A proof of Itô's formula using a discrete Itô's formula [1]

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Abstract. In this paper, we will prove Itô's formula for Brownian motion in the case of $f \in C^2(\boldsymbol{R})$, using a discrete Itô's formula.

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1 Introduction

Let $\{Z_t; t = 0, 1, \cdots\}$ be a $\boldsymbol{Z}$-valued symmetric random walk, that is, $Z_0 = 0$, $Z_t = \xi_1 + \xi_2 + \cdots + \xi_t$ where $\xi_1, \xi_2, \cdots$ are independently and identically distributed with $P[\xi_1 = 1] = P[\xi_1 = -1] = 1/2$. We have the following.

Lemma (Discrete Itô's Formula)

For any $f : \boldsymbol{Z} \to \boldsymbol{R}$ and any nonnegative integer t, it holds that

$$\begin{aligned} f(Z_{t+1}) - f(Z_t) = &\frac{f(Z_t + 1) - f(Z_t - 1)}{2}(Z_{t+1} - Z_t) \\ &+ \frac{f(Z_t + 1) - 2f(Z_t) + f(Z_t - 1)}{2} . \end{aligned} \tag{1}$$

This is called a discrete Itô's formula. It was discovered by the first author. The proof is very easy. We only have to consider the difference between the left-hand side(henceforth, abbreviated LHS) and the second term of the right-hand side(henceforth, abbreviated RHS) of the above equation. For the details of this discrete Itô's formula, see [1],[2],[3]. In the next section, we will give a proof of Itô's formula for Brownian motion in the case of $f \in C^2(\boldsymbol{R})$ using the above discrete Itô's formula. It seems natural that Itô differential formula can be approximated by the discrete Itô (Itô difference) formula. In the proof, it is important that how we approximate Brownian motion by random walks. For the approximation method, the reader is referred to Itô and Mckean [4] section 1.10 .

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2 The proof

Let $\{W_s; s \geqq 0\}$ be a standard Brownian motion and f be in $C^2(\boldsymbol{R})$. In the sequel, we are going to prove the following statement:

$$P\Big[f(W_t) - f(0) = \int_0^t f'(W_s)\,dW_s + \frac{1}{2}\int_0^t f''(W_s)\,ds \quad \text{for } {}^\forall t \geqq 0\Big] = 1\,. \tag{2}$$

Let us begin the proof. As it is well-known to anyone who has proved Itô's formula, it is sufficient that we show

$$P\Big[f(W_t) - f(0) = \int_0^t f'(W_s)\,dW_s + \frac{1}{2}\int_0^t f''(W_s)\,ds\Big] = 1\,, \tag{3}$$

where $f \in C^2(\boldsymbol{R})$ has a compact support and $t > 0$. So we suppose that $f \in C^2(\boldsymbol{R})$ has a compact support and fix $t > 0$. We will introduce an approximation to Brownian motion by random walks. For $n = 1, 2, \cdots$, define that

$$\begin{aligned} \tau_{n,0} &:= 0\,, \\ \tau_{n,i} &:= \inf\Big\{s > \tau_{n,i-1}; |W_s - W_{\tau_{n,i-1}}| = \frac{\sqrt{t}}{2^n}\Big\} \qquad \text{for } i = 1, 2, \cdots. \end{aligned}$$

Then by the strong Markov property of Brownian motion, $\tau_{n,i} - \tau_{n,i-1}$ $(i = 1, 2, \cdots)$ are independently and identically distributed. And so are $W_{\tau_{n,i}} - W_{\tau_{n,i-1}}$ $(i = 1, 2, \cdots)$. In addition to that, it holds that

$$\left.\begin{aligned} E[W_{\tau_{n,1}}] &= 0\,, \\ E[\tau_{n,1}] &= \tfrac{t}{2^{2n}}\,, \\ E[{\tau_{n,1}}^2] &= \tfrac{5}{3}\tfrac{t^2}{2^{4n}}\,. \end{aligned}\right\} \tag{4}$$

Actually, $\{W_s; s \geqq 0\}$, $\{{W_s}^2 - s; s \geqq 0\}$ and $\{{W_s}^4 - 6s{W_s}^2 + 3s^2; s \geqq 0\}$ are martingales. And also, by Doob's optional sampling theorem, $\{W_{s\wedge\tau_{n,1}}; s \geqq 0\}$, $\{{W_{s\wedge\tau_{n,1}}}^2 - (s \wedge \tau_{n,1}); s \geqq 0\}$ and $\{{W_{s\wedge\tau_{n,1}}}^4 - 6(s \wedge \tau_{n,1}){W_{s\wedge\tau_{n,1}}}^2 + 3(s \wedge \tau_{n,1})^2; s \geqq 0\}$ are martingales. Thus, it follows that

$$\begin{aligned} E[W_{s\wedge\tau_{n,1}}] &= 0\,, \\ E[s \wedge \tau_{n,1}] &= E[{W_{s\wedge\tau_{n,1}}}^2]\,, \\ E[(s \wedge \tau_{n,1})^2] &= -\frac{1}{3}E[{W_{s\wedge\tau_{n,1}}}^4] + 2E[(s \wedge \tau_{n,1}){W_{s\wedge\tau_{n,1}}}^2]\,. \end{aligned}$$

Then if we let $s \to \infty$, we can obtain (4). Furthermore, by the above-mentioned facts, $\{\tau_{n,i} - i\frac{t}{2^{2n}}; i = 0, 1, 2, \cdots\}$ is a martingale. Thus, by (4) and the submartingale inequality, it holds that

$$\begin{aligned} P\Big[\sup_{1\leqq i\leqq 2^{2n}} |\tau_{n,i} - i\frac{t}{2^{2n}}| > \varepsilon\Big] &\leqq \varepsilon^{-2}E[(\tau_{n,2^{2n}} - t)^2] \\ &= \varepsilon^{-2}\frac{2}{3}\frac{t^2}{2^{2n}}\,. \end{aligned} \tag{5}$$

So if we apply Borel-Cantelli lemma, we obtain that

$$P\Big[\lim_{n\to\infty}\sup_{1\leqq i\leqq 2^{2n}}|\tau_{n,i}-i\frac{t}{2^{2n}}|=0\Big]=1\,. \tag{6}$$

Then by (6) and the uniform continuity of a continuous function defined on a compact interval, it follows that

$$P\Big[\lim_{n\to\infty}\sup_{1\leqq i\leqq 2^{2n}}|W(\tau_{n,i})-W(i\frac{t}{2^{2n}})|=0\Big]=1\,. \tag{7}$$

Here by the discrete Itô formula and appropriate scaling, we obtain that

$$\begin{aligned}f(W(\tau_{n,2^{2n}}))-f(0)=&\sum_{i=0}^{2^{2n}-1}\frac{f(W_{\tau_{n,i}}+\frac{\sqrt{t}}{2^n})-f(W_{\tau_{n,i}}-\frac{\sqrt{t}}{2^n})}{2\frac{\sqrt{t}}{2^n}}(W_{\tau_{n,i+1}}-W_{\tau_{n,i}})\\&+\frac{1}{2}\sum_{i=0}^{2^{2n}-1}\{f(W_{\tau_{n,i}}+\frac{\sqrt{t}}{2^n})-2f(W_{\tau_{n,i}})+f(W_{\tau_{n,i}}-\frac{\sqrt{t}}{2^n})\}.\end{aligned} \tag{8}$$

First, by (7), the LHS of (8) converges to $f(W_t)-f(0)$ almost surely as $n\to\infty$. Next, we can show that the second term of the RHS of (8) converges to $\frac{1}{2}\int_0^t f''(W_s)\,ds$ a.s. as $n\to\infty$. In fact, we have that

$$\begin{aligned}&\Big|\sum_{i=0}^{2^{2n}-1}\{f(W_{\tau_{n,i}}+\frac{\sqrt{t}}{2^n})-2f(W_{\tau_{n,i}})+f(W_{\tau_{n,i}}-\frac{\sqrt{t}}{2^n})\}-\int_0^t f''(W_s)\,ds\Big|\\&\leqq\ \Big|\sum_{i=0}^{2^{2n}-1}\{f(W_{\tau_{n,i}}+\frac{\sqrt{t}}{2^n})-2f(W_{\tau_{n,i}})+f(W_{\tau_{n,i}}-\frac{\sqrt{t}}{2^n})\}\\&\qquad-\sum_{i=0}^{2^{2n}-1}f''(W(i\frac{t}{2^{2n}}))\frac{t}{2^{2n}}\Big|\\&\qquad+\Big|\sum_{i=0}^{2^{2n}-1}f''(W(i\frac{t}{2^{2n}}))\frac{t}{2^{2n}}-\int_0^t f''(W_s)\,ds\Big|\,.\end{aligned}$$

The second term of the RHS of the above inequality converges to zero a.s. as $n\to\infty$, because $f''(W_s)$ is Riemann integrable on $[0,t]$. As for the first term of the RHS, when we put it as A_n and represent second order remainder terms of Taylor expansion in integral forms, we have the following.

$$\begin{aligned} A_n &\leqq \sum_{i=0}^{2^{2n}-1} \Big| \int_{W_{\tau_{n,i}}}^{W_{\tau_{n,i}}+\frac{\sqrt{t}}{2^n}} f''(s)(W_{\tau_{n,i}}+\frac{\sqrt{t}}{2^n}-s)\,ds - \frac{1}{2}f''(W_{\tau_{n,i}})\frac{t}{2^{2n}}\Big| \\ &\quad + \sum_{i=0}^{2^{2n}-1} \Big| \int_{W_{\tau_{n,i}}}^{W_{\tau_{n,i}}-\frac{\sqrt{t}}{2^n}} f''(s)(W_{\tau_{n,i}}-\frac{\sqrt{t}}{2^n}-s)\,ds - \frac{1}{2}f''(W_{\tau_{n,i}})\frac{t}{2^{2n}}\Big| \\ &\quad + t\,\sup\Big\{|f''(u)-f''(v)|; |u-v| \leqq \sup_{1\leqq i\leqq 2^{2n}} |W(\tau_{n,i}) - W(i\frac{t}{2^{2n}})|\Big\}\,. \end{aligned}$$

Here it holds that for $x, y \in \boldsymbol{R}$,

$$\Big| \int_x^y f''(s)(y-s)\,ds - \frac{1}{2}f''(x)(y-x)^2\Big| \leqq (y-x)^2 \sup_{|u-v|\leqq|y-x|} |f''(u)-f''(v)|\,.$$

So we obtain that

$$\begin{aligned} A_n &\leqq 2t\,\sup\Big\{|f''(u)-f''(v)|; |u-v| \leqq \frac{\sqrt{t}}{2^n}\Big\} \\ &\quad + t\,\sup\Big\{|f''(u)-f''(v)|; |u-v| \leqq \sup_{1\leqq i\leqq 2^{2n}} |W(\tau_{n,i}) - W(i\frac{t}{2^{2n}})|\Big\}\,. \end{aligned}$$

By the uniform continuity of f'' and (7), the RHS of the above inequality converges to zero a.s. as $n \to \infty$. Therefore, the second term of the RHS of (8) converges to $\frac{1}{2}\int_0^t f''(W_s)\,ds$ a.s. as $n \to \infty$.

Last, let us show that the first term of the RHS of (8) converges to $\int_0^t f'(W_s)\,dW_s$ in probability as $n \to \infty$. We define that

$$H_n(s) := \sum_{i=0}^{2^{2n}-1} \frac{f(W_{\tau_{n,i}}+\frac{\sqrt{t}}{2^n}) - f(W_{\tau_{n,i}}-\frac{\sqrt{t}}{2^n})}{2\frac{\sqrt{t}}{2^n}} 1_{(\tau_{n,i},\tau_{n,i+1}]}(s)\,.$$

Then the first term of the RHS of (8) can be written as $\int_0^{\tau_{n,2^{2n}}} H_n(s)\,dW_s$. Let

ε, δ be strictly positive. First, we have

$$
\begin{aligned}
p_n \quad := \quad & P\Big[\Big|\int_0^{\tau_{n,2^{2n}}} H_n(s)\,dW_s - \int_0^t f'(W_s)\,dW_s\Big| > \delta\Big] \\
\leqq \quad & P\Big[\Big|\int_0^{\tau_{n,2^{2n}}} (H_n(s) - f'(W_s))\,dW_s\Big| > \frac{\delta}{2} \cap |\tau_{n,2^{2n}} - t| \leqq \varepsilon\Big] \\
& +2P[|\tau_{n,2^{2n}} - t| > \varepsilon] \\
& +P\Big[\Big|\int_t^{\tau_{n,2^{2n}}} f'(W_s)\,dW_s\Big| > \frac{\delta}{2} \cap 0 \leqq \tau_{n,2^{2n}} - t \leqq \varepsilon\Big] \\
& +P\Big[\Big|\int_t^{\tau_{n,2^{2n}}} f'(W_s)\,dW_s\Big| > \frac{\delta}{2} \cap -\varepsilon \leqq \tau_{n,2^{2n}} - t < 0\Big] \\
\leqq \quad & P\Big[\sup_{0 \leqq r \leqq t+\varepsilon} \Big|\int_0^r (H_n(s) - f'(W_s))\,dW_s\Big| > \frac{\delta}{2}\Big] \\
& +2P[|\tau_{n,2^{2n}} - t| > \varepsilon] \\
& +P\Big[\sup_{t \leqq r \leqq t+\varepsilon} \Big|\int_t^r f'(W_s)\,dW_s\Big| > \frac{\delta}{2}\Big] \\
& +P\Big[\Big|\int_{t-\varepsilon}^t f'(W_s)\,dW_s\Big| > \frac{\delta}{4}\Big] + P\Big[\sup_{t-\varepsilon \leqq r \leqq t} \Big|\int_{t-\varepsilon}^r f'(W_s)\,dW_s\Big| > \frac{\delta}{4}\Big].
\end{aligned}
$$

Here, $\{\int_0^r (H_n(s) - f'(W_s))\,dW_s; r \geqq 0\}$, $\{\int_t^r f'(W_s)\,dW_s; r \geqq t\}$ and $\{\int_{t-\varepsilon}^r f'(W_s)\,dW_s; r \geqq t-\varepsilon\}$ are continuous martingales and these Itô integrals have the Itô isometry because f' is bounded. So by the submrtingale inequality, Chebyshev's inequality, Jensen's inequality and Itô integral's isometry, it holds that

$$
\begin{aligned}
p_n \quad \leqq \quad & \frac{2}{\delta}E\Big[\Big|\int_0^{t+\varepsilon} (H_n(s) - f'(W_s))\,dW_s\Big|\Big] + \frac{2}{\varepsilon}E[|\tau_{n,2^{2n}} - t|] \\
& +\frac{2}{\delta}E\Big[\Big|\int_t^{t+\varepsilon} f'(W_s)\,dW_s\Big|\Big] + \frac{8}{\delta}E\Big[\Big|\int_{t-\varepsilon}^t f'(W_s)\,dW_s\Big|\Big] \\
\leqq \quad & \frac{2}{\delta}\Big\{E\Big[\int_0^{t+\varepsilon} (H_n(s) - f'(W_s))^2\,ds\Big]\Big\}^{1/2} + \frac{2}{\varepsilon}\{E[(\tau_{n,2^{2n}} - t)^2]\}^{1/2} \\
& +\frac{2}{\delta}\Big\{E\Big[\int_t^{t+\varepsilon} f'(W_s)^2\,ds\Big]\Big\}^{1/2} + \frac{8}{\delta}\Big\{E\Big[\int_{t-\varepsilon}^t f'(W_s)^2\,ds\Big]\Big\}^{1/2}.
\end{aligned}
$$

Furthemore, letting M be the maximum of $|f'|$, we have the following.

$$
\begin{aligned}
p_n \quad \leqq \quad & \frac{2}{\delta}\Big\{E\Big[\int_0^{t+\varepsilon} (H_n(s) - f'(W_s))^2\,ds\Big]\Big\}^{1/2} + \frac{2}{\varepsilon}\{E[(\tau_{n,2^{2n}} - t)^2]\}^{1/2} \\
& +\frac{10}{\delta}M\varepsilon^{1/2}. \qquad (9)
\end{aligned}
$$

As for the first term of the RHS of (9), we have the following upper bound by the mean value theorem(with $\theta = \theta(W_{\tau_{n,i}})$ and $|\theta| < 1$), Hölder's inequality and(4):

$$
\begin{aligned}
&E\Big[\int_0^{t+\varepsilon}(H_n(s)-f'(W_s))^2\,ds\Big] \\
&\quad= E\Big[\int_0^{t+\varepsilon}\Big(\sum_{i=0}^{2^{2n}-1} f'(W_{\tau_{n,i}}+\theta\frac{\sqrt{t}}{2^n})1_{(\tau_{n,i},\tau_{n,i+1}]}(s)-f'(W_s)\Big)^2\,ds\Big] \\
&\quad= \sum_{i=0}^{2^{2n}-1} E\Big[\int_0^{t+\varepsilon}(f'(W_{\tau_{n,i}}+\theta\frac{\sqrt{t}}{2^n})-f'(W_s))^2 1_{(\tau_{n,i},\tau_{n,i+1}]}(s)\,ds\Big] \\
&\qquad +E\Big[\int_0^{t+\varepsilon} f'(W_s)^2 1_{(\tau_{n,2^{2n}},\infty)}(s)\,ds\Big] \\
&\quad\leqq \sum_{i=0}^{2^{2n}-1} E\Big[\Big(\sup_{\tau_{n,i}\leqq s\leqq\tau_{n,i+1}}|f'(W_{\tau_{n,i}}+\theta\frac{\sqrt{t}}{2^n})-f'(W_s)|\Big)^2(\tau_{n,i+1}-\tau_{n,i})\Big] \\
&\qquad +M^2E[|\tau_{n,2^{2n}}-t-\varepsilon|]
\end{aligned}
$$

$$
\begin{aligned}
&\leqq \sum_{i=0}^{2^{2n}-1}\Big\{E\Big[\Big(\sup_{\tau_{n,i}\leqq s\leqq\tau_{n,i+1}}|f'(W_{\tau_{n,i}}+\theta\frac{\sqrt{t}}{2^n})-f'(W_s)|\Big)^4\Big] \\
&\qquad\qquad \times E[(\tau_{n,i+1}-\tau_{n,i})^2]\Big\}^{1/2} \\
&\quad +M^2\{E[(\tau_{n,2^{2n}}-t)^2]\}^{1/2}+M^2\varepsilon \\
&\leqq \sqrt{\frac{5}{3}}t\Big\{E\Big[\Big(\sup\Big\{|f'(u)-f'(v)|;|u-v|\leqq \sup_{\substack{0\leqq i\leqq 2^{2n}-1\\ \tau_{n,i}\leqq s\leqq\tau_{n,i+1}}}|W_{\tau_{n,i}}+\theta\frac{\sqrt{t}}{2^n}-W_s|\Big\}\Big)^4\Big]\Big\}^{1/2} \\
&\quad +M^2\{E[(\tau_{n,2^{2n}}-t)^2]\}^{1/2}+M^2\varepsilon\,. \qquad (10)
\end{aligned}
$$

Here by (6), it holds that with probability one,

$$
\begin{aligned}
\sup_{\substack{0\leqq i\leqq 2^{2n}-1\\ \tau_{n,i}\leqq s\leqq\tau_{n,i+1}}}(s-\tau_{n,i}) &= \sup_{0\leqq i\leqq 2^{2n}-1}(\tau_{n,i+1}-\tau_{n,i}) \\
&\leqq 2\sup_{1\leqq i\leqq 2^{2n}}|\tau_{n,i}(\omega)-i\frac{t}{2^{2n}}|+\frac{t}{2^{2n}} \\
&\to 0 \quad (n\to\infty)\,.
\end{aligned}
$$

From this fact, the uniform continuity of a continuous function defined on a compact interval and the uniform continuity of f', it holds that with probability one,

$$\lim_{n\to\infty}\sup\Big\{|f'(u)-f'(v)|;|u-v|\leqq \sup_{\substack{0\leqq i\leqq 2^{2n}-1\\ \tau_{n,i}\leqq s\leqq\tau_{n,i+1}}}|W_{\tau_{n,i}}+\theta\frac{\sqrt{t}}{2^n}-W_s|\Big\}=0\,.$$

So, by dominated convergence theorem, the first term of the last RHS of (10) converges to zero as $n\to\infty$. In addition, by (5), the second term of the last RHS of (10) and the second term of the RHS of (9) converges to zero as $n\to\infty$. Therefore, we have that

$$\limsup_{n\to\infty} p_n \leqq \frac{12}{\delta}M\varepsilon^{1/2}\to 0 \quad (\varepsilon\to 0)\,.$$

This means that the first term of the RHS of (8) converges to $\displaystyle\int_0^t f'(W_s)\,dW_s$ in probability as $n\to\infty$. So we obtain (3). (Q.E.D.)

Remark

- In [5], Szabados obtained another type of discrete Itô formula as the following.

$$g(Z_t)=\sum_{i=0}^{t-1}f(Z_i)(Z_{i+1}-Z_i)+\frac{1}{2}\sum_{i=0}^{t-1}\frac{f(Z_{i+1})-f(Z_i)}{Z_{i+1}-Z_i}\,, \qquad (11)$$

  where g is defined as

$$g(k)=\mathrm{sgn}(k)\Big\{\frac{1}{2}f(0)+\sum_{j=1}^{|k|-1}f(j\,\mathrm{sgn}(k))+\frac{1}{2}f(k)\Big\}\,.$$

  Furthermore, for $f\in C^1(\boldsymbol{R})$, he derived a new representation of $\int_0^t f(W_s)\,dW_s$, using Itô's formula, his discrete Itô's formula and the same approximation method of Brownian motion by random walks. Even if we use his discrete Itô's formula instead of (1), we can prove Itô's formula. Therefore (1) and (11) are not different in the limit case. But in the discrete case, they are different in that though (1) gives Doob-Meyer decomposition, (11) does not generally do so.

- We can prove Itô's formula for $f\in C^{1,2}(\boldsymbol{R}_+\times\boldsymbol{R})$, using discrete Itô's formula in the explicitly time-dependent case:

$$\begin{aligned} g(t+1,Z_{t+1})-g(t,Z_t) &= \frac{g(t+1,Z_t+1)-g(t+1,Z_t-1)}{2}(Z_{t+1}-Z_t)\\ &\quad+\frac{g(t+1,Z_t+1)-2g(t+1,Z_t)+g(t+1,Z_t-1)}{2}\\ &\quad+g(t+1,Z_t)-g(t,Z_t)\,, \end{aligned}$$

where $g : \boldsymbol{Z}_+ \times \boldsymbol{Z} \to \boldsymbol{R}$ and t is a nonnegative integer. In fact, we have the following from this discrete Itô's formula and appropriate scaling.

$$
\begin{aligned}
&f(t, W(\tau_{n,2^{2n}})) - f(0,0) \\
&\qquad = \sum_{i=0}^{2^{2n}-1} \frac{f((i+1)\frac{t}{2^{2n}}, W_{\tau_{n,i}} + \frac{\sqrt{t}}{2^n}) - f((i+1)\frac{t}{2^{2n}}, W_{\tau_{n,i}} - \frac{\sqrt{t}}{2^n})}{2\frac{\sqrt{t}}{2^n}} \\
&\qquad\qquad \times (W_{\tau_{n,i+1}} - W_{\tau_{n,i}}) \\
&\qquad\quad + \frac{1}{2}\sum_{i=0}^{2^{2n}-1} \{f((i+1)\frac{\sqrt{t}}{2^{2n}}, W_{\tau_{n,i}} + \frac{\sqrt{t}}{2^n}) - 2f((i+1)\frac{\sqrt{t}}{2^{2n}}, W_{\tau_{n,i}}) \\
&\qquad\qquad\qquad + f((i+1)\frac{\sqrt{t}}{2^{2n}}, W_{\tau_{n,i}} - \frac{\sqrt{t}}{2^n})\} \\
&\qquad\quad + \sum_{i=0}^{2^{2n}-1} \{f((i+1)\frac{t}{2^{2n}}, W_{\tau_{n,i}}) - f(i\frac{t}{2^{2n}}, W_{\tau_{n,i}})\},
\end{aligned}
$$

where $f \in C^{1,2}(\boldsymbol{R}_+ \times \boldsymbol{R})$ has a compact support and $t > 0$ is fixed. Here we only have to consider the limit of each terms when we let $n \to \infty$.

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