**Graduate School of Economics and Institute of Economic Research
Hitotsubashi University**

**An Extension of Arrow's Lemma
with Economic Applications**

Kotaro Suzumura

(Institute of Economic Research, Hitotsubashi University)

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Institute of Economic Research, Hitotsubashi University

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Abstract

One of the most powerful analytical tools in the theory of individual and social choice is Szpilrajn's extension theorem to the effect that any quasi-ordering has an ordering extension. In the context of examining if the individualistic assumptions used in economic environments may help in exorcising the general impossibility theorem, Arrow proved a lemma that extends Szpilrajn's theorem. We strengthen Arrow's lemma and exemplify the use and usefulness of this extended lemma in the context of tradeoff between no-envy equity and Pareto efficiency, on the one hand, and the logical conflict between social welfare and individual rights, on the other.

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Address for Correspondence

Professor Kotaro Suzumura
Institute of Economic Research
Hitotsubashi University
Naka 2-1, Kunitachi, Tokyo 186-8603, Japan
Phone & Fax: 81-42-580-8353
E-mail: suzumura@ier.hit-u.ac.jp

1 Introduction

One of the most powerful analytical tools in the theory of individual and social choice is Edward Szpilrajn's (1930) extension theorem to the effect that a quasi-ordering can be coherently extended into an ordering. As Kenneth Arrow (1951, p.64) observed, "[t]he theorem is trivial in any particular application; nevertheless, it is not trivial in its full generality."

There are several directions into which this basic theorem can be extended. The first direction is to seek for the necessary and sufficient condition(s) on the class of binary relations under which any relation in this class can be coherently extended into an ordering. It was Kotaro Suzumura (1976, Theorem 3; 1983, Appendix to Chapter 1, Theorem A(5)) who answered this question in terms of a newly coined concept of *consistency*.[1] The second direction, which is relevant in the theory of consumer's behavior and resource allocation mechanisms, is to seek for the condition(s) under which a continuous binary relation can be extended into a continuous ordering. This question was first posed and answered by Jean-Yves Jaffrays (1974); a recent paper by Walter Bossert, Yves Sprumont, and Kotaro Suzumura (2002) identified finer sufficiency condition(s) for the existence of a continuous ordering extension of a continuous relation. The third direction, which was explored by Arrow (1951, pp.64-68) in one of the earliest attempts to extend Szpilrajn's theorem, is to show the following: "Suppose that, of all possible pairs of alternatives, the choices among some pairs are fixed in advance, and in a consistent way, so that if x is fixed in advance to be chosen over y and y fixed in advance to be chosen over z, then x is fixed in advance to be chosen over z. Suppose, however, that there is a set S of alternatives such that the choice between no pair of them is prescribed in advance. Then ..., given *any* ordering of the elements in S, there is a way of ordering all the alternatives which will be compatible both with the given ordering in S and with the choice made in advance. In other words, if we know there is some ordering and we know some of the choices implied by that ordering but the known choices do not give any direct information as to choices between elements in a subset S, then there is also no indirect information as to the choices in S, i.e., the ordering of all the alternatives is compatible with any ordering in S." Recollect that Arrow's lemma was the basis of his justly famous general impossibility theorem under the individualistic assumptions.

The purpose of the present paper is two-fold. The first purpose is to extend Arrow's lemma further, just as Suzumura (1976; 1983, Appendix to Chapter 1) extended Szpilrajn's extension theorem. The second purpose is to exemplify the use and usefulness of the extended lemma in the context of tradeoff between no-envy concept of equity and Pareto efficiency, on the one hand, and in the context of logical conflict between social welfare and individual rights, on the other.

The structure of the paper is as follows. In Section 2, we introduce our notations and definitions, and formulate Szpilrajn's extension theorem, Arrow's lemma, and our extension thereof. Section 3 exemplifies the services rendered by the extended Arrow's lemma in the context of equity-efficiency tradeoff, whereas Section 4 presents the parallel illustration in the context of welfare and rights. Section 5 concludes this paper with a few final remarks. The proof of the extended Arrow's lemma is presented in Section 6.

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[1] For further generalizations, see, among others, John Duggan (1999).

2 Notations, Definitions, and an Extension of Arrow's Lemma

Let X be the universal set of alternatives. A *binary relation* on X is a subset Q of the Cartesian product $X \times X$. For any binary relation $Q \subseteq X \times X$, the *asymmetric part* of Q and the *symmetric part* of Q are defined, respectively, by

(1)
$$P(Q) = \{(x, y) \in X \times X \mid (x, y) \in Q \ \& \ (y, x) \notin Q\}$$

and

(2)
$$I(Q) = \{(x, y) \in X \times X \mid (x, y) \in Q \ \& \ (y, x) \in Q\}.$$

When $(x, y) \in Q$ means that, according to the judgements of the decision-maker, an alternative x is judged at least as good as another alternative y, then $P(Q)$ and $I(Q)$ mean, respectively, the strict preference relation and the indifference relation.

A binary relation $Q \subseteq X \times X$ is said to be

(a) *reflexive* if and only if $(x, x) \in Q$ holds for all $x \in X$;
(b) *transitive* if and only if $(x, y) \in Q$ and $(y, z) \in Q$ imply $(x, z) \in Q$ for all $x, y, z \in X$;
(c) *complete* if and only if at least one of $(x, y) \in Q$ and $(y, x) \in Q$ holds for all $x, y \in X$ such that $x \neq y$;
(d) *a quasi-ordering* if it satisfies reflexivity and transitivity; and
(e) *an ordering* if it is a complete quasi-ordering.

For any binary relation $Q \subseteq X \times X$, a binary relation $R \subseteq X \times X$ is said to be an *extension* of Q if and only if $Q \subseteq R$ and $P(Q) \subseteq P(R)$ hold. By definition, an extension R of Q preserves whatever pairwise information which are already contained in Q, and supplements them with additional pairwise information. In particular, an extension R of Q which is an ordering on X is called an *ordering extension* of Q.

We are now ready to state the classical extension theorem due to Szpilrajn (1930) in the reformulated form due to Arrow (1951, p.64) and Bengt Hansson (1968).

Szpilrajn's Theorem
If Q is a quasi-ordering on X, then there is an ordering extension $R \subseteq X \times X$ of Q.

Arrow's (1951, pp.64-68) generalization of Szpilrajn's theorem reads as follows.

Arrow's Lemma
Let Q be a quasi-ordering on X, S a subset of X such that, if $x \neq y$ and $x, y \in S$, then $(x, y) \notin Q$, and T an ordering on S. Then there exists an ordering extension R of Q such that the restriction of R on S coincides with T.[2]

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[2] For example, suppose that Q is the Pareto quasi-ordering on X which keeps silence over a subset S of X. Then, given *any* ordering T on the set S of Pareto non-comparable alternatives, there exists a Pareto-compatible ordering R on X which coincides with T over S.

Arrow's lemma implies Szpilrajn's theorem as a special case where $S = \emptyset$. Note that Szpilrajn's theorem as well as Arrow's lemma assume that the binary relation Q is a quasi-ordering, i.e., transitive. There are at least two reasons to generalize these classical propositions with respect to this common property of transitivity. In the first place, Q being transitive, its symmetric part, viz., the indifference relation $I(Q)$, is transitive too. Ever since Duncan Luce's (1956, p.179) famous coffee example, however, it is well-known that the assumption of transitive indifference is insidious, as it means that the decision-maker has an extraordinary ability to discriminate even an infinitesimally minute difference between alternatives from the viewpoint of their effect on preference satisfaction. In the second place, Q being transitive is nothing other than a sufficient condition for the existence of an ordering extension thereof. It is worthwhile to explore the necessary and sufficient condition(s) on Q for the existence of an ordering extension thereof. This is precisely what we are going to accomplish in the rest of this paper.

A few auxiliary steps are in order. To begin with, the *diagonal binary relation* on X is defined by

$$(3) \qquad \Delta = \{(x, x) \in X \times X \mid x \in X\}.$$

In the second place, given any two binary relations $Q^1, Q^2 \subseteq X \times X$, define their *composition* $Q^1 \circ Q^2$ by

$$(4) \qquad Q^1 \circ Q^2 = \{(x, y) \in X \times X \mid \exists z \in X : (x, z) \in Q^1 \,\&\, (z, y) \in Q^2\}.$$

Given any binary relation $Q \subseteq X \times X$, define an infinite sequence of binary relations on X by $\{Q^{(\tau)}\}_{\tau=1}^{\infty}$ by $Q^{(1)} = Q, Q^{(\tau)} = Q \circ Q^{(\tau-1)}$ ($2 \leq \tau < +\infty$). Then the *transitive closure* $TC(Q)$ of Q is defined by

$$(5) \qquad TC(Q) = \cup_{\tau=1}^{\infty} Q^{(\tau)},$$

which is the smallest transitive superset of Q. Also, Q is transitive if and only if $Q = TC(Q)$ holds. See, for example, Suzumura (1983, Theorem A(1)) for these and other properties of transitive closure. In the third place, some weaker variants of the transitivity property play a crucial role in what follows. A binary relation $Q \subseteq X \times X$ is said to be

(f) *quasi-transitive* if and only if $P(Q)$ is transitive;
(g) *P-acyclic* if and only if there exists no $x \in X$ such that $(x, x) \in TC(P(Q))$; and
(h) *consistent* if and only if, for any natural number $t \geq 3$, there exists no finite sequence $x^1, x^2, \ldots, x^t$ such that $(x^1, x^2) \in P(Q), (x^2, x^3) \in Q, \ldots, (x^{t-1}, x^t) \in Q$ and $(x^t, x^1) \in Q$ hold.

By definition, transitivity implies quasi-transitivity, which in turn implies acyclicity, but each one of these logical implications cannot be reversed in general. Likewise, transitivity implies consistency, which in turn implies acyclicity, but each one of these logical implications cannot be reversed in general. Furthermore, there is no logical implication between consistency and quasi-transitivity.

The condition of consistency is crucial in generalizing Szpilrajn's theorem as well as Arrow's lemma. Indeed, we have the following two propositions.

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Suzumura's Theorem [Suzumura (1976, Theorem 3; 1983, Chapter 1, Theorem A(5))]

A binary relation Q on X has an ordering extension R if and only if Q satisfies consistency.

Main Theorem

Let Q be a binary relation on X, S a subset of X such that, if $x \neq y$ and $x, y \in S$, then $(x, y) \notin TC(Q)$, and T an ordering on S. Then there exists an ordering extension R of Q such that the restriction of R on S coincides with T if and only if Q satisfies consistency.

It should be clear that the Main Theorem subsumes all the propositions mentioned in this paper as special cases. This is logically satisfactory, but the usefulness of an extension of the existing results lies in the extra mileage it can secure in generating new insights when it is applied. Thus, relegating the proof of the Main Theorem into Section 6, the next two sections will exemplify the use and usefulness thereof in a couple of economic contexts.

3 First Economic Application: Equity-Efficiency Trade-off

The first context is the tradeoff relationship which holds between Pareto efficiency and no-envy equity à la Duncan Foley (1967), Serge Kolm (1971) and Hal Varian (1974; 1975).[3] In the neat parlance of Hal Varian (1975, pp.240-241), "the theory of fairness ... is founded in the notion of 'extended sympathy' and in the idea of 'symmmetry' in the treatment of agents .... In effect, we are asking each agent to put himself in the position of each of the other agents to determine if that is a better or worse position than the one he is now in."

To substantiate this foundation of the theory of fairness, let X and $N = \{1, 2, \ldots, n\}$ ($2 \leq n < +\infty$) stand, respectively, for the set of all conceivable social states and the set of individuals in the society. To formulate the possibility of imaginary exchange of circumstances among individuals, each individual $i \in N$ is assumed to have an ordering $\mathbb{R}_i$ on $X \times N$. The intended meaning of $((x, j), (y, k)) \in \mathbb{R}_i$ is that, according to i's judgements, being in the position of j when the social state x prevails is at least as good as being in the position of k when the social state y prevails. Let R_i on X be defined by $(x, y) \in R_i$ holds if and only if $((x, i), (y, i)) \in \mathbb{R}_i$ holds. We may now define the *efficiency criterion R^f* by

$$(6) \qquad R^f = \cap_{i \in N} R_i,$$

which embodies the famous Pareto principle. Turning to the equity side of the evaluative exercise, for each social state $x \in X$, let a set $H(x) \subseteq N \times N$ be defined by

$$(7) \qquad H(x) = \{(i, j) \in N \times N \mid ((x, j), (x, i)) \in P(R_i)\}.$$

By definition, $H(x)$ gathers all instances of interpersonal envy at $x \in X$, which motivates us to define the *equity criterion* R^q by

$$(8) \qquad R^q = \{(x,y) \in X \times X | H(x) \subseteq H(y)\}.$$

It is clear that the efficiency criterion R^f and the equity crietrion R^q define quasi-orderings on X.

Remark 1

There are several methods to extend the equity quasi-ordering R^q into a complete ordering. One frequently invoked method is the following:

$$(9) \qquad R^q_* = \{(x,y) \in X \times X \mid \#H(x) \le \#H(y)\},$$

where $\#S$ for any set S is the number of elements in S. It is clear that R^q_* is an ordering extension of R^q. By converting the set-theoretical inclusion into the comparison of the numbers of included alternatives, R^q_* goes beyond R^q and defines a complete ordering on X. However, this wider applicability of R^q_* than that of R^q is bought at a high price. Suppose that $H(x) = \{(i,j)\}$ and $H(y) = \{(g,h),(k,l)\}$, where g, h, i, j, k and l are all distinct. Then we have

$$(10) \qquad (x,y),(y,x) \notin R^q; (x,y) \in P(R^q_*).$$

Those who support the adoption of R^q_* may claim that x is more equitable than y as the instances of interpersonal envy is smaller at x than at y. However, those who support the adoption of R^q may cast the following doubt: why should i be left solely envious of j at x simply to save g ank k from envying h and l, respectively, at y in the name of fairness? It is in view of this possible argument against R^q_* that we adopt R^q in the main and refer to R^q_* only when doing so will bring about something of special value. $\|$

How can we combine the efficiency quasi-ordering and the equity quasi-ordering to arrive at the overall social welfare judgements on X? It is well known that the mere juxtaposition of these criteria would not do, as there are frequent instances of conflict between them. To crystallize this problem of equity-efficiency tradeoff, we have only to consider the following:

Example 1 [Suzumura (1983, p.129)]

Let $X = \{x,y\}$ and $N = \{1,2\}$. Suppose that a profile of individual preference orderings $(\mathbb{R}_1, \mathbb{R}_2)$ is such that[4]

$$\mathbb{R}_1 : (y,2),(y,1),(x,1),(x,2)$$
$$\mathbb{R}_2 : (y,2),(y,1),(x,2),(x,1).$$

It is clear that $R^f = \Delta \cup \{(y,x)\}$. It is also clear that $H(x) = \emptyset \subsetneq H(y) = \{(1,2)\}$, so that we have $R^q = \Delta \cup \{(x,y)\}$. Thus, the efficiency quasi-ordering and the equity quasi-ordering are in direct conflict with each other. $\|$

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[4] Preference orderings are written horizontally with the more preferred alternative to the left of the less preffered, the indifferent alternatives (if any) being embraced within square brackets.

Confronted with the equity-efficiency tradeoff, one possible stance to take is to invoke some hierarchic or lexicographic structures between the conflicting principles or judgements. This step was followed in many works, most notably by Koichi Tadenuma (2002).[5] In our present context, there are two lexicographic combinations of R^q and R^f. The *efficiency-first principle* is defined by

$$(11) \qquad\qquad R^{fq} = R^f \cup \{N(R^f) \cap R^q\},$$

whereas the *equity-first principle* is defined by

$$(12) \qquad\qquad R^{qf} = R^q \cup \{N(R^q) \cap R^f\},$$

where $N(R)$ for any binary relation R on X denotes the *non-comparability relation* defined by

$$(13) \qquad\qquad (x,y) \in N(R) \text{ if and only } (x,y) \notin R \text{ and } (y,x) \notin R.$$

It follows from (11) and (12) that

$$(14) \qquad\qquad P(R^{fq}) = P(R^f) \cup \{N(R^f) \cap P(R^q)\}$$

and

$$(15) \qquad\qquad P(R^{qf}) = P(R^q) \cup \{N(R^q) \cap P(R^f)\},$$

respectively. In simple words, $(x,y) \in R^{fq}$ holds if and only if x is at least as efficient as y, or x and y are Pareto non-comparable and x is at least as equitable in the no-envy sense as y. The interpretation of $(x,y) \in R^{qf}$ is similar save for the change of order between equity and efficiency.

How would these lexicographic combinations of no-envy equity criterion and Pareto efficiency criterion fare in the logical arena of extendability into an ordering? Concerning the fate of the efficiency-first principle, consider the following:

Example 2

Let $X = \{x, y, z, w\}$ and $N = \{1, 2\}$. Suppose that a profile of individual preference orderings $(\mathbb{R}_1, \mathbb{R}_2)$ is such that

$$\mathbb{R}_1 : (y,2), (w,2), (w,1), [(z,1), (z,2)], (y,1), [(x,1), (x,2)]$$
$$\mathbb{R}_2 : (y,2), (x,2), (w,2), (z,2), (z,1), (w,1), (x,1), (y,1).$$

It is easy to check that $R^f = \Delta \cup \{(w,z), (y,x)\}$. It is also easy to check that $H(x) = \emptyset, H(y) = \{(1,2)\}, H(z) = \emptyset$ and $H(w) = \{(1,2)\}$. In view of (14), we may then obtain $(x,w) \in P(R^{fq})$ by virtue of equity, $(w,z) \in P(R^{fq})$ by virtue of efficiency, $(z,y) \in P(R^{fq})$ by virtue of equity, and $(y,x) \in P(R^{fq})$ by virtue of efficiency. Thus, R^{fq} is *not* acyclic, hence is *not* consistent. ∥

Turning to the fate of the equity-first principle, consider the following:

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[5] See also Kotaro Suzumura (1983a) and Reiko Gotoh, Kotaro Suzumura and Naoki Yoshihara (2004).

Example 3

Let $X = \{x, y, z\}$ and $N = \{1, 2\}$. Suppose that a profile of individual preference orderings $(\mathbb{R}_1, \mathbb{R}_2)$ is such that

$$\mathbb{R}_1 : (x, 1), (x, 2), (y, 2), (y, 1), (z, 1), (z, 2)$$
$$\mathbb{R}_2 : (x, 1), (x, 2), (y, 2), (y, 1), (z, 1), (z, 2).$$

It is easy to check that $R^f = \Delta \cup \{(x, y), (y, z), (x, z)\}$. It is also easy to check that $H(x) = \{(2, 1)\}, H(y) = \{(1, 2)\}$, and $H(z) = \{(2, 1)\}$. By virtue of (15), we then have $(x, y) \in P(R^{qf})$ and $(y, z) \in P(R^{qf})$, whereas (12) entails that $(z, x) \in I(R^{qf})$. Thus, R^{qf} is acyclic, but it is *not* consistent. ‖

Gathering all pieces together, the inevitable conclusion seems to be as follows. The equity-efficiency tradeoff, which is crystallized in terms of **Example 1**, cannot be exorcized by the lexicographic combinations of the equity-as-no-envy criterion, on the one hand, and the Pareto-efficiency criterion, on the other, whichever criterion may be given priority over the other in defining their lexicographic combination.

Remark 2

Our verdict on the lexicographic combinations of equity-efficiency criteria may be construed to be contradictory with that of Tadenuma (2002, p.463), according to whom " the efficiency-first relation may have a cycle, whereas the equity-first relation is transitive." This apparent difference can be easily accounted for. In the first place, Tadenuma used R^q_* instead of R^q and his proof that "the equity-first relation is transitive" hinges squarely on the fact that R^q_* is a complete ordering. See **Remark 1** for our reservation on the use of this equity criterion. In the second place, what Tadenuma (2002, Proposition 2) proved is the transitivity of $P(R^{qf}_*)$, where R^{qf}_* is defined by

$$(16) \qquad\qquad\qquad R^{qf}_* = R^q_* \cup \{N(R^q_*) \cap R^f\}$$

viz., the quasi-transitivity of R^{qf}_*. Recollect that the quasi-transitivity of R^{qf}_* is neither necessary nor sufficient for the consistency thereof. Thus, there is no logical contradiction between Tadenuma's result and our verdict on the performance of equity-first criterion. ‖

4   Second Economic Application: Welfare and Rights

It was Amartya Sen (1970a; 1970b/1979, Chapter 6*; 1992) who made a pioneering attempt to introduce the value of liberty as one of the inviolable individual rights among other essential values in social choice theory. His intuition may be neatly summarized as follows: "[T]here ought to exist a certain minimum area of personal freedom which must on no account be violated; for if it is overstepped, the individual will find himself in an area too narrow for even the minimum development of his natural faculties which alone makes it possible to pursue, and even to conceive, the various ends which men hold good or right or sacred [Isaiah Berlin (1969, p.124)]." This intuition seems appealing, but Sen has shown that there exists a basic conflict between the value of individual rights in this sense and the value of public welfare in

the weak sense of the Pareto principle. This conflict was christened the *impossibility of a Paretian liberal*, which caused a stir in the profession. The essence of this impossibility result can be illustrated in terms of the following example.[6]

Example 4 : Lady Chatterley's Lover Case

There is a copy of *Lady Chatterley's Lover* which is available to Mr. P (the Prude) and Mr. L (the Lewd) for reading. Everything else being the same, there are four social alternatives: Mr. P alone reading it, viz., (r, n), Mr. L alone reading it, viz., (n, r), both Mr. P and Mr. L reading it, viz., (r, r), and no one reading it, viz., (n, n), where r stands for reading the book, whereas n stands for not reading the book. Mr. P prefers (n, n) most ("This is an awful book; it should not be read by anybody"), (r, n) is his second best which is better than (n, r) ("I will take the damage upon myself rather than exposing the lascivious Mr. L to the imminent danger of reading such a book"), and finally (r, r) ("What a terrible mistake to let Mr. L and myself to face such a muck!"). Mr. L, on his part, prefers (r, r) most ("That would be useful to open Mr. P's obstinate mind to the reality of human life"), (r, n) is his second best which is better than (n, r) ("I will enjoy reading it for sure, but I am willing to sacrifice my joy if I can educate Mr. P for that!"), and lastly (n, n) ("What a terrible waste of a great literary work!"). The situation can be summarized as follows:

<div align="center">

Mr. L's choice

$$\begin{array}{cc} & n \qquad\quad r \end{array}$$

Mr. P's choice $\begin{array}{c} r \\ n \end{array} \left[\begin{array}{cc} (r,n) & (r,r) \\ (n,n) & (n,r) \end{array} \right]$

</div>

$$R_P : (n,n), (r,n), (n,r), (r,r)$$
$$R_L : (r,r), (r,n), (n,r), (n,n)$$

Note that both persons prefer (r, n) to (n, r), and there exist no other pair of social alternatives over which preferences of both persons concur. Thus, the only constraint imposed by the Pareto principle is that (r, n) should be socially preferred to (n, r).

Consider now the pair of social alternatives $((n, n), (r, n))$. The only difference between these social alternatives is that Mr. P does not read this book in (n, n) and he does read it in (r, n), whereas Mr. L does not read it whichever alternative may materialize. In this sense, $((n, n), (r, n))$ lies in Mr. P's private sphere over which Mr. P's preferences should be socially respected if the social choice procedure is minimally libertarian in the parlance of Sen, viz., (n, n) should be socially preferred to (r, n).[7]

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[6] This example is due essentially to Sen (1970a; 1970b/1979, Chapter 6), but Sen's original example is modified for the sake of facilitating our subsequent analysis as well as of making the relationship between Sen's impossibility of a Paretian liberal and the classical prisoner's dilemma explicit.

[7] There is a serious debate on Sen's (1970a; 1970b/1979, Chapter6; 1992) articulation of libetarian rights in terms of the individual's local decisiveness in social choice, and Robert Sugden (1985), Wulf Gaertner, Prasanta Pattanaik and Kotaro Suzumura (1992) developed an alternative articulation in terms of game forms. Interested readers are referred to Suzumura (2004) for a self-contained exposition on the social choice theoretic analysis of individual rights.

Likewise, the only difference between (n, r) and (n, n) is that Mr. L reads this book in (n, r) and he does not read it in (n, n). Thus, the minimally libertarian society should prefer (n, r) to (n, n).

Gathering all piecemeal information together, we must conclude that (r, n) is socially preferred to (n, r), (n, r) is socially preferred to (n, n), and (n, n) is socially preferred to (r, n), competing a strict social preference cycle. Thus, there exists no social ordering in the Paretian and minimally libertarian society. $\|$

To see the robustness of the logical conflict between the claim of individual rights and the claim of public welfare in the form of the Pareto principle, and to see the efficacy of lexicographic combination(s) of these moral principles in resolving the impossibility of a Paretian liberal, note first that the Pareto quasi-ordering R^f is given by

$$R^f : (r, n), (n, r).$$

Second, observe that the pair of sets $\mathbb{D} = (D_P, D_L)$, where

$$D_P = \{((r, n), (n, n)), ((n, n), (r, n)), ((r, r), (n, r)), ((n, r), (r, r))\}$$
$$D_L = \{((r, n), (r, r)), ((r, r), (r, n)), ((n, n), (n, r)), ((n, r), (n, n))\},$$

deserves to be christened the *libertarian rights-system in the sense of Sen*. Let us define the *rights-oriented social preferences* by

$$R^r = \{D_P \cap P(R_P)\} \cup \{D_L \cap P(R_L)\}$$
$$= \{((n, n), (r, n)), ((n, r), (r, r)), ((r, r), (r, n)), ((n, r), (n, n))\}.$$

We have seen in Example 4 that the mere juxtaposition of R^f and R^r would prevent us from defining an overall social welfare ordering, as there exists a strict preference cycle in $R^f \cup R^r$.

Consider now the lexicographic combinations of the two values, viz., the *efficiency-first principle*

(17) $$R^{fr} = R^f \cup \{N(R^f) \cap R^r\}$$

and the *rights-first principle*

(18) $$R^{rf} = R^r \cup \{N(R^r) \cap R^f\},$$

where the non-comparability relations $N(R^f)$ and $N(R^r)$ are defined in accordance with (13). Unfortunately, both lexicographic combinations are ineffective in resolving Sen's impossibility theorem. Indeed, in the case of R^{fr}, we have a strict preference cycle

$$((r, n), (n, r)) \in P(R^{fr}), ((n, r), (r, r)) \in P(R^{fr}), ((r, r), (r, n)) \in P(R^{fr}),$$

whereas in the case of R^{rf}, we have a strict preference cycle

$$((n, r), (n, n)) \in P(R^{rf}), ((n, n), (r, n)) \in P(R^{rf}), ((r, n), (n, r)) \in P(R^{rf}),$$

vindicating this ineffectiveness.

Remark 3

It should be clear that there exists a strong resemblance between the impossibility of a Paretian liberal and the classical prisoner's dilemma. Indeed, our expositon of the former in Example 4 shows that n (resp. r) is the dominant strategy for Mr. P (resp. Mr. L) and the dominant strategy equilibrium (n, r) is Pareto-dominated by (r, n). This resemblance was first pointed out in print by Ben Fine (1975). ‖

Remark 4

It was Robert Nozick (1974, p.166) who suggested that the conflict between the claim of libertarian rights and that of Pareto principle is to be resolved by assigning quite different roles to these two requirements: "Individual rights are co-possible; each person may exercise his rights as he chooses. The exercise of these rights fixes some features, a choice may be made by a social choice mechanism based upon a social ordering; if there are any choices left to make! Rights do not determine a social ordering but instead set the constraints within which a social choice is to be made, by excluding certain alternatives, fixing others, and so on ... *How else can one cope with Sen's result?*" The upshot of this proposed resolution is that (r, n) and (r, r) are vetoed by the exercise of Mr. P's libertarian right, whereas (n, n) is vetoed by the exercise of Mr. L's libertarian right, leaving (n, r) as the only viable candidate for social choice. This may be construed to be an early proposal of the rights-first principle. Note, however, that Nozick is not concerned with the construction of social welfare ordering embodying the exercise of libertarian rights at all. ‖

5 Concluding Remarks

In many contexts of social and individual choice, a crucial role is often played by the existence of an ordering extension of a binary relation. This paper generalized an extension lemma due to Arrow (1951, Chapter VI). Our main theorem subsumes not only Arrow's lemma itself, but also Suzumura's extension theorem (1976; 1983, Chapter 1) based on the crucial concept of consistency, both of which generalize Szpilrajn's (1930) classical ordering extension theorem. To exemplify the use and usefulness of our main theorem, we have shown how we can apply it to the concrete economic problems such as the equity-efficiency tradeoff, on the one hand, and the impossibility of a Paretian liberal, on the other.

In concluding this paper, a few remarks are in order. In the first place, the crucial condition of consistency implies acyclicity, and it is implied by transitivity, but the converse implications are not true in general, whereas it does not imply quasi-transitivity, neither is it implied by quasi-transitivity. Although quasi-transitivity is widely construed to be a natural weakening of the stringent condition of transitivity, it is worthwhile to emphasize that it has nothing to do with the condition for the existence of an ordering extension of a binary relation. In the second place, the fact that Arrow's lemma, as well as the main theorem of this paper which generalizes Arrow's lemma, admits the existence of a set over which the binary relation cannot say anything directly in the case of Arrow 's lemma, and directly as well as indirectly in the case of our main theorem, is important; it allows us to accommodate considerations

other than the considerations embodied in the binary relation at hand. To illustrate this point in terms of the applications, there are considerations other than equity, efficiency and individual rights in forming the fully-fledged social evaluation ordering; it is important to secure the existence of an ordering extension of the binary relation in question which is compatible with any outside specification concerning these other considerations. In the third place, the method of mixing multiple moral principles lexicographically is often invoked if and when these multiple moral principles conflict squarely with each other. Our applications in the two chosen contexts show that there is no sure-fire guarantee that such a lexicographic mixture of moral principles serves as a resolvent of the conflict among these component principles. Something further is needed other than the lexicographic mixture of moral principles pure and simple. In another work, viz., Reiko Gotoh, Kotaro Suzumura and Naoki Yoshihara (2004), we have explored one channel through which the lexicographic mixture of moral principles subject to the circumscribed conditions can be made logically compatible.

6 Proof of the Main Theorem[8]

Proof of Necessity: Suppose that there exists an ordering extension R of Q such that the restriction of R on S coincide with the given ordering T on S. Let t be any natural number, and suppose that

(19) $$(x^1, x^2) \in P(Q), (x^2, x^3) \in Q, \ldots, (x^{t-1}, x^t) \in Q$$

hold for some $\{x^1, x^2, \ldots x^t\} \subseteq X$. R being an ordering extension of Q, it follows from (19) that

(20) $$(x^1, x^2) \in P(R), (x^2, x^3) \in R, \ldots, (x^{t-1}, x^t) \in R,$$

which implies $(x^1, x^t) \in P(R)$ by virtue of the transitivity of R. Then we have $(x^t, x^1) \notin R$, which implies $(x^t, x^1) \notin Q$ in view of $Q \subseteq R$. Thus, Q must be consistent.

Proof of Sufficiency: Starting from Q, define a binary relation Q^* by

(21) $$Q^* = \Delta \cup TC(Q).$$

It is clear that Q^* is reflexive. To show that Q^* is transitive, let $(x, y) \in Q^*$ and $(y, z) \in Q^*$ for some $x, y, z \in X$. If $(x, y) \in TC(Q)$ and $(y, z) \in TC(Q)$, we obtain $(x, z) \in TC(Q) \subseteq Q^*$. If $(x, y) \in \Delta$ [resp. $(y, z) \in \Delta$], we obtain $x = y$ [resp. $y = z$], so that $(x, z) \in Q^*$ follows from $(y, z) \in Q^*$ [resp. $(x, y) \in Q^*$]. Thus, Q^* is a quasi-ordering.

Using this Q^* and the ordering T on S, we now define

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[8] The structure of the following proof is due essentially to Ken-Ichi Inada (1954), who constructed a simple alternative proof of Arrow's Lemma.

$$(22) \qquad Q^{**} = Q^* \cup (Q^* \circ T) \cup (Q^* \circ T \circ Q^*) \cup (T \circ Q^*) \cup T.$$

Step 1: Q^{**} is a quasi-ordering on X.

It follows from (20) and (21) that Q^{**} is reflexive. To show that Q^{**} is transitive, suppose that $(x, y) \in Q^{**}$ and $(y, z) \in Q^{**}$ for some $x, y, z \in X$.

(a) If $(x, y) \in Q^*$ and $(y, z) \in Q^*$, then $(x, z) \in Q^* \subseteq Q^{**}$ by virtue of the transitivity of Q^*.

(b) If $(x, y) \in Q^*$ and $(y, z) \in Q^* \circ T$, there exists an $s \in X$ such that $(x, y) \in Q^*, (y, s) \in Q^*$ and $(s, z) \in T$. By virtue of the transitivity of Q^*, we have $(x, s) \in Q^*$ and $(s, z) \in T$, viz., $(x, z) \in Q^* \circ T \subseteq Q^{**}$.

(c) If $(x, y) \in Q^*$ and $(y, z) \in Q^* \circ T \circ Q^*$, there exist $s, t \in X$ such that $(x, y) \in Q^*, (y, s) \in Q^*, (s, t) \in T$ and $(t, z) \in Q^*$. Invoking the transitivity of Q^*, we then obtain $(x, s) \in Q^*, (s, t) \in T$ and $(t, z) \in Q^*$, viz., $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$.

(d) If $(x, y) \in Q^*$ and $(y, z) \in T \circ Q^*$, we obtain $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$.

(e) If $(x, y) \in Q^*$ and $(y, z) \in T$, we obtain $(x, z) \in Q^* \circ T \subseteq Q^{**}$.

(f) If $(x, y) \in Q^* \circ T$ and $(y, z) \in Q^*$, we obtain $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$.

(g) If $(x, y) \in Q^* \circ T$ and $(y, z) \in Q^* \circ T$, there exist $s, t \in X$ such that $(x, s) \in Q^*, (s, y) \in T, (y, t) \in Q^*$ and $(t, z) \in T$. It follows from $(s, y) \in T$ and $(t, z) \in T$ that both y and t belong to S. Coupled with $(y, t) \in Q^*$, this fact implies $y = t$. It then follows that $(s, z) \in T$, which implies $(x, z) \in Q^* \circ T \subseteq Q^{**}$ in view of $(x, s) \in Q^*$

(h) If $(x, y) \in Q^* \circ T$ and $(y, z) \in Q^* \circ T \circ Q^*$, there exist $s, t, u \in X$ such that $(x, s) \in Q^*, (s, y) \in T, (y, t) \in Q^*, (t, u) \in T$ and $(u, z) \in Q^*$. It follows from $(s, y) \in T$ and $(t, u) \in T$ that both y and t belong to S. Coupled with $(y, t) \in Q^*$, this fact implies that $y = t$. It then follows that $(s, u) \in T$, which implies $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$ in view of $(x, s) \in Q^*$ and $(u, z) \in Q^*$.

(i) If $(x, y) \in Q^* \circ T$ and $(y, z) \in T \circ Q^*$, $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$ holds.

(j) If $(x, y) \in Q^* \circ T$ and $(y, z) \in T$, there exists $s \in X$ such that $(x, s) \in Q^*, (s, y) \in T$ and $(y, z) \in T$ hold. It follows that $(x, s) \in Q^*$ and $(s, z) \in T$, so that $(x, z) \in Q^* \circ T \subseteq Q^{**}$.

(k) If $(x, y) \in Q^* \circ T \circ Q^*$ and $(y, z) \in Q^*$, there exist $s, t \in X$ such that $(x, s) \in Q^*, (s, t) \in T, (t, y) \in Q^*$ and $(y, z) \in Q^*$ hold. It follows that $(x, s) \in Q^*, (s, t) \in T$ and $(t, z) \in Q^*$, so that $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$.

(l) If $(x, y) \in Q^* \circ T \circ Q^*$ and $(y, z) \in Q^* \circ T$, there exist $s, t, u \in X$ such that $(x, s) \in Q^*, (s, t) \in T, (t, y) \in Q^*, (y, u) \in Q^*$ and $(u, z) \in T$. It follows from $(t, y) \in Q^*$ and $(y, u) \in Q^*$ that $(t, u) \in Q^*$, whereas $(s, t) \in T$ and $(u, z) \in T$ imply that both t and u belong to S. It follows that $t = u$, so that we obtain $(x, z) \in Q^* \circ T \subseteq Q^{**}$.

(m) If $(x, y) \in Q^* \circ T \circ Q^*$ and $(y, z) \in Q^* \circ T \circ Q^*$, there exist $s, t, u, v \in X$ such that $(x, s) \in Q^*, (s, t) \in T, (t, y) \in Q^*, (y, u) \in Q^*$ and $(u, v) \in T$ and $(v, z) \in Q^*$. It follows from $(t, y) \in Q^*$ and $(y, u) \in Q^*$ that $(t, u) \in Q^*$, whereas $(s, t) \in T$ and $(u, v) \in T$ imply that both t and u belong to S. It follows that $t = u$, so that we obtain $(x, s) \in Q^*, (s, v) \in T$ and $(v, z) \in Q^*$, where we invoked the transitivity of T. Thus, $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$.

(n) If $(x, y) \in Q^* \circ T \circ Q^*$ and $(y, z) \in T \circ Q^*$, there exist $s, t, u \in X$ such that $(x, s) \in Q^*, (s, t) \in T, (t, y) \in Q^*, (y, u) \in T$ and $(u, z) \in Q^*$. It follows from $(s, t) \in T$

and $(y, u) \in T$ that both t and y belong to S which, coupled with $(t, y) \in Q^*$, implies $t = y$. We then have $(x, s) \in Q^*$, and $(s, u) \in T$ and $(u, z) \in Q^*$, where we invoked the transitivity of T. It follows that $(x, z) \in Q^* \circ T \circ Q^* \subseteq Q^{**}$.

(o) If $(x, y) \in Q^* \circ T \circ Q^*$ and $(y, z) \in T$, there exist $s, t \in X$ such that $(x, s) \in Q^*, (s, t) \in T, (t, y) \in Q^*$ and $(y, z) \in T$. It follows from $(s, t) \in T$ and $(y, z) \in T$ that both t and y belong to S which, coupled with $(t, y) \in Q^*$, implies $t = y$. We then have $(x, s) \in Q^*$, $(s, t) \in T$ and $(t, z) \in T$, from which we can conclude that $(x, z) \in Q^* \circ T \subseteq Q^{**}$, where use is made of the transitivity of T.

(p) If $(x, y) \in T \circ Q^*$ and $(y, z) \in Q^*$, we may obtain $(x, z) \in T \circ Q^* \subseteq Q^{**}$ in view of the transitivity of Q^*.

(q) If $(x, y) \in T \circ Q^*$ and $(y, z) \in Q^* \circ T$, there exist $s, t \in X$ such that $(x, s) \in T, (s, y) \in Q^*, (y, t) \in Q^*$ and $(t, z) \in T$. By virtue of the transitivity of Q^*, we have $(x, s) \in T, (s, t) \in Q^*$ and $(t, z) \in T$. It follows from $(x, s) \in T$ and $(t, z) \in T$ that both s and t belong to S which, coupled with $(s, t) \in Q^*$, implies $s = t$. Thus, we have $(x, s) \in T$ and $(s, z) \in T$, which further implies $(x, z) \in T \subseteq Q^{**}$ by virtue of the transitivity of T.

(r) If $(x, y) \in T \circ Q^*$ and $(y, z) \in Q^* \circ T \circ Q^*$, there exist $s, t, u \in X$ such that $(x, s) \in T, (s, y) \in Q^*, (y, t) \in Q^*, (t, u) \in T$ and $(u, z) \in Q^*$. By virtue of the transitivity of Q^*, we have $(s, t) \in Q^*$, whereas $(x, s) \in T$ and $(t, u) \in T$ imply that both s and t belong to S. It follows that $s = t$, so that we obtain $(x, u) \in T$ in view of the transitivity of T. Thus, $(x, z) \in T \circ Q^* \subseteq Q^{**}$.

(s) If $(x, y) \in T \circ Q^*$ and $(y, z) \in T \circ Q^*$, there exist $s, t \in X$ such that $(x, s) \in T, (s, y) \in Q^*, (y, t) \in T$, and $(t, z) \in Q^*$. It follows from $(x, s) \in T$ and $(y, t) \in T$ that both s and y belong to S. In view of $(s, y) \in Q^*$, we then obtain $s = y$, so that we obtain $(x, t) \in T$ and $(t, z) \in Q^*$, where use in made of the transitivity of T. Thus, $(x, z) \in T \circ Q^* \in Q^{**}$.

(t) If $(x, y) \in T$ and $(y, z) \in Q^*$, we have $(x, z) \in T \circ Q^* \in Q^{**}$.

(u) If $(x, y) \in T$ and $(y, z) \in Q^* \circ T$, we have $(x, y) \in T, (y, s) \in Q^*$, and $(s, z) \in T$ for some $s \in X$. It follows from $(x, y) \in T$ and $(s, z) \in T$ that both y and s belong to S, which implies $y = s$ in view of $(y, s) \in Q^*$. Taking the transitivity of T into consideration, we then obtain $(x, z) \in T \subseteq Q^{**}$.

(v) If $(x, y) \in T$ and $(y, z) \in Q^* \circ T \circ Q^*$, there exist $s, t \in X$ such that $(y, s) \in Q^*, (s, t) \in T$ and $(t, z) \in Q^*$. It follows from $(x, y) \in T$ and $(s, t) \in T$ that both y and s belong to S, so that $(y, s) \in Q^*$ implies that $y = s$. By virtue of the transitivity of T, we then obtain $(x, t) \in T$ and $(t, z) \in Q^*$, viz., $(x, z) \in T \circ Q^* \subseteq Q^{**}$.

(w) If $(x, y) \in T$ and $(y, z) \in T \circ Q^*$, there exists $s \in X$ such that $(y, s) \in T$ and $(s, z) \in Q^*$. T being transitive, we then obtain $(x, s) \in T$ and $(s, z) \in Q^*$, viz., $(x, z) \in T \circ Q^* \subseteq Q^{**}$.

(x) If $(x, y) \in T$ and $(y, z) \in T$, we obtain $(x, z) \in T \subseteq Q^{**}$ by virtue of the transitivity of T.

Step 2: Q^{**} is an extension of Q^*.

Since $Q^* \subseteq Q^{**}$ holds by definition, we have only to prove that $(x, y) \in P(Q^*)$ must imply $(y, z) \notin Q^{**}$. Suppose that $(x, y) \in P(Q^*)$ and $(y, x) \in Q^{**}$ for some $x, y \in X$. We show that this is actually impossible.

(a) Suppose that $(y, x) \in Q^*$. This is incompatible with $(x, y) \in P(Q^*)$.

(b) Suppose that $(y, x) \in Q^* \circ T$, viz., $(y, s) \in Q^*$ and $(s, x) \in T$ for some $s \in X$. Q^* being transitive, we then obtain $(x, s) \in Q^*$. Since $(s, x) \in T$, both s and x belong to S, so that $s = x$ in view of $(x, s) \in Q^*$. Then we have $(x, y) \in P(Q^*)$ and $(y, x) \in Q^*$, which is impossible.

(c) Suppose that $(y, x) \in Q^* \circ T \circ Q^*$. Then there exist $s, t \in X$ such that $(y, s) \in Q^*, (s, t) \in T$ and $(t, x) \in Q^*$. It follows from $(x, y) \in Q^*$ and $(y, s) \in Q^*$ that $(x, s) \in Q^*$, which implies $(t, s) \in Q^*$ in view of $(t, x) \in Q^*$. Since both s and t belong to S in view of $(s, t) \in T, (t, s) \in Q^*$ implies $t = s$. Then $(y, s) \in Q^*$ and $(t, x) \in Q^*$ imply $(y, x) \in Q^*$ in contradiction with $(x, y) \in P(Q^*)$.

(d) Suppose that $(y, x) \in T \circ Q^*$. Then there exists $s \in X$ such that $(y, s) \in T$ and $(s, x) \in Q^*$. It follows from $(x, y) \in Q^*$ and $(s, x) \in Q^*$ that $(s, y) \in Q^*$. In view of $(y, s) \in T$, which impies that both y and s belong to S, $(s, y) \in Q^*$ implies $s = y$. But $(x, y) \in P(Q^*)$, $(s, x) \in Q^*$ and $s = y$ are contradictory.

(e) Suppose that $(y, x) \in T$, which implies that both x and y belong to S. Since $(x, y) \in Q^*$ holds, it follows that $x = y$. But this is imcompatible with $(x, y) \in P(Q^*)$.

Step 3: For all $x, y \in S, (x, y) \notin T$ implies $(x, y) \notin Q^{**}$.

Suppose that $(x, y) \notin T$ and $(x, y) \in Q^{**}$ for some $x, y \in S$.

(a) If $(x, y) \in Q^*$, then $x = y$ must be true in contradiction with $(x, y) \notin T$ and the reflexivity of T.

(b) If $(x, y) \in Q^* \circ T$, there exits $s \in X$ such that $(x, s) \in Q^*$ and $(s, y) \in T$, which implies that $x = s$ in view of $x \in S$. But $(x, y) \notin T$, $(s, y) \in T$ and $x = s$ are contradictory.

(c) If $(x, y) \in Q^* \circ T \circ Q^*$, there exist $s, t \in X$ such that $(x, s) \in Q^*$, $(s, t) \in T$ and $(t, y) \in Q^*$. It follows from $(s, t) \in T$ that both s and t belong to S. Since $x, y, \in S$ by definition, It follows from $(x, s) \in Q^*$ and $(t, y) \in Q^*$ that $x = s$ and $t = y$. Thus $(s, t) \in T$ implies $(x, y) \in T$ in contradiction with $(x, y) \notin T$.

(d) If $(x, y) \in T \circ Q^*$, there exists $s \in X$ such that $(x, s) \in T$ and $(s, y) \in Q^*$. It follows from $(x, s) \in T$ that both x and s belong to S, whereas $y \in S$ by definition. Thus, $(s, y) \in Q^*$ cannot but imply $s = y$, which in turn implies $(x, y) \in T$ in contradiction with $(x, y) \notin T$.

(e) If $(x, y) \in T$, this is directly contradictory with $(x, y) \notin T$.

Step 4: There exists an ordering extension R of Q.

Q^{**} being a quasi-ordering by virtue of Step 1, there exists an ordering extension R thereof by virtue of Szpilrajn's Theorem. Q^{**} being an extension of Q^* by virtue of Step 2, R is an ordering extension of Q^*. To complete the proof of this step, we have only to prove that Q^* is an extension of Q, viz., $Q \subseteq Q^*$ and $P(Q) \subseteq P(Q^*)$. The former is obvious by definition of Q^*, viz., (21). To prove the latter, assume that $(x, y) \in P(Q)$, viz., $(x, y) \in Q$ and $(y, x) \notin Q$. It follows from $(x, y) \in Q$ that $(x, y) \in Q^*$. Assume that $(y, z) \in Q^*$. Clearly $(y, x) \notin \Delta$, since otherwise $(x, y) \in P(Q)$ could not be true. Thus, we must obtain $(y, x) \in TC(Q)$. When $(x, y) \in P(Q)$ is added to this, we have a contradiction with the consistency of Q.

Step 5: For all $x, y \in S, (x, y) \in R$ holds if and only if $(x, y) \in T$.

By virtue of (22) and Step 4, we have $T \subseteq Q^{**} \subseteq R$. Thus, we have only to show that $(x, y) \in R$ implies $(x, y) \in T$. Suppose to the contrary that $(x, y) \notin T$ for some

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$x, y \in S$. Thanks to Step 3, we then obtain $(x, y) \notin Q^{**}$. T being complete on S, $(y, x) \in T$ holds which implies $(y, x) \in Q^{**}$ by virtue of (22). R being an ordering extension of Q^{**}, we then obtain $(y, x) \in R$ and $(x, y) \notin R$. This completes the proof. $\parallel$

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