - \sum_{i=0}^{t-1} h_1(Z_i)(Z_{i+1} - Z_i)$$

holds.

Here we will show that $L_t = \sum_{i=0}^{t-1} h_2(Z_i) - \sum_{i=0}^{t-1} h_1(Z_i)(Z_{i+1} - Z_i)$ by induction.

For t=1, clearly, $L_1 = h_2(Z_0) - h_1(Z_0)(Z_1 - Z_0) = \frac{1+Z_1}{2}$ holds.

Assuming t,

$$\sum_{i=0}^{t} h_2(Z_i) - \sum_{i=0}^{t} h_1(Z_i)(Z_{i+1} - Z_i) = L_t + h_2(Z_t) - h_1(Z_t)(Z_{t+1} - Z_t)$$

$$= L_t + 1_{(Z_t=0 \cap Z_{t+1}=1) \cup (Z_t=1 \cap Z_{t+1}=0)} = L_{t+1}.$$

So we have that

$$\lceil Z_t \rceil = \sum_{i=0}^{t-1} \mathrm{sgn}(Z_i)(Z_{i+1} - Z_i) + L_t \quad \text{(Discrete Tanaka formula)}$$

.

Putting $\hat{Z}_t = \sum_{i=0}^{t-1} \mathrm{sgn}(Z_i)(Z_{i+1} - Z_i)$, we remark that $\hat{Z}.$ is clearly a simple symmetric random walk.[2]

On the other hand, we have that

$$M_t - Z_t = -Z_t + M_t$$

holds.

So the uniqueness of following discrete Skorokhod Equation gives a proof of this theorem.

(q.e.d)

Lemma 2.(Discrete Skorokhod Lemma) (For a proof of the continuous versions, see [6])

Let us define the following three path spaces:

$$\Omega_1 = \{f | f : \mathrm{Z}_+ \to \mathrm{Z},\ f(0) = 0, \text{ and } \forall x \in \mathrm{Z}_+, f(x+1) - f(x) = \pm 1\}$$

[2]If we use gambling terminology, a simple symmetric random walk Z_t is the amount of money which a gambler A makes after t times "red or black" play with equal probability if his each bet is 1 \$. Let us assume that the gambler A continue to play "red" continuously. Consider another gambler B whose i-th play is "black " if $Z_{i-1} \leqq 0$, "red" if $Z_{i-1} \geqq 1$ with his each 1 \$ bet. Then $\hat{Z}_t$ is the amount of money which the gambler B makes after t times play and clearly it is also a simple symmetric random walk.

$$\Omega_2 = \{f | f : \mathrm{Z}_+ \to \mathrm{Z}_+ \text{ and } \forall x \in \mathrm{Z}_+, f(x+1) - f(x) = 0 \text{ or } \pm 1\}$$

$$\Omega_3 = \{f | f : \mathrm{Z}_+ \to \mathrm{Z}_+ \text{ and } \forall x \in \mathrm{Z}_+, f(x+1) - f(x) = 0 \text{ or } 1, \text{and }, f(0) = 0\}$$

where $Z_+ = \{x | x \geqq 0, x \in \mathrm{Z}\}$.

Given $f \in \Omega_1$ and $x \in \mathrm{Z}$, there exist unique $g \in \Omega_2$ and $h \in \Omega_3$ such that

1. $g(t) = x + f(t) + h(t)$,
2. $h(t+1) - h(t) > 0$ only if $g(t) = 0$ i.e. $h(t)$ increases only on g(t)=0.

Proof

Set $g(t) = x+f(t) - \min_{0 \leqq s \leqq t}(\min(x+f(s),0)$, $h(t) = -\min_{0 \leqq s \leqq t}(\min(x+f(s),0))$. We can easily see that $g(t)$ and $h(t)$ satisfy both conditions above. We will prove the uniqueness. Suppose $\hat{g}(t)$,$\hat{h}(t)$ satisfy the above conditions. Then $g(t) - \hat{g}(t) = h(t) - \hat{h}(t)$ for all $t \geqq 0$. If there exists $t_1 \in \mathrm{N}$ such that $g(t_1) - \hat{g}(t_1) > 0$, we set $t_2 = \max\{t < t_1 | g(t) - \hat{g}(t) = 0, t \in \mathrm{N} \cup \{0\}\}$ Then $g(t) > \hat{g}(t) \geqq 0$ for all $t_2 < t \leqq t_1$ and hence by the above conditions $h(t_1) = h(t_2) = 0$. Since $\hat{h}$ is increasing, we have that $0 < g(t_1) - \hat{g}(t_1) = h(t_1) - \hat{h}(t_1) \leqq h(t_2) - \hat{h}(t_2) = g(t_2) - \hat{g}(t_2) = 0$. This is a contradiction. Here $\hat{g} \equiv g$ and $\hat{h} \equiv h$.

(q.e.d)

Remark 3. This precise random walk analogue of Lévy's theorem was already obtained by G. Simons([14]). He gave a proof of this theorem by similar discussions but without a discrete stochastic calculus.

Remark 4.

Saisho and Tanemura([13]) displayed similar discrete Skorokhod equations through their research about Pitman type theorem for one dimensional diffusions.

Similarly, we have the following facts.

We put

$$L_t^- = \sharp\{i | i = 0, \ldots t-1, Z_i = 0 \cap Z_{i+1} = 1\}$$

and

$$L_t^+ = \sharp\{i | i = 0, \ldots t-1, Z_i = 1 \cap Z_{i+1} = 0\}.$$

Then

$$\max(Z_t - 1, 0) = \sum_{i=0}^{t-1} 1_{(Z_i \geqq 1)}(Z_{i+1} - Z_i) + L_t^+$$

$$\max(0, -Z_t) = -\sum_{i=0}^{t-1} 1_{(Z_i \leqq 0)}(Z_{i+1} - Z_i) + L_t^-$$

$$= -\sum_{i=0}^{t-1} 1_{(Z_i \leqq 0)}(Z_{i+1} - Z_i) + (1/2)\sum_{i=0}^{t-1} 1_{\{0\}}(Z_i) - (1/2)\sum_{i=0}^{t-1} 1_{\{0\}}(Z_i)(Z_{i+1} - Z_i).$$

$$\max(Z_t, 0) = \sum_{i=0}^{t-1} 1_{(Z_i \leqq 0)}(Z_{i+1} - Z_i) + (1/2)\sum_{i=0}^{t-1} 1_{\{0\}}(Z_i) + (1/2)\sum_{i=0}^{t-1} 1_{\{0\}}(Z_i)(Z_{i+1} - Z_i).$$

So we have that

$$|Z_t| = \sum_{i=0}^{t-1}(1_{(Z_i \geqq 0)} - 1_{(Z_i \leqq 0)})(Z_{i+1} - Z_i) + \sum_{i=0}^{t-1} 1_{\{0\}}(Z_i)$$

$$= \sum_{i=0}^{t-1}(\mathrm{sgn}(Z_i)(Z_{i+1} - Z_i) + \sum_{i=0}^{t-1} 1_{\{0\}}(Z_i) + \sum_{i=0}^{t-1} 1_{\{0\}}(Z_i)(Z_{i+1} - Z_i)$$

$$= \sum_{i=0}^{t-1} \mathrm{sgn}(Z_i)(Z_{i+1} - Z_i) + 2L_t^-.$$

Then we have also the following theorem.

Theorem 2.

$$|Z_\cdot| - 2L_\cdot^- = Z_\cdot \quad \text{in law.}$$

Remark 4.
Csáki, Csörgő and Révész ([1],[2],[11])considered a local time of random walk in the following way:

$$\zeta(x, t) = \sharp\{i | i \leqq t, Z_i = x\}$$

and then they showed that

$$\zeta(x,t) = |Z_t - x| - |x| - \sum_{i=0}^{t-1} \hat{\text{sgn}}(Z_i - x)\xi_{i+1}$$

where

$$\hat{\text{sgn}}(x) = \begin{cases} 1 & \cdots x \geqq 1 \\ 0 & \cdots x = 0 \\ -1 & \cdots x \leqq -1 \end{cases} .$$

Csáki and Révész([1]) obtained a "nearly true " Lévy's theorem for a simple symmetric random walk using $\zeta(x,t)$. We remark that their version of a discrete Tanaka Meyer formula is also different from this paper's version but we point out that applying this paper's version of discrete Itô formula to $f(y) = |y - x|$, this version of discrete Tanaka Meyer formula is very easily obtained. Here we also note that Miyazaki and Tanaka ([10],[16]) also researched a random walk analogue of Pitman' s theorem.

Remark 5.

This kind of problem is also related to the so called Lévy's transformation:

$$W. \to \hat{W}. = \int_0^{\cdot} \text{sgn}(W_s) dW_s,$$

which is measure-preserving on path space. Whether this transformation is ergodic or not, a question raised by D.Revuz and M.Yor([12]), is still open. Dubins and Smorodinsky([3]) gave a proof of ergodicity in the modified, discrete and infinite time horizon case. Roughly speaking, their definition of Lévy transform is Z'_t= the one that skipping the flat path from $\sum_{i=0}^{t-1} \hat{\text{sgn}}(Z_i)\xi_{i+1}$. Also Dubins, Emery and Yor([4]) made some considerations about Lévy transformation on some continuous martingales. We note that for original Lévy's transformation problem, Malric([9]) obtained recently that the union of zero points of iterated Lévy's transforms is a.s. dense in R_+. This paper's version of Lévy's transformation is the following natural generalization of the continuous Lévy's transformation:

Definition

$$T(Z.) = \hat{Z}. = \sum_{i=0}^{\cdot - 1} \text{sgn}(Z_i)(Z_{i+1} - Z_i)$$

is called discrete Lévy transformation.

Observation

This Lévy's transformation $T : \Omega_1^M \to \Omega_1^M$ is not ergodic if $M \geqq 4$ where

$\Omega_1^M = \{f|f : \{0, 1, \ldots M\} \to \mathrm{Z},\ f(0) = 0, \text{ and} \forall x \in \mathrm{N} \cup \{0\}, f(x+1) - f(x) = \pm 1\}$.

If we take $M = 4$, we observe very explicitly that $T^8 = id$. Also we can observe that when $M = 5, T^{16} = id$, when $M = 6,\ T^{32} = id$, when $M = 7,\ T^{32} = id$, $\ldots$.
We denote the path $Z_0 = 0, Z_1 = 1, Z_2 = 2, Z_3 = 3, Z_4 = 4$ as $++++$, the path $Z_0 = 0, Z_1 = -1, Z_2 = -2, Z_3 = -1, Z_4 = -2$ as $--+-$. Then we observe that $T(++++) = -+++,\quad T(-+++) = +--+,\quad T(+--+) = --+-,\quad T(--+-) = ++-+,\quad T(++-+) = -+-+,\quad T(-+-+) = +-+-,\quad T(+-+-) = ----,\quad T(----) = ++++$
and $T(+++-) = -++-,\quad T(-++-) = +---,\quad T(+---) = --++,\quad T(--++) = ++--,\quad T(++--) = -+--,\quad T(-+--) = +-++,\quad T(+-++) = ---+,\quad T(---+) = +++-$. So putting $\Omega_1^{4'} = \{++++, -+++, +--+, --+-, ++-+, -+-+, +-+-, ----\}$ and $\Omega_1^{4''} = \{+++-, -++-, +---, --++, ++--, -+--, +-++, , ---+\}$, $\Omega_1^4 = \Omega_1^{4'} \cup \Omega_1^{4''}$ holds, i.e. Ω_1^4 has two ergodic components.
Also denoting $Z_0 = 0, Z_1 = 1, Z_2 = 2, Z_3 = 3, Z_4 = 4, \ldots$ as $++++*$, putting $\Omega_1^{4'}* = \{++++*, -+++*, +--+*, --+-*, ++-+*, -+-+*, +-+-*, ----*\}$ and $\Omega_1^{4''}* = \{+++-*, -++-*, +---*, --++*, ++--*, -+--*, +-++*, ---+*\}$, we obtain $\Omega_1 = \Omega_1^{4'}* \cup \Omega_1^{4''}*$holds i.e. Ω_1 has at least two ergodic components. Defining $\phi(M) = \inf\{i|T : \Omega_1^M \to \Omega_1^M,\quad T^i = id\}$, we note that generally $\phi(M)$ is so far not known.

Last we give the Lévy's theorem for a simple nonsymmetric random walk $Z_t^{(p)}$.

Theorem 3.(Lévy's theorem for $Z_t^{(p)}$)

$$(Z_\cdot^{(p)} - \min_{0 \leqq s \leqq \cdot} Z_s^{(p)}, -\min_{0 \leqq s \leqq \cdot}(Z_s^{(p)}) = (\lceil J_\cdot^{(p)} \rceil, L_\cdot^{J^{(p)}}) \quad \text{in law}$$

where $J_t^{(p)}(J_0^{(p)} = 0)$ is a discrete bang-bang process which is defined as follows:

the transition probability $p(x, y)$ of $J_t^{(p)}$ is the following:

For $x \geqq 1$, $p(x, y) = \begin{cases} p & \cdots y = x+1 \\ 1-p & \cdots y = x-1 \end{cases}$.

For $x \leqq 0$, $p(x, y) = \begin{cases} 1-p & \cdots y = x+1 \\ p & \cdots y = x-1 \end{cases}$

We note that $L_t^{J^{(p)}}$ is the local time of $J_t^{(p)}$ at 0 up to the time $t := \sharp\{i|i = 0, \ldots t-1,\ (J_i^{(p)} = 0 \cap J_{i+1}^{(p)} = 1) \cup (J_i^{(p)} = 1 \cap J_{i+1}^{(p)} = 0)\}$.

Proof of Theorem 3.

We consider the following stochastic difference equation:

$$X_{t+1} - X_t = \mathrm{sgn}(X_t)(Z^{(p)}_{t+1} - Z^{(p)}_t)\,(X_0 = 0)$$

.

By the definition of $J^{(p)}_t$, X_t=$J^{(p)}_t$.

For $\lceil J^{(p)}_t \rceil$, applying the discrete Itô formula,

we get that

$$\lceil J^{(p)}_t \rceil = \sum_{i=0}^{t-1} \mathrm{sgn}(J^{(p)}_i)(J^{(p)}_{i+1} - J^{(p)}_i) + L^{J^{(p)}}_t$$

.

Here we note that $\sum_{i=0}^{t-1} \mathrm{sgn}(J^{(p)}_i)(J^{(p)}_{i+1} - J^{(p)}_i) = \sum_{i=0}^{t-1}(\mathrm{sgn}(J^{(p)}_i))^2(Z^{(p)}_{i+1} - Z^{(p)}_i) = Z^{(p)}_t$.

Then by Discrete Skorokhod Lemma 2. and $\lceil J^{(p)}_t \rceil = Z^{(p)}_t + L^{J^{(p)}}_t$, we get that

$$L^{J^{(p)}}_t = -\min_{0 \leqq s \leqq .} Z^{(p)}_s,\ \lceil J^{(p)}_t \rceil = Z^{(p)}_{.} - \min_{0 \leqq s \leqq .} Z^{(p)}_s.$$

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