Edokko Options:A New Framework of Barrier Options [1]

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Abstract. In this paper, we will give a new framework of barrier options to generalize 'Parisian Option' and 'Delayed Barrier Option'. Take a stopping time τ as the caution time. When τ occurs, derivatives are given 'Caution'. After τ, if K.O. time $\sigma = \sigma(\tau)$ occurs, derivative contracts vanish. We simply say that first 'Caution' second 'K.O.'. Using this framework, designs of barrier options become more flexible than before and new risk management will be possible. New barrier options in this category are called *Edokko Options or Tokyo Options*.

Keywords: barrier option, α-percentile option, Parisian option, Delayed Barrier option, Black-Scholes model, option pricing, Edokko option.

1 Introduction

Barrier options are useful and popular derivatives in over the-counter markets because they are less expensive than plain vanilla contracts. Usual barrier options are so called 'one touch options' i.e. the contracts of which are knocked out when the price of the underlying asset S_t hits a prespecified level (K.O. barrier) from above or below. In this barrier option, the option writer might see that the underlying asset approaches the bar and could try to sell the underlying asset intentionally and escape payment. It might be unfair that this kind of price manipulation is possible. So far, 'Parisian Option(Chesney, Jeanblanc-Picque

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and Yor[2])‘ and ‘Cumulative Parisian Option((Chesney, Jeanblanc-Picque and Yor[2])=Delayed Barrier Option(Linetsky[9]) are exotic barrier options which make this price manipulation difficult.

The main purpose of this paper is to generalize ‘Parisian Option‘ and ‘Delayed Barrier Option‘ and give a new framework of barrier options in order to save options from intentional knock out and price new derivatives in this category.

First, the period from a present time to a maturity time is classified into the following maximum three periods. Let τ be a stopping time. Though it is possible to take τ some other exotic stopping times, for simplicity, in this paper, we take for τ as the first hitting time τ_A of the underlying asset S_t to the bar(threshold) A. We call this τ_A *a Caution Time or a 1-st Trigger Time* and $R_S = \{t|0 \leqq t < \tau\}$ *a Safety Region.* As far as derivatives belong to this Safety Region, we decide that derivatives are secured and derivative contracts never vanish. We call $R_C = \{t|t \geqq \tau\}$ *the Caution Region.* If derivatives belong to R_C, they are given ‘Caution‘ and might be knocked out. Once derivatives are given ‘Caution‘, we usually assume that this caution will never disappear till expiry. But we may make other contracts which recover from ‘Caution‘ as we mention later. Whether contracts are knocked out or not is determined by the following. We take σ as *a K.O.Time or a 2-nd Trigger Time.* Let us assume that

1. $\sigma \geqq \tau$.
2. $\sigma = \sigma(\tau), \sigma$ is a random variable which may depend on τ.
3. σ is a $\mathcal{F}_T$ measurable random variable, not necessarily a stopping time. Actually, in later examples we may take a last exit time for σ.

$R_{K.O.} = \{t|T \geqq t \geqq \sigma\}$ is called *Knock Out Region* and if $R_{K.O.} \neq \emptyset$, derivative contracts should be knocked out. In other words, if σ occurs before the maturity time T, contracts vanish. We simply say that first ‘Caution‘ second ‘K.O.‘. Using this framework and adjusting a Caution Threshold A, a Caution Time τ, a K.O. Time σ, K.O. Region $R_{K.O.}$, designs of barrier options become more flexible than before. We may give derivative holders an extra option such that derivatives can escape from ‘Caution‘ by paying some extra money when the Caution Time τ occurs. This extra option of the contract is thought of the same as ‘Insurance‘ in ‘Black Jack‘ game.

New barrier options that belong to this framework are called *Edokko Option, Edokko Barrier Option or Tokyo Option* where ‘Edo‘ means the old name of Tokyo and ‘ko‘ means people. Also this first ‘Caution‘ and second ‘K.O.‘ framework is called *Edokko framework.* In later examples, the K.O. time σ is completely dependent on the caution time τ especially, Remaining Caution Time ($T - \tau$) but it is possible to consider many other cases.

2 Examples

In this section, we give many examples of barrier options which belong to this framework. As the first example, we give the usual one touch barrier option that is considered as $\tau_A = \sigma$.

Example 2.1 *One Touch Option (usual barrier option)*

$R_C = R_{K.O.}$ i.e. 'Caution' immediately gives 'K.O'.

¿From now on, we give examples of exotic barrier options that $\tau_A \neq \sigma$. All the following examples have features such that

- It is more difficult to make price manipulation.
- In the Black Scholes model, there exists closed form expressions of the prices.
- The contents of derivative contracts are easily understood.

We assume that any Caution Time is of the form $\tau_A = \inf\{t | S_t = A\}$.

Example 2.2 *Delayed Barrier Option(Linetsky[10])=Cumulative Parisian Option(Chesney, Jeanblanc-Picque and Yor [2])*

This option is a down-and -out option that is knocked out when the occupation time below the barrier A exceeds a given fraction $\alpha, 0 < \alpha < 1$ of the maturity time T. Using our framework, for $\alpha(0 < \alpha < 1)$,

$$R_{K.O.} = \{t | \int_0^t 1_{(-\infty, A]}(S_u) du \geqq \alpha T\}$$

In other words, we remark that the condition which the α percentile of the underlying asset $S_u(0 \leqq u \leqq T)$ becomes less than A is equivalent to this K.O. condition.

Example 2.3 *Cumulative Parisian Edokko Option*

This option is a down-and -out option that is knocked out when the occupation time below the barrier A exceeds a given fraction $\alpha, 0 < \alpha < 1$ of the remaining caution time $T - \tau_A$. Using our framework,

for $\alpha(0 < \alpha < 1)$,

$$R_{K.O.} = \{t | \int_{\tau_A}^t 1_{(-\infty, A]}(S_u) du \geqq \alpha(T - \tau_A)\}$$

Remark 2.1 $\frac{\int_{\tau_A}^t 1_{(-\infty,A]}(S_u)du}{T-\tau_A} \geqq \alpha \iff \alpha -$ *percentile of* $S_u(\tau_A \leqq u \leqq T) \leqq A$

In other words, we remark that the condition which α percentile of the underlying asset $S_u(\tau_A \leqq u \leqq T)$ becomes less than A is equivalent to this K.O. condition.

Example 2.4 *Parisian Option(Chesney, Jeanblanc-Picque and Yor [2])*

A Parisian option becomes worthless if the underlying asset reaches a prespecified level A and remains continuously below this level for a time interval longer than a fixed number D. Specifying $R_{K.O.}$, for a positive constant D,
$R_{K.O.}$={t|the length of the current excursion below under the level A straddling $t \geqq D$}.

Example 2.5 *Parisian Edokko Option*

A Parisian Edokko option becomes worthless if the underlying asset reaches a prespecified level A and remains continuously below this level for a time interval longer than a fixed number $\alpha(T - \tau_A)$ for $\alpha(0 < \alpha < 1)$. Specifying $R_{K.O.}$, for $\alpha(0 < \alpha < 1)$,
$R_{K.O.}$={t|the length of the current excursion below under the level A straddling $t \geqq \alpha(T - \tau_A)$}

Example 2.6 *Two Touch Option*

The K.O. condition of 'two touch option' is that taking two numbers T_0 and T_1 $(T_0 \leqq T_1 \leqq T)$, $\tau_A < T_0$ and there exists some $t(t \geqq T_1)$ such that $S_t = A$. That is, equivalently, putting g:=the last exit time from A before T, the payoff of 'two touch option' at T is $(1 - 1_{(\tau_A \leqq T_0)} 1_{(g \geqq T_1)}) f(S_T)$ where the payoff of the derivative without K.O. condition is $f(S_T)$. Giving R_C and $R_{K.O.}$,

$$R_C = \begin{cases} \{t | t \geqq \tau_A\} & \text{if} \quad \tau_\mathrm{A} \leqq \mathrm{T}_0 \\ \varnothing & \text{if} \quad \tau_\mathrm{A} > \mathrm{T}_0. \end{cases}$$

$$R_{K.O.} = \begin{cases} \{t \geqq t_0 | {}^{\exists} t_0 > T_1, S_{t_0} \leqq A\} & \text{if} \quad \tau_\mathrm{A} \leqq \mathrm{T}_0 \\ \varnothing & \text{if} \quad \tau_\mathrm{A} > \mathrm{T}_0. \end{cases}$$

Remark 2.2

This 'two touch option' is knocked out if the underlying asset reaches A more than two times. But unconditional two touch option is mathematically meaningless because once Brownian motion or Brownian motion with drift reaches A, it reaches A infinitely many times near that time. For this 'two touch option', if there exists a touch after T_1, this option should be knocked out. Saying more exactly, we should call it 'More Than Two Touch Option'.

Example 2.7 *Two Touch Edokko Option*

The K.O. condition of 'two touch Edokko option' is that taking $\alpha(0 < \alpha < 1)$, there exists some $t(t \geqq (1 - \alpha)\tau_A + \alpha T)$ such that $S_t = A$. That is, the payoff of 'two touch Edokko option'at T is $(1 - 1_{(g \geqq (1-\alpha)\tau_A + \alpha T)}) f(S_T)$ where the payoff of the derivative without K.O. condition is $f(S_T)$. Giving R_C and $R_{K.O.}$, $R_C = \{t | t \geqq \tau_A\}$.

$$R_{K.O.} = \{t \geqq t_0 | {}^{\exists} t_0 > (1 - \alpha)\tau_A + \alpha T), S_{t_0} \leqq A\}$$

Example 2.8 *Monitoring Barrier Option*

After T_0 of the caution time τ_A, if the underlying asset is less than another bar B, a derivative should be knocked out and we call this derivative *Monitoring Barrier Option*. At the time $\tau_A + T_0$, the underlying asset is monitored. That is, for fixed $T_0(< T)$ and B,

$$\tau_A + T_0 \leqq T \quad and \quad S_{\tau_A+T_0} \leqq B \to K.O.$$
$$otherwise \to O.K.(\text{does not vanish})$$

Example 2.9 *Edokko Monitoring Option*

At the time $(1-\alpha)\tau_A + \alpha T (0 < \alpha < 1)$, if the underlying asset is less than another bar B, a derivative should be knocked out and we call this derivative *Edokko Monitoring Option*. At the time $(1-\alpha)\tau_A + \alpha T$, the underlying asset is monitored. That is, for $\alpha(0 < \alpha < 1)$ and B,

$$S_{(1-\alpha)\tau_A+\alpha T} \leqq B \to K.O.$$

$$S_{(1-\alpha)\tau_A+\alpha T} > B \to O.K.$$

Example 2.10 *Simple Parisian Like Option*

After the caution time τ_A, if it takes more than $\alpha T, 0 < \alpha < 1)$ for the underlying asset to return to another bar B, a derivative should be knocked out. In other words,
for fixed $B(> A)$ and $\alpha(0 < \alpha < 1)$,

$$\tau_B^{'} \leqq \alpha T \to K.O.$$

$$\tau_B^{'} > \alpha T \to O.K.$$

where、$\tau_B^{'} = \inf\{t > \tau_A | S_t = B\}$

Remark 2.3

Parisian Option monitors the length of any excursions. On the other hand this derivative monitors the length of the first excursion including the caution time.

Example 2.11 *Simple Parisian Like Edokko Option*

After the caution time τ_A, if it takes more than $\alpha(T-\tau_A),(0<\alpha<1)$ for the underlying asset to return to another bar B, a derivative should be knocked out. In other words,

for fixed $B(>A)$ and $\alpha(0<\alpha<1)$,

$$\tau_B^{'} \leqq (1-\alpha)\tau_A + \alpha T \rightarrow K.O.$$

$$\tau_B^{'} > (1-\alpha)\tau_A + \alpha T \rightarrow O.K.$$

where、$\tau_B^{'} = \inf\{t > \tau_A | S_t = B\}$

Remark 2.4

We obtain the extra money which makes derivatives escape from 'Caution' when the 'Caution Time' $\tau_A = u$ and we notice that it is easily caluculated in all above examples. .

the extra money which makes derivatives escape from 'Caution' when the 'Caution Time' $\tau_A = u$

$$= \quad e^{-r(T-u)}E(1_{(\sigma(u)\leqq T)}f(S_T)|\tau_A = u)$$

The following example is not included in 'Edokko Options' but using this option we can have that smooth hedging of usual barrier options. Linetsky([10]) dealt with similar options using occupation times.

Example 2.12 *Remaining Caution Time Discounted Option*

The payoff of this option at T $=e^{-\lambda(T-\tau_A)}f(S_T)$ for a positive constant λ. We remark that if λ approaches ∞, the payoff of this option approaches the payoff of a usual one touch barrier option.

3 Pricing

We can obtain closed form expressions of the prices of the above-mentioned examples in Black Scholes model. In this section, choosing 'cumulative Parisian Edokko Option' and 'two touch Edokko option' we will show these pricing formulae.

Let $X(t)$ be a continuous stochastic process.
We put $A_X(t,x) = \frac{1}{t}\int_0^t 1_{(-\infty,x]}(X(s))ds$, where

$$1_{(-\infty,x]}(y) = \begin{cases} 1 & \text{if} \quad y \leq x \\ 0 & \text{if} \quad y \geq x \end{cases}$$

Since $A_X(t,\cdot)$ is increasing, the inverse function $m_X(t,\cdot)$ exists i.e.

$$A_X(t, m_X(t,\alpha)) = \alpha\,(0<\alpha<1),\; m_X(t, A_X(t,x)) = x,$$

$$A_X(t,x) > \alpha \rightleftharpoons m_X(t,\alpha) < x$$

hold. Miura([10]) called options related to $m_X(t,\alpha)$ *α-percentile options*. Seeing that $m_X(t,1/2)=$ the median of $X(s)\,(0 \le s \le t)$ and $m_X(t,1-0) = \max_{0\le s\le t} X(s)$, we can observe that *$\alpha$-percentile options* are based on order statistics and have merits that are hardly affected by extreme values. For pricing of *α-percentile options*, see([1],[3],[4],[5],[6],[11]). We use this α-percentile as stopping conditions of derivative contracts. In this sense, we may call **example 2.2 and example 2.3** *α-percentile barrier options*.

Let W_t denote a standard Brownian motion. First we prepare the following theorem about the joint density of Brownian motion and its occupation time. This formula is obtained by ([5]) to price the α percentile option with a payoff $Max(S_T - m_S(T,\alpha), 0)$. This result is equivalent to an occupation time law of Pinned Brownian motion(In the Brownian bridge case, this law is known as "Uniform law"). Actually, we recover the Arcsine law of usual Brownian motions as a marginal distribution.

Theorem 3.1

$$P(W_t \in da, \int_0^t 1_{(-\infty,0]}(W_s)ds \in du)$$
$$= \begin{cases} (\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2(t-s)}} ds)dadu \cdots for\, a > 0 \\ (\int_0^u \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2s}} ds)dadu \cdots for\, a < 0 \end{cases}$$

proof

We put that $f(t,x) = E[1_{[a,+\infty)}(x+W_t)e^{-\beta\int_0^t 1_{(-\infty,0]}(x+W_s)ds}]\,(for\, a>0, \beta > 0)$.

Using the Feynman-Kac Theorem, we have :

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} - \beta 1_{(-\infty,0]}(x)f \quad f(0,x) = 1_{[a,+\infty)}(x).$$

Taking Laplace transforms of both sides, and denoting:$\hat{f}(\xi,x) = \int_0^{+\infty} dt e^{-\xi t} f(t,x)$, we obtain:

$$-1_{[a,+\infty)}(x) + \xi\hat{f} = \frac{1}{2}\frac{\partial^2 \hat{f}}{\partial x^2} - \beta 1_{(-\infty,0]}(x)\hat{f}.$$

Solving this ordinary differential equation and considering boundary conditions at 0 and a, we obtain

$$\hat{f}(0) = \frac{e^{-\sqrt{2\xi}a}}{\sqrt{\xi}(\sqrt{\xi}+\sqrt{\xi+\beta}).}$$

Then, we see

$$\begin{aligned}
\frac{-\partial \hat{f}(0)}{\partial a} &= \sqrt{2}\frac{e^{-\sqrt{2\xi}a}}{\sqrt{\xi}+\sqrt{\xi+\beta}} \\
&= \left(\widehat{\frac{e^{-\beta t}-1}{(-\beta)\sqrt{2\pi t^3}}}\right)\left(\widehat{\frac{\sqrt{2}a}{2\sqrt{\pi t^3}}e^{-\frac{(\sqrt{2}a)^2}{4t}}}\right) \\
&= \left(\widehat{\frac{1-e^{-\beta t}}{\sqrt{2\pi t^3}\beta} * \frac{a}{\sqrt{2\pi t^3}}e^{-\frac{a^2}{2t}}}\right) \\
&= \left(\widehat{\int_0^t \frac{a}{\sqrt{2\pi s^3}}\frac{1-e^{-\beta s}}{\beta}\frac{a}{\sqrt{2\pi (t-s)^3}}e^{-\frac{a^2}{2(t-s)}}ds}\right) \\
&= \left(\widehat{\int_0^t e^{-\beta u}du\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}}e^{-\frac{a^2}{2(t-s)}}ds}\right).
\end{aligned}$$

This shows that for $a > 0$,
$P(W_t \in da,\, \int_0^t 1_{(-\infty,0]}(W_s)ds \in du) = (\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}}e^{\frac{-a^2}{2(t-s)}}ds)dadu.$
Similarily we obtain the joint density function for $a < 0$. (q.e.d.)

Remark 3.1

Chesney, Jeanblanc-Picque and Yor([2]) got the same results by another approach. Also Karatzas and Shreve([8] pp 423, Prop. 3.9) obtained the similar results of this Theorem.

We denote the joint density function of $(W_t, \int_0^t 1_{(-\infty,0]}(W_s)ds)$ by $f_{(W_t,\int_0^t 1_{(-\infty,0]}(W_s)ds)}(a,u)$.
Applying Girsanov's theorem, we get that the joint density function $g_{(X_t^{\mu,\sigma},\int_0^t 1_{(-\infty,0]}(X_s^{\mu,\sigma})ds)}(a,x)$ for a Brownian motion with drift $(\sigma W_t + \mu t = X_t^{\mu,\sigma})$ is:

$$g_{(X_t^{\mu,\sigma},\int_0^t 1_{(-\infty,0]}(X_s^{\mu,\sigma})ds)}(a,x) = e^{-\frac{\mu^2 t}{2\sigma^2}}e^{\frac{\mu a}{\sigma^2}}(1/\sigma)f_{(W_t,\int_0^t 1_{(-\infty,0]}(W_s)ds)}(\frac{a}{\sigma},x).$$

¿From we can determine the price of Cumulative Parisian Edokko Option under the Black-Scholes model.

Under the risk neutral measure in the Black-Scholes model, we take the S.D.E. which the underlying asset price $S(t)$ satisfies as follows:

$$dS_t = rS_tdt + \sigma S_t dW_t, \quad S_0 = S$$

where r= the instantaneous risk free rate, σ= the volatility.

Then we know that

$$S_t = Se^{\sigma W_t + (r-\frac{1}{2}\sigma^2)t}$$

$$= Se^{X^{r-\frac{1}{2}\sigma^2,\sigma}(t)}$$

.

We denote a payoff at maturity T as $f(S_T)$. Considering stopping condition $1_{m_X(T,\alpha)\leqq A}$, we have that a payoff of Cumulatine Parisian Edokko Option at the maturity T $(1-1_{m_X(T-\tau_A,\alpha)\leqq A})f(S_T)$ where we assume that $A<S$. Then the price of Cumulative Parisian Edokko Option ($=C(T,S,\alpha,x)$)is obtained by $C(T,S,\alpha,A)=E(e^{-rT}(1-1_{m_X(T-\tau_A,\alpha)\leqq A})f(S_T))$.

Remarking that $E(e^{-rT}f(S_T))=e^{-rT}\int_{-\infty}^{+\infty}f(Se^{(r-(1/2)\sigma^2)T+\sigma x})\frac{e^{(-1/2)x^2}}{\sqrt{2\pi}}dx=C_1(T,S)$=usual $B.S.$, it is enough to calculate that $E(e^{-rT}1_{m_X(T-\tau_A,\alpha)\leqq x}f(S_T))=C_2(T,S,\alpha,x)$.
So, it is sufficient to obtain the joint density function of $(S_T,\int_0^T 1_{(-\infty,A]}(S_s)ds,\tau_A)$.

$$(S_T,\int_0^T 1_{(-\infty,A]}(S_s)ds)$$

$$=(Se^{(r-(1/2)\sigma^2)T+\sigma W_T},\int_0^T 1_{(-\infty,A]}(Se^{(r-(1/2)\sigma^2)s+\sigma W_s})ds)$$

$$=(Se^{X_T^{(r-(1/2)\sigma^2),\sigma}},\int_0^T 1_{(-\infty,\log{(A/S)}]}(X_s^{(r-(1/2)\sigma^2),\sigma})ds)$$

We put $\tau_A=\inf\{t|S_t={}_A\}=\inf\{t|X_t^{(r-(1/2)\sigma^2),\sigma}=\log A/S\}$.
Then, conditioning $\tau_A=u$, we have

$$(S_T,\int_0^T 1_{(-\infty,A]}(S_s)ds)|_{\tau_A=u}$$

$$=(Ae^{X_T^{(r-(1/2)\sigma^2),\sigma}-X_u^{(r-(1/2)\sigma^2),\sigma}},\int_u^T 1_{(-\infty,A]}(Ae^{X_s^{(r-(1/2)\sigma^2),\sigma}-X_u^{(r-(1/2)\sigma^2),\sigma}})ds)|_{\tau_A=u}$$

$$=(Ae^{\hat{X}_{T-u}^{(r-(1/2)\sigma^2),\sigma}},\int_u^T 1_{(-\infty,0]}(\hat{X}_{s-u}^{(r-(1/2)\sigma^2),\sigma})ds)|_{\tau_A=u}$$

$$=(Ae^{\hat{X}_{T-u}^{(r-(1/2)\sigma^2),\sigma}},\int_0^{T-u} 1_{(-\infty,0]}(\hat{X}_{s}^{(r-(1/2)\sigma^2),\sigma})ds)|_{\tau_A=u}$$

where we put $\hat{X}_t=X_t-X_u$ and we remark that $\hat{X}_t$ is independent $\mathcal{F}_u=\sigma\{X_s;s\leqq u\}$.

So, we see that

$$C_2(T,S,\alpha,A)=E(e^{-rT}1_{m_S(T-\tau_A,\alpha)\leqq A}f(S_T))$$

$$=e^{-rT}E(E(1_{m_S(T-\tau_A,\alpha)\leqq A}f(S_T)|\tau_A))$$

$$= e^{-rT}\int_0^T E(1_{m_S(T-u,\alpha)\leqq A} f(S_T)|\tau_A = u)h_{\tau_A}(u)du$$

$$= e^{-rT}\int_0^T \int\int_{b\geqq A(T-u)} f(Ae^a, b)g_{(X_{T-u}^{(r-(1/2)\sigma^2),\sigma}, \int_0^{T-u} 1_{(-\infty,0]}(X_s^{(r-(1/2)\sigma^2),\sigma})ds)}(a,b)dadbh_\tau(u)du$$

where we recall the known result $h_{\tau_A}(s)$=the density function of $h_{\tau_A} = \frac{|\log A/S|}{\sigma\sqrt{2\pi s^3}} e^{\frac{-(\log A/S-(r-(1/2)\sigma^2)s)^2}{2s\sigma^2}}$.

That is,
the Price of Cumulative Parisian Edokko Option

$$= \quad B.S. - C_2(T, S, \alpha, A).$$

Next we would like to obtain the price of two touch Edokko option.

First we prepare that some joint density results from Brownian Motion.

$$P(W_1 \in du, \int_0^1 1_{(-\infty,0]}(W_s)ds \in da,\ g \in ds)$$

$$= \begin{cases} \frac{u}{\pi\sqrt{s^3(1-s)^3}} e^{\frac{-u^2}{2(1-s)}} dsdadu \cdots for \quad 0 < a < s < 1 \quad and \quad 0 < u < \infty \\ \frac{-u}{\pi\sqrt{s^3(1-s)^3}} e^{\frac{-u^2}{2s}} dsdadu \cdots for \quad 0 < a < s < 1 \quad and \ -\infty < u < 0 \end{cases}$$

$$where \quad g = \sup\{u < 1 | W_u = 0\}.$$

Decomposing the Brownian Path before and after g, we obtain the proof of this formula which appeared on ([2],[8]).

By Girsanov's Theorem and time scale change, we have that

$$P(W_T + \mu T \in du, \int_0^T 1_{(-\infty,0]}(W_s + \mu s)ds \in da,\ g_T \in ds)$$

$$= \begin{cases} e^{\mu u-(1/2)\mu^2 T} \frac{u}{\pi\sqrt{s^3(T-s)^3 T^3}} e^{\frac{-u^2}{2(T-s)}} dsdadu \cdots for \quad 0 < a < s < T \quad and \quad 0 < u < \infty \\ e^{\mu u-(1/2)\mu^2 T} \frac{-u}{\pi\sqrt{s^3(T-s)^3 T^3}} e^{\frac{-u^2}{2s}} dsdadu \cdots for \quad 0 < a < s < T \quad and \quad -\infty < u < 0 \end{cases}$$

$$where \quad g_T = \sup\{u < T | W_u + \mu u = 0\}$$

Price of two touch Edokko option

$$
\begin{aligned}
&= e^{-rT}E[(1-1_{((1-\alpha)\tau_A+\alpha T<g))})f(S_T)] \\
&= C_1(T,S) - e^{-rT}\int_0^T h_{\tau'_A}(t)dt \\
&\quad \int\int\int_{0<a<s<T-t,-\infty<u<\infty,(1-\alpha)t+\alpha(T-t)<s} f(Ae^{\sigma u}) \\
&\quad f_{(W_{T-t}+(1/\sigma)(r-(1/2)\sigma^2)(T-t),\, \int_0^{T-t}1_{(-\infty,0]}(W_s+(1/\sigma)(r-(1/2)\sigma^2)s)ds,\, g_{T-t})}(u,a,s)dudads \\
&= B.S. - e^{-rT}\int_0^T h_{\tau'_A}(t)dt\int_{\alpha(T-t)}^{T-t}ds\int_0^{\infty} f(Ae^{\sigma u}) \\
&\quad e^{(1/\sigma)(r-(1/2)\sigma^2)u-(1/2)(1/\sigma)^2(r-(1/2)\sigma^2)^2(T-t)}\frac{u}{\pi\sqrt{s(T-t-s)^3(T-t)^3}}e^{\frac{-u^2}{2(T-t-s)}}du \\
&\quad -e^{-rT}\int_0^T h_{\tau'_A}(t)dt\int_{\alpha(T-t)}^{T-t}ds\int_{-\infty}^{0} f(Ae^{\sigma u}) \\
&\quad e^{(1/\sigma)(r-(1/2)\sigma^2)u-(1/2)(1/\sigma)^2(r-(1/2)\sigma^2)^2(T-t)}\frac{-u}{\pi\sqrt{s(T-t-s)^3(T-t)^3}}e^{\frac{-u^2}{2s}}du \\
&\quad \textit{where} \quad A' = (1/\sigma)\log(A/S) \quad \textit{and} \quad h_{\tau'_A}(s) = \frac{|A'|}{\sqrt{2\pi s^3}}e^{\frac{-(A'-(1/\sigma)^2(r-(1/2)\sigma^2)s)^2}{2s}}.
\end{aligned}
$$

References

1. Akahori, J.(1995) Some formulae for a new type of path-dependent option. *Ann. Appl. Probab.* **5**, 383-388.

2. Chesney,M.,M. Jeanblanc-Picque, and M. Yor (1997) Brownian excursions and Parisian barrier options, *Adv.Appl. Prob.***29**, 165-184.

3. Dassios, A.(1995) The distribution of the quantile of a Brownian motion with drift and the pricing related path-dependent options. *Ann. Appl. Probab.* **5**, 389-398.

4. Embrechts, P., Rogers,L. C. G., and Yor, M (1995) A proof of Dassios's representation of the α-quantile of Brownian motion with drift. *Ann. Appl. Probab.* **5** 757-767.

5. Fujita, T. (1997) On the price of the α-percentile options. *Hitotsubashi University Faculty of Commerce Working Paper Series* No.24.

6. Fujita,T., (2000)A Note on the Joint Distribution of α, β-Percentiles and Its Application to the Option Pricing, *Asia-Pacific Financial Markets*, **7**(4) December , 339-344.

7. Karatzas,I. and Shreve,S.E.,(1987) A decomposition of the Brownian Path, *Staistics & Probability Letters*, **5**, 87-93.

8. Karatzas,I. and Shreve,S.E.,(1991) Brownian Motion and Stochastic Calculus, Springer.

9. Linetsky,V.,(1999) Step Options, *Mathematical Finance*,**9**,No.1, 55-96.

10. Miura, R. (1992) A note on look-back option based on order statistics. *Hitotsubashi Journal of Commerce and Management* **27**, 15-28.

11. Yor, M (1995) The distribution of Brownian quantiles. *J.Appl.Prob.* **2**, 405-416.