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ON THE OPTIMAL STATIONARY STATE FOR THE QUASI-STATIONARY MODEL OF CAPITAL ACCUMULATION UNDER UNCERTAINTY: THE CHARACTERIZATION OF THE DISCOUNTED GOLDEN-RULE STATE BY PRICES*

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Abstract

A general quasi-stationary model of capital accumulation under uncertainty is considered and the discounted golden-rule state is characterized by prices. A support price for the discounted golden-rule state, which is a finitely additive vector-valued measure, is proved to exist. By using the support price, the discounted golden-rule state is shown to be an optimal stationary state. Also, under the assumption of monotonicity, support prices are proved to be integrable functions.

Key words: Optimal stationary state; Quasi-stationary model; Uncertainty; Support price.
JEL classification: C02, C61, D80, D90, O41

I. *Introduction*

In this paper we shall prove the existence of a support price for the optimal stationary state in a general quasi-stationary model of capital accumulation under uncertainty. The support price is useful in proving the existence and the turnpike property of optimal programs of capital accumulation.

The economic model presented in this paper is a reduced stochastic model of capital accumulation. We assume that the economy is stationary in that probability distributions of production technologies and instantaneous utility functions are the same at all periods in time. However, our model is quasi-stationary since future utilities are discounted. Stationary models in which future utilities are not discounted were considered by Radner (1973), Dana (1973), Evstigneev (1974), Jeanjean (1974), Zilcha (1976-a), and so on. Also, quasi-stationary models have been considered in many literatures such as McKenzie (1986) in a deterministic case, Zilcha (1976-b), Brock & Majumdar (1978), Brock & Magill (1979), Takekuma (1988), and

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Yano (1989) in stochastic models.

In this paper we shall characterize the discounted golden-rule state by prices in a stochastic quasi-stationary model. The discounted golden-rule state was originally defined by Khan & Mitra (1986) in a deterministic model. We shall prove that the discounted golden-rule state is supported by a finitely additive vector-valued measure. By using its support price, the discounted golden-rule state is easily shown to be an optimal stationary state. Furthermore, under the assumption of monotonicity, the support price for the optimal stationary state will be proved to be an integrable function. Also, if the discounted golden-rule state is in "interior", it will be shown that its support price must be an integrable function. In our argument we shall use the technique originated by Bewley (1972) in equilibrium analysis and Zilcha (1976-a) in growth theory.

This paper is formulated in the following fashion. In section 2 we shall discuss the case of a deterministic quasi-stationary model, which makes it easier to consider the case of stochastic models. In section 3 a general quasi-stationary model under uncertainty is presented and in section 4 a support price for the discounted golden-rule state will be found in the space of finitely additive vector-valued measures. In section 5 the support price for the optimal stationary state will be shown to belong to the space of integrable functions.

II. *A Quasi-Stationary Model in the Deterministic Case*

Let $T=\{1, 2, 3, \cdots\}$ be the space of time. To describe an economy in the period between any two adjacent points in time, we use a subset $X \subset R^n_+ \times R^n_+ \times R$, where R^n_+ is the non-negative orthant of an n-dimensional Euclidean space R^n and $R=R^1$. Let x and y denote amounts of capital stock and u be a level of instantaneous utility. Then, $(x, y, u) \in X$ means that the amount x of capital stock at time $t-1$ can be transformed into the amount y of capital stock at time t and the level u of social welfare can be attained in the period between time $t-1$ and time t.

A *feasible program* of capital accumulation starting from an initial amount $x_0 \in R^n_+$ of capital stock is described by a sequence $\{(x_t, u_t) | t \in T\}$ such that $(x_{t-1}, x_t, u_t) \in X$ for all $t \in T$.

We assume the following conditions on the economy.

(*A.2.1*) (*convexity*): The set X is a convex subset of R^{2n+1}.
(*A.2.2*) (*boundedness*): There exist numbers $\bar{b}$ and $\bar{u}$ such that $(x, y, u) \in X$ implies that $|y| \leq \max\{\bar{b}, |x|\}$ and $|u| \leq \max\{\bar{u}, |x|\}$.
(*A.2.3*) (*expansibility*): There exists $(\bar{x}, \bar{y}, \bar{u}) \in X$ such that $\bar{x} - \delta\bar{y} < 0$.
(*A.2.4*) (*monotonicity*): If $(x, y, u) \in X$ and $x \leq x'$, then $(x', y, u) \in X$.

Lemma 2.1: Any feasible program is bounded.

Proof: Let $\{(x_t, u_t) | t \in T\}$ be a feasible program starting from $x_0 \in R^n_+$. Then, by (A.2.2), $|x_t| \leq \max\{\bar{b}, |x_0|\}$ and $|u_t| \leq \max\{\bar{u}, |x_0|\}$ for all $t \in T$. Q.E.D.

Let δ be a discount rate which is a fixed number such that $0 < \delta < 1$. For a feasible program $\{(x_t, u_t) | t \in T\}$ starting from $x_0 \in R^n_+$, the infinite sum, $\sum_{t=1}^{+\infty} \delta^{t-1} u_t$, can be defined. Thus, under assumptions (A.2.1) and (A.2.2), we can consider the following optimization problem:

$$\text{Max} \sum_{t=1}^{\infty} \delta^{t-1} u_t \text{ subject to } (x_{t-1}, x_t, u_t) \in X \text{ for all } t \in T \text{ and } x_0 \text{ is given.}$$

A feasible program $\{(x_t, u_t) \mid t \in T\}$ which is a solution for the above problem is said to be *an optimal program* starting from $x_0 \in R^n_+$.

A pair (x^*, u^*) of an amount x of capital stock and a level u^* of social welfare is called *a stationary state* if $(x^*, x^*, u^*) \in X$. For a stationary state (x^*, u^*), by letting $x_t = x^*$ and $u_t = u^*$ for each $t \in T$, a feasible program $\{(x_t, u_t) \mid t \in T\}$ starting from x^* can be defined. Such a program is called a *stationary program*. The stationary state (x^*, u^*) is said to be *optimal* if the stationary program starting from x^* is optimal.

A vector $p^* \in R^n$ is called *a support price* for stationary state (x^*, u^*) if

$$u^* - p^* \cdot (1-\delta) x^* \geq u - p^* \cdot (x - \delta y) \text{ for all } (x, y, u) \in X.$$

Lemma 2.2: If there is a support price for a stationary state (x^*, u^*), then (x^*, u^*) is an optimal stationary state, i.e., the stationary program starting from x^* is optimal.

Proof: Let p^* be a support price for a stationary state (x^*, u^*). Then, for any feasible program $\{(x_t, u_t) \mid t \in T\}$ starting from x^*, we have

$$u^* - p^* \cdot (1-\delta) x^* \geq u_t - p^* \cdot (x_{t-1} - \delta x_t) \text{ for each } t \in T.$$

Therefore,

$$\sum_{t-1}^{s} \delta^{t-1} u^* - p \cdot x^* + \delta^s p \cdot x^* \geq \sum_{t-1}^{s} \delta^{t-1} u_t - p^* \cdot x^* + \delta^s p^* \cdot x_s \text{ for each } s \in T.$$

By letting s go to infinity, we have

$$\sum_{t=1}^{\infty} \delta^{t-1} u^* \geq \sum_{t=1}^{\infty} \delta^{t-1} u_t,$$

which implies that the stationary program starting from x^* is optimal. Q.E.D.

According to the definition of Khan & Mitra (1986), a stationary state (x^*, u^*) is called *a discounted golden-rule state* if

$$u^* \geq u \quad \text{for all } (x, y, u) \in X \text{ with } x - \delta y = (1-\delta) x^*.$$

A discounted golden-rule state is illustrated in Figure 1.

Theorem 2.1: A stationary state (x^*, u^*) is a discounted golden-rule state if and only if there exists a support price for (x^*, u^*).

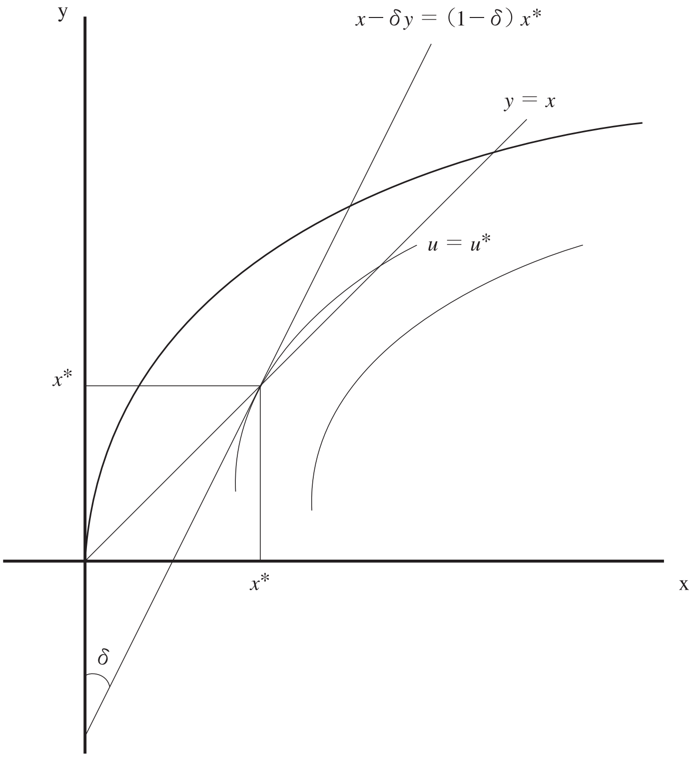
Proof: Let (x^*, u^*) be a discounted golden-rule state. Define a convex subset of $R \times R^n$ by

$$A = \{(a, z) \mid u > a, z = x - \delta y \text{ for some } (x, y, u) \in X\}.$$

Then, $(u^*, (1-\delta) x^*) \notin A$. Therefore, by a separation theorem, there exists a non-zero vector $(c, -p^*) \in R \times R^n$ such that

$$c u^* - p^* \cdot (1-\delta) x^* \geq c a - p^* \cdot z \text{ for all } (a, z) \in A.$$

FIG. 1



From the shape of set A and (A.2.4), it follows that $c \geq 0$ and $p^* \geq 0$. Also, if $c=0$, then, by (A. 2.3), we have $0 \geq -p^* \cdot (1-\delta)x^* \geq -p^* \cdot (\bar{x}-\delta\bar{y}) > 0$, which is a contradiction. Hence, we can assume that $c=1$. Thus, p^* is a support price for (x^*, u^*).

Conversely, let p^* be a support price for (x^*, u^*), i.e.,

$$u^* - p^* \cdot (1-\delta)x^* \geq u - p^* \cdot (x-\delta y) \text{ for all } (x, y, u) \in X.$$

Clearly, if $(x, y, u) \in X$ and $u > u^*$, then $(1-\delta)x^* \neq x - \delta y$, which implies that (x^*, u^*) is a discounted golden-rule state. Q.E.D.

By Lemma 2.2 and Theorem 2.1, we immediately have the following corollary.

Corollary 2.1: Any discounted golden-rule state is an optimal stationary state.

Also, by Theorem 2.1, we have the following corollary.

Corollary 2.2: A stationary state (x^*, u^*) is a discounted golden-rule state if and only if there exists a vector p^* such that

$$p^* \cdot (1-\delta)x^* < p^* \cdot (x-\delta y) \quad \text{for all } (x, y, u) \in X \quad \text{with } u > u^*.$$

Proof: If (x^*, u^*) is a discounted golden-rule state, then, by Theorem 2.1, there exists a support price p^* for (x^*, u^*), i.e.,

$$u^* - p^* \cdot (1-\delta)x^* \geq u - p^* \cdot (x - \delta y) \text{ for all } (x, y, u) \in X.$$

Clearly, when $(x, y, u) \in X$ and $u > u^*$, $p^* \cdot (1-\delta)x^* < p^* \cdot (x - \delta y)$.

Conversely, assume that there is a vector p^* such that

$$p^* \cdot (1-\delta)x^* < p^* \cdot (x - \delta y) \quad \text{for all } (x, y, u) \in X \quad \text{with } u > u^*.$$

Clearly, if $(x, y, u) \in X$ and $x - \delta y = (1-\delta)x^*$, then $u^* \geq u$. Thus, (x^*, u^*) is a discounted golden-rule state. Q.E.D.

III. *A Quasi-Stationary Stochastic Model*

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Each element in Ω denotes a possible state of nature, which may be interpreted as a stream of environments in all past, present, and future periods. Family $\mathscr{F}$ is the set of all possible events and P denotes the probability distribution of states. $\mathscr{F}_0$ is a σ-subfield of $\mathscr{F}$ which is the information about states that is known at time 0. The information that will become known up to time $t \in T$ is described by a filtration $\{\mathscr{F}_t \mid t \in T\}$, i.e., $\mathscr{F}_t$ is a σ-subfield of $\mathscr{F}$ and $\mathscr{F}_{t-1} \subset \mathscr{F}_t$ for all $t \in T$. We assume that probability space $(\Omega, \mathscr{F}_t, P)$ is complete for each $t \in T$.

We assume the stationarity of information and probability distribution over time.

(A.3.1) (*stationarity of uncertainty*): There is a mapping $\tau: \Omega \to \Omega$ such that, for each $t \in T$, mapping $\tau: (\Omega, \mathscr{F}_t, P) \to (\Omega, \mathscr{F}_{t-1}, P)$ is measurability- and measure-preserving:
(1) Mapping $\tau: (\Omega, \mathscr{F}_t) \to (\Omega, \mathscr{F}_{t-1})$ is one to one and onto, and both τ and its inverse τ^{-1} are measurable.
(2) $P(\tau^{-1}(E)) = P(E)$ for all $E \in \mathscr{F}_{t-1}$.

The above assumption means that two probability spaces $(\Omega, \mathscr{F}_t, P)$ and $(\Omega, \mathscr{F}_{t-1}, P)$ are isomorphic to each other under mapping τ. In fact, $\tau(\mathscr{F}_t) = \mathscr{F}_{t-1}$ and $P(\tau(E)) = P(E)$ for all $E \in \mathscr{F}_t$. Hence, $(\Omega, \mathscr{F}_t, P)$ and $(\Omega, \mathscr{F}_{t-1}, P)$ are equivalent.

Under assumption (A.3.1), for any $\mathscr{F}_{t-1}$-measurable function $\mathbf{f}: \Omega \to R^n$, by (1) of (A.3.1), there is an $\mathscr{F}_t$-measurable function $\mathbf{g}: \Omega \to R^n$ defined by $\mathbf{g} = \mathbf{f} \circ \tau$. Conversely, for any $\mathscr{F}_t$-measurable function $\mathbf{g}: \Omega \to R^n$, there is an $\mathscr{F}_{t-1}$-measurable function $\mathbf{f}: \Omega \to R^n$ defined by $\mathbf{f} = \mathbf{g} \circ \tau^{-1}$. In addition, by (2) of (A.3.1), we have

$$P(\mathbf{g}^{-1}(B)) = P((\mathbf{f} \circ \tau)^{-1}(B)) = P(\tau^{-1}(\mathbf{f}^{-1}(B)) = P(\mathbf{f}^{-1}(B))$$

for any Borel subset B of R^n. Therefore, functions $\mathbf{f}$ and $\mathbf{g}$ can be regarded as the same random variable only except that periods in time are different.

Let $\omega \in \Omega$ and $\omega' = \tau(\omega)$. Then, state ω' can be regarded as state ω, except that state ω' happens one period earlier. Thus, mapping τ is a time forward-shifting operator.

To show the structure of production technology and social welfare in the economy in period from time $t-1$ to time t, we use a relation, $\Psi_t: \Omega \to R^n \times R^n \times R$, whose graph is

$$G(\Psi_t) = \{(x, y, u, \omega) \mid (x, y, u) \in \Psi_t(\omega)\},$$

is $\mathscr{B}(R^n) \otimes \mathscr{B}(R^n) \otimes \mathscr{B}(R) \otimes \mathscr{F}_t$-measurable, where $\mathscr{B}(R^n)$ is the family of all Borel subsets of

R^n. For $\omega \in \Omega$ and $(x, y, u) \in R^n \times R^n \times R$, $(x, y, u) \in \Psi_t(\omega)$ means that, under state ω, the amount x of capital stock at time $t-1$ can be transformed into the amount y of capital stock at time t and the level u of social welfare can be attained in the period between time $t-1$ and time t.

We assume the stationarity of relation Ψ_t.

(A.3.2) (*stationarity of production technology and social welfare*): For each $t \in T$, $\Psi_t = \Psi_t \circ \tau^{t-1}$, i.e.,

$$\Psi_t(\omega) = \Psi_1(\tau^{t-1}(\omega)) \text{ for all } \omega \in \Omega.$$

Here, τ^{t-1} denotes the $(t-1)$-time composite of mapping τ.

We assume the convexity and boundedness of the economy. Since the model is stationary by (A.3.2), we only have to assume conditions on relation Ψ_1. In what follows, we write Ψ instead of Ψ_1.

(A.3.3) (*convexity*): For each $\omega \in \Omega$, $\Psi(\omega)$ is a convex subset of $R^n \times R^n \times R$.
(A.3.4) (*boundedness*): There exist numbers $\bar{b}$ and $\bar{u}$ such that $(x, y, u) \in \Psi(\omega)$ implies that $|y| \leq \max\{\bar{b}, |x|\}$ and $|u| \leq \max\{\bar{u}, |x|\}$.

Let us denote the set of all essentially bounded $\mathscr{F}_t$-measurable functions on $(\Omega, \mathscr{F}_t, P)$ to R^n by $\mathscr{L}_\infty^n(\mathscr{F}_t)$. When $n=1$, we write $\mathscr{L}_\infty(\mathscr{F}_t)$ for $\mathscr{L}_\infty^1(\mathscr{F}_t)$. A feasible program of capital accumulation can be described by a stochastic process $\{(\mathbf{k}_t, \mathbf{u}_t) | t \in T\}$, where $\mathbf{k}_t \in \mathscr{L}_\infty^n(\mathscr{F}_t)$ and $\mathbf{u}_t \in \mathscr{L}_\infty(\mathscr{F}_t)$. $\mathbf{k}_t(\omega)$ and $\mathbf{u}_t(\omega)$ denote the quantities of capital stock and the level of utility at time t in state ω respectively. By (A.3.1), $\mathbf{k}_t \circ \tau^{-t} \in \mathscr{L}_\infty^n(\mathscr{F}_0)$ and $\mathbf{u}_t \circ \tau^{-t} \in \mathscr{L}_\infty^n(\mathscr{F}_0)$ for all $t \in T$, and therefore any program is described by a sequence of $\mathscr{F}_0$-measurable functions.

A process $\{(\mathbf{k}_t, \mathbf{u}_t) | t \in T\}$ is called *a feasible program starting from* $\mathbf{k}_0 \in \mathscr{L}_\infty^n(\mathscr{F}_0)$ if $(\mathbf{k}_{t-1}(\omega), \mathbf{k}_t(\omega), \mathbf{u}_t(\omega)) \in \Psi_t(\omega)$ a. s. for each $t \in T$. The structure of the economy in period from time 0 to time 1 is defined by

$$\mathscr{X} = \{(\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{L}_\infty^n(\mathscr{F}_0) \times \mathscr{L}_\infty^n(\mathscr{F}_0) \times \mathscr{L}_\infty(\mathscr{F}_0) \,|\, (f(\omega), g(\tau(\omega)), u(\tau(\omega)) \in \Psi(\omega) \; a.s.\}$$

Then, a process $\{(\mathbf{k}_t, \mathbf{u}_t) | t \in T\}$ is *a feasible program starting from* $\mathbf{k}_0$ if and only if $(\mathbf{k}_{t-1} \circ \tau^{1-t}, \mathbf{k}_t \circ \tau^{-t}, \mathbf{u}_t \circ \tau^{-t}) \in \mathscr{X}$ for all $t \in T$.

Lemma 3.1: Any feasible program is bounded.

Proof: It is easy to prove this lemma by (A.3.4). Q.E.D.

For any feasible program $\{(\mathbf{k}_t, \mathbf{u}_t) | t \in T\}$ starting from $\mathbf{k}_0 \in \mathscr{L}_\infty^n(\mathscr{F}_0)$, by Lemma 3.1, the infinite sum of expected utilities,

$$\sum_{t=1}^{+\infty} \delta^{t-1} \int_\Omega \mathbf{u}_t dP,$$

can be defined. Given $\mathbf{k}_0 \in \mathscr{L}_\infty^n(\mathscr{F}_0)$, a feasible program $\{(\mathbf{k}_t, \mathbf{u}_t) | t \in T\}$ starting from $\mathbf{k}_0$ that maximizes the above infinite sum is said to be *an optimal program starting from* $\mathbf{k}_0$.

Let $(\mathbf{k}, \mathbf{u}) \in \mathscr{L}_\infty^n(\mathscr{F}_0) \times \mathscr{L}_\infty(\mathscr{F}_0)$ be a pair of functions such that $(\mathbf{k}, \mathbf{k}, \mathbf{u}) \in \mathscr{X}$. Then, a program $\{(\mathbf{k} \circ \tau^t, \mathbf{u} \circ \tau^t) | t \in T\}$ is a feasible program starting from $\mathbf{k}$. Such a program is called *a stationary program* and $(\mathbf{k}, \mathbf{u})$ is called *a stationary state*. In a stationary program, an identical plan is repeated forever, i.e., the same amount of capital stock is planned to be accumulated and the same level of utility is planned to be attained in the same environment independently of time. A stationary state $(\mathbf{k}, \mathbf{u})$ is called *an optimal state* if the stationary program $\{(\mathbf{k}\circ\tau^t, \mathbf{u}\circ\tau^t) \mid t\in T\}$ is optimal.

IV. *Support Prices in* $ba^n(\mathscr{F}_0)$

Let $\mathscr{L}^n_\infty(\mathscr{F}_0)^*$ denote the dual space of $\mathscr{L}^n_\infty(\mathscr{F}_0)$, i.e., the set of all continuous linear functions on $\mathscr{L}^n_\infty(\mathscr{F}_0)$ to R.

Remark 4.1: Let $ba^n(\mathscr{F}_0)$ be the set of all bounded finitely additive n-dimensional vector-valued measures on $\mathscr{F}_0$ absolutely continuous with respect to P. By a theorem [Dunford & Schwarts (1964), Thm.IV. 8.16, p.296], $ba^n(\mathscr{F}_0)$ can be regarded as the dual space of $\mathscr{L}^n_\infty(\mathscr{F}_0)$. That is, for each $\boldsymbol{\pi}\in ba^n(\mathscr{F}_0)$, a continuous linear functional on $\mathscr{L}^n_\infty(\mathscr{F}_0)$ to R is defined by

$$\mathbf{f}\in\mathscr{L}^n_\infty(\mathscr{F}_0) \quad\rightarrow\quad \boldsymbol{\pi}\cdot\mathbf{f}=\sum_{i=1}^{n}\int_\Omega f_i d\pi_i\in R$$

and $\mathscr{L}^n_\infty(\mathscr{F}_0)^*$ can be regarded as the set of such functionals.

A vector-valued measure $\boldsymbol{\pi}^*\in ba^n(\mathscr{F}_0)$ is called *a support price* for stationary state $(\mathbf{k}^*, \mathbf{u}^*)$ if

$$\int_\Omega \mathbf{u}^* dP-\boldsymbol{\pi}^*\cdot(1-\delta)\mathbf{k}^*\geq\int_\Omega \mathbf{u}dP-\boldsymbol{\pi}^*\cdot(\mathbf{f}-\delta\mathbf{g}) \text{ for all } (\mathbf{f}, \mathbf{g}, \mathbf{u})\in\mathscr{X}.$$

Theorem 4.1: If there is a support price for a stationary state $(\mathbf{k}, \mathbf{u})$, then $(\mathbf{k}, \mathbf{u})$ is an optimal stationary state, i.e., the program, $\{(\mathbf{k}\circ\tau^t, \mathbf{u}\circ\tau^t) \mid t\in T\}$, starting from $\mathbf{k}$ is optimal.

Proof: Let $\boldsymbol{\pi}\in ba^n(\mathscr{F}_0)$ be a support price for $(\mathbf{k}, \mathbf{u})$. Then, for any feasible program $\{(\mathbf{k}_t, \mathbf{u}_t) \mid t\in T\}$ starting from $\mathbf{k}_0=\mathbf{k}$, we have

$$\int_\Omega \mathbf{u}dP-\boldsymbol{\pi}\cdot(1-\delta)\mathbf{k}\geq\int_\Omega \mathbf{u}_t\circ\tau^{-t}dP-\boldsymbol{\pi}\cdot(\mathbf{k}_{t-1}\circ\tau^{1-t})+\delta\boldsymbol{\pi}\cdot(\mathbf{k}_t\circ\tau^{-t})$$

for all $t\in T$. Therefore,

$$\sum_{t=1}^{s}\delta^{t-1}\int_\Omega \mathbf{u}dP-\boldsymbol{\pi}\cdot\mathbf{k}+\delta^s\boldsymbol{\pi}\cdot\mathbf{k}\geq\sum_{t=1}^{s}\delta^{t-1}\int_\Omega \mathbf{u}_t\circ\tau^{-t}dP-\boldsymbol{\pi}\cdot\mathbf{k}+\delta^s\boldsymbol{\pi}\cdot(\mathbf{k}_s\circ\tau^{-s})$$

for all $s>0$. By letting s go to infinity, it follows that

$$\sum_{t-1}^{+\infty}\delta^{t-1}\int_\Omega \mathbf{u}dP\geq\sum_{t-1}^{+\infty}\delta^{t-1}\int_\Omega \mathbf{u}_t\circ\tau^{-t}dP, \text{ i.e.,}$$

$$\sum_{t=1}^{+\infty}\delta^{t-1}\int_\Omega \mathbf{u}\circ\tau^t d(P\circ\tau^{-t})\geq\sum_{t=1}^{+\infty}\delta^{t-1}\int_\Omega \mathbf{u}_t d(P\circ\tau^{-t}).$$

Hence, by (A.3.1), we have

$$\sum_{t-1}^{+\infty}\delta^{t-1}\int_{\Omega}\mathbf{u}\circ\tau^{t}dP\geq\sum_{t-1}^{+\infty}\delta^{t-1}\int_{\Omega}\mathbf{u}_{t}dP,$$

which implies that program $\{(\mathbf{k}\circ\tau^t, \mathbf{u}\circ\tau^t \mid t\in T)\}$ is optimal. Q.E.D.

In order to have support prices, we need the following extra assumptions.

(*A.4.1*) (*monotonicity*): If $(x, y, u)\in\Psi(\omega)$ and $x\leq x'$, then $(x', y, u)\in\Psi(\omega)$.
(*A.4.2*) (*expansibility*): There exists $(\bar{\mathbf{f}}, \bar{\mathbf{g}}, \bar{\mathbf{u}})\in\mathscr{X}$ such that $\bar{\mathbf{f}}-\delta\bar{\mathbf{g}}<0$.

A stationary state $(\mathbf{k}^*, \mathbf{u}^*)$ is called *a discounted golden-rule state* if

$$\int_{\Omega}\mathbf{u}^*dP\geq\int_{\Omega}\mathbf{u}dP \quad \text{for all } (\mathbf{f}, \mathbf{g}, \mathbf{u})\in\mathscr{X} \quad \text{with } \mathbf{f}-\delta\mathbf{g}=(1-\delta)\mathbf{k}^*.$$

Theorem 4.2: A stationary state $(\mathbf{k}^*, \mathbf{u}^*)$ is a discounted golden-rule state if and only if there exists a support price for $(\mathbf{k}^*, \mathbf{u}^*)$.

Proof: Let $(\mathbf{k}^*, \mathbf{u}^*)$ be a discounted golden-rule state. Define a convex subset of $R\times\mathscr{L}^n_\infty(\mathscr{F}_0)$ by

$$\mathscr{A}=\{(a, \mathbf{h})\mid\int_{\Omega}\mathbf{u}dP>a, \mathbf{h}=\mathbf{f}-\delta\mathbf{g} \text{ for some } (\mathbf{f}, \mathbf{g}, \mathbf{u})\in\mathscr{X}\}.$$

Then, assumption (A.4.1) implies that set $\mathscr{A}$ has a non-empty interior. Also, $(\int_{\Omega}\mathbf{u}^*dP, (1-\delta)\mathbf{k}^*)\notin\mathscr{A}$. Therefore, by a separation theorem [Dunford & Schwartz (1964), Thm. V.2.8, p.417], there exists a non-zero vector $(c, -\boldsymbol{\pi}^*)\in R\times ba^n(\mathscr{F}_0)$ such that

$$c\int_{\Omega}\mathbf{u}^*dP-\boldsymbol{\pi}^*\cdot(1-\delta)\mathbf{k}^*\geq ca-\boldsymbol{\pi}^*\cdot\mathbf{h} \text{ for all } (a, \mathbf{h})\in\mathscr{A}.$$

From the shape of set $\mathscr{A}$ and (A.4.1), it follows that $c\geq0$ and $\boldsymbol{\pi}^*\geq0$. Also, if $c=0$, then, by (A.4.2), we have $0\geq-\boldsymbol{\pi}^*\cdot(1-\delta)\mathbf{k}^*\geq-\boldsymbol{\pi}^*\cdot(\bar{\mathbf{f}}-\delta\bar{\mathbf{g}})>0$, which is a contradiction. Hence, we can assume that $c=1$. Thus, $\boldsymbol{\pi}^*$ is a support price for $(\mathbf{k}^*, \mathbf{u}^*)$.

Conversely, let $\boldsymbol{\pi}^*$ be a support price for $(\mathbf{k}^*, \mathbf{u}^*)$, i.e.,

$$\int_{\Omega}\mathbf{u}^*dP-\boldsymbol{\pi}^*\cdot(1-\delta)\mathbf{k}^*\geq\int_{\Omega}\mathbf{u}dP-\boldsymbol{\pi}^*\cdot(\mathbf{f}-\delta\mathbf{g}) \text{ for all } (\mathbf{f}, \mathbf{g}, \mathbf{u})\in\mathscr{X}.$$

Clearly, if $\int_{\Omega}\mathbf{u}dP>\int_{\Omega}\mathbf{u}^*dP$ for some $(\mathbf{f}, \mathbf{g}, \mathbf{u})\in\mathscr{X}$, then $(1-\delta)\mathbf{k}^*\neq\mathbf{f}-\delta\mathbf{g}$, which implies that $(\mathbf{k}^*, \mathbf{u}^*)$ is a discounted golden-rule state. Q.E.D.

By Theorems 4.1 and 4.2, we immediately have the following corollary.

Corollary 4.1: Any discounted golden-rule state is an optimal stationary state.

Also, Theorem 4.2 can be written in the fashion of the following corollary.

Corollary 4.2: A stationary state $(\mathbf{k}^*, \mathbf{u}^*)$ is a discounted golden state if and only if there exists $\boldsymbol{\pi}^*\in ba^n(\mathscr{F}_0)$ such that

$$\boldsymbol{\pi}^*\cdot(\mathbf{f}-\delta\mathbf{g})>\boldsymbol{\pi}^*\cdot(1-\delta)\mathbf{k}^* \text{ for all } (\mathbf{f}, \mathbf{g}, \mathbf{u})\in\mathscr{X} \text{ with } \int_{\Omega}\mathbf{u}dP>\int_{\Omega}\mathbf{u}^*dP.$$

Proof: If $(\mathbf{k}^*, \mathbf{u}^*)$ is a discounted golden-rule state, then, by Theorem 4.2, there exists a support price $\boldsymbol{\pi}^* \in \mathscr{ba}^n(\mathscr{F}_0)$ for $(\mathbf{k}^*, \mathbf{u}^*)$, i.e.,

$$\int_\Omega \mathbf{u}^* dP - \boldsymbol{\pi}^* \cdot (1-\delta)\mathbf{k}^* \geq \int_\Omega \mathbf{u} dP - \boldsymbol{\pi}^* \cdot (\mathbf{f} - \delta \mathbf{g}) \text{ for all } (\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{X}.$$

Clearly, when $(\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{X}$ and $\int_\Omega \mathbf{u} dP > \int_\Omega \mathbf{u}^* dP$, $\boldsymbol{\pi}^* \cdot (\mathbf{f} - \delta \mathbf{g}) > \boldsymbol{\pi}^* \cdot (1-\delta)\mathbf{k}^*$.

Conversely, assume that there is $\boldsymbol{\pi}^* \in \mathscr{ba}^n(\mathscr{F}_0)$ such that

$$\boldsymbol{\pi}^* \cdot (\mathbf{f} - \delta \mathbf{g}) > \boldsymbol{\pi}^* \cdot (1-\delta)\mathbf{k}^* \text{ for all } (\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{X} \text{ with } \int_\Omega \mathbf{u} dP > \int_\Omega \mathbf{u}^* dP.$$

Clearly, if $(\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{X}$ and $\mathbf{f} - \delta \mathbf{g} = (1-\delta)\mathbf{k}^*$, then $\int_\Omega \mathbf{u}^* dP \geq \int_\Omega \mathbf{u} dP$. Thus, $(\mathbf{k}^*, \mathbf{u}^*)$ is a discounted golden-rule state. Q.E.D.

V. *Support Prices in* $\mathscr{L}_1^n(\mathscr{F}_0)$

Let $\mathscr{L}_1^n(\mathscr{F}_0)$ be the set of all integrable functions defined on $(\Omega, \mathscr{F}_0, P)$ to R^n. The following theorem implies that for any discounted golden-rule state we can find a support price which is an element in $\mathscr{L}_1^n(\mathscr{F}_0)$.

Theorem 5.1: For any discounted golden-rule state $(\mathbf{k}^*, \mathbf{u}^*)$, there is $\mathbf{p}^* \in \mathscr{L}_1^n(\mathscr{F}_0)$ such that

$$\int_\Omega [\mathbf{u}^* - \mathbf{p}^*(1-\delta)\mathbf{k}^*] dP \geq \int_\Omega [\mathbf{u} - \mathbf{p}^*(\mathbf{f} - \delta \mathbf{g})] dP \text{ for all } (\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{X}.$$

Proof: By Theorem 4.2, there is $\boldsymbol{\pi}^* \in \mathscr{ba}^n(\mathscr{F}_0)$ such that

$$\int_\Omega \mathbf{u}^* dP - \boldsymbol{\pi}^* \cdot (1-\delta)\mathbf{k}^* \geq \int_\Omega \mathbf{u} dP - \boldsymbol{\pi}^* \cdot (\mathbf{f} - \delta \mathbf{g}) \text{ for all } (\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{X}.$$

Since $\boldsymbol{\pi}^* \geq 0$, measure $\boldsymbol{\pi}^*$ can be decomposed into two measures by a theorem [Yosida & Hewitt (1952), Thm.1.23, p.52], that is

$$\boldsymbol{\pi}^* = \mathbf{v}_c + \mathbf{v}_p,$$

where $\mathbf{v}_c$ is a non-negative countably additive measure on $(\Omega, \mathscr{F}_0)$ which is absolutely continuous with respect to P and $\mathbf{v}_p$ is a non-negative purely finitely additive measure on $(\Omega, \mathscr{F}_0)$. Therefore, by Radon-Nikodym theorem there is a unique $\mathbf{p}^* \in \mathscr{L}_1^n(\mathscr{F}_0)$ such that

$$\int_\Omega \mathbf{f} d\mathbf{v}_c = \int_\Omega \mathbf{p}^* \cdot \mathbf{f} dP \quad \text{for all } \mathbf{f} \in \mathscr{L}_\infty^n(\mathscr{F}_0).$$

Moreover, by a theorem [Yosida & Hewitt (1952), Thm.1.22, p.52], there is a sequence $\{E_i\}_{i=1}^\infty$ such that $E_i \subset E_{i+1}$ and $\mathbf{v}_p(E_i) = 0$ for all i and $\lim_{i \to \infty} P(E_i) = 1$.

Let $(\mathbf{f}, \mathbf{g}, \mathbf{u}) \in \mathscr{X}$ and, for each i, define $\mathbf{f}_i$, $\mathbf{g}_i$ and $\mathbf{u}_i$ by

$$\mathbf{f}_i(\omega)=\begin{bmatrix}\mathbf{f}(\omega) & \text{for each } \omega\in E_i \\ \mathbf{k}^*(\omega) & \text{otherwise}\end{bmatrix},$$

$$\mathbf{g}_i(\omega)=\begin{bmatrix}\mathbf{g}(\omega) & \text{for each } \omega\in E_i \\ \mathbf{k}^*(\omega) & \text{otherwise}\end{bmatrix},$$

and

$$\mathbf{u}_i(\omega)=\begin{bmatrix}\mathbf{u}(\omega) & \text{for each } \omega\in E_i \\ \mathbf{u}^*(\omega) & \text{otherwise}\end{bmatrix}.$$

Then, since $\boldsymbol{\pi}^*$ be a support price for $(\mathbf{k}^*, \mathbf{u}^*)$,

$$\int_\Omega \mathbf{u}^* dP-\boldsymbol{\pi}^*\cdot(1-\delta)\mathbf{k}^*\geq\int_\Omega \mathbf{u}_i dP-\boldsymbol{\pi}^*\cdot(\mathbf{f}_i-\delta\mathbf{g}_i).$$

Therefore, by the definition of $\mathbf{f}_i$, $\mathbf{g}_i$, and $\mathbf{u}_i$, we have

$$\int_{E_i}\mathbf{u}^* dP-(1-\delta)\int_{E_i}\mathbf{p}^*\cdot\mathbf{k}^* dP-(1-\delta)\int_{\Omega\setminus E_i}\mathbf{k}^* d\mathbf{v}_p$$
$$\geq\int_{E_i}\mathbf{u}dP-\int_{E_i}\mathbf{p}^*\cdot(\mathbf{f}-\delta\mathbf{g})dP-(1-\delta)\int_{\Omega\setminus E_i}\mathbf{k}^* d\mathbf{v}_p$$

for all i. In the limit, we have this theorem. Q.E.D.

Now, let us consider a stationary state $(\mathbf{k}^*, \mathbf{u}^*)$ which satisfies the following interiority condition.

(A.5.1): There exist numbers $\gamma>0$ and $\beta>0$ such that $(\mathbf{k}^*-\gamma\mathbf{1}, \mathbf{k}^*, \mathbf{u}^*-\beta)\in\mathscr{X}$.

Theorem 5.2: Let $(\mathbf{k}^*, \mathbf{u}^*)$ be a stationary state supported by $\boldsymbol{\pi}^*\in ba^n(\mathscr{F}_0)$. If $(\mathbf{k}^*, \mathbf{u}^*)$ satisfies the interiority condition, $\boldsymbol{\pi}^*$ must be in $\mathscr{L}_1^n(\mathscr{F}_0)$.

Proof: Suppose that $\boldsymbol{\pi}^*$ were not countably additive. Then, there is a sequence $\{E_i\}_{i=1}^\infty$ such that $E_i\subset E_{i+1}$, $\bigcup_i E_i=\Omega$, and $\lim_{i\to\infty}\boldsymbol{\pi}^*(E_i)\cdot\mathbf{1}<\boldsymbol{\pi}^*(\Omega)\cdot\mathbf{1}$. Define $\mathbf{f}_i$ and $\mathbf{u}_i$ by

$$\mathbf{f}_i(\omega)=\begin{bmatrix}\mathbf{k}^*(\omega) & \text{for each } \omega\in E_i \\ \mathbf{k}^*(\omega)-\gamma\mathbf{1} & \text{otherwise}\end{bmatrix}$$

and

$$\mathbf{u}_i(\omega)=\begin{bmatrix}\mathbf{u}^*(\omega) & \text{for each } \omega\in E_i \\ \mathbf{u}^*(\omega)-\beta & \text{otherwise}\end{bmatrix}.$$

Then, by the interiority condition, for some $\gamma>0$ and $\beta>0$, $(\mathbf{f}_i, \mathbf{k}^*, \mathbf{u}_i)\in\mathscr{X}$. Therefore, since $\boldsymbol{\pi}^*$ be a support price for $(\mathbf{k}^*, \mathbf{u}^*)$, we have

$$\int_\Omega \mathbf{u}^* dP-\boldsymbol{\pi}^*\cdot(1-\delta)\mathbf{k}^*\geq\int_\Omega \mathbf{u}_i dP-\boldsymbol{\pi}^*\cdot(\mathbf{f}_i-\delta\mathbf{k}^*).$$

Hence, by the definition of $\mathbf{f}_i$ and $\mathbf{u}_i$, we have

$$\boldsymbol{\pi}^* \cdot (\mathbf{f}_i - \mathbf{k}^*) \geq \int_{\Omega} (\mathbf{u}_i - \mathbf{u}^*) dP,$$

i.e., $$-\gamma \boldsymbol{\pi}^*(\Omega \backslash E_i) \cdot \mathbf{1} \geq -\int_{\Omega \backslash E_i} \beta dP,$$

i.e., $$\gamma(\boldsymbol{\pi}^*(E_i) \cdot \mathbf{1} - \boldsymbol{\pi}^*(\Omega) \cdot \mathbf{1}) \geq -\int_{\Omega \backslash E_i} \beta dP \quad \text{for all } i.$$

In the above inequality, as i goes to infinity, L.H.S. is strictly negative and R.H.S. becomes arbitrarily close to 0, which is a contradiction. Q.E.D.

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