Class and Exploitation in General Convex Cone Economies

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Abstract

In this paper, we reexamine the mathematical analysis of Marxian exploitation theory. First, we reexamine the validity of the two types of Marxian labor exploitation, Morishima’s (1974) type and Roemer’s type (1982), in the argument of Class-Exploitation Correspondence Principle (CECP) [Roemer (1982)]. We show that CECP does not hold true in general convex cone economies if the formulation of labor exploitation is given by the Morishima type (resp. the Roemer type). Thus, we propose two new definitions of labor exploitation that preserve CECP as a theorem even in general convex cone economies. Furthermore, we characterize the necessary and sufficient condition for plausible formulations of Marxian exploitation to preserve CECP as a theorem. Finally, we examine the performance of these new definitions in terms of Fundamental Marxian Theorem (FMT).

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1 Introduction

Marxian exploitation of labor is the difference between labor hours an individual provides and labor hours necessary to produce a consumption bundle the individual can purchase via his income. This concept is used as an index of ‘unjust’ distribution. That is, the existence of labor exploitation should reflect the existence of ‘unjust’ distribution in some sense.

During the 1970’s and 1980’s, there were remarkable developments in the debate about this concept in mathematical Marxian economics. Fundamental Marxian Theorem (FMT) was originally proved by Okishio (1963) and later named as such by Morishima (1973). FMT showed a correspondence between the existence of positive profit and the existence of exploitation. It gives us a useful characterization for non-trivial equilibria, where a trivial equilibrium is one such that its social production point is zero.\(^1\)\(^2\)

There has also been a substantial development in works on exploitation of labor: “General Theory of Exploitation and Class” promoted by Roemer (1982, 1986). It argues that in a capitalist economy if the labor supplied by agents is inelastic with respect to their wealth (that is, the value of their own capital), then Class-Exploitation Correspondence Principle (CECP) can hold. This argument implies that under identical preferences of agents with the above mentioned inelastic labor supply condition, class and exploitation status in the capitalist economy accurately reflect inequality in the distribution of wealth. That is, the wealthier agents are exploiters, and they can rationally choose from all of classes in society to belong to the capitalist class. In contrast, the least wealthy agents are exploited, and they cannot

\(^1\)Note that FMT was originally considered to prove the classical Marxian argument that the exploitation of labor is the source of positive profits in the capitalist economy. However, it does not follow from FMT that the exploitation of labor is the unique source of positive profits. The reason is that any commodity can be shown to be exploited in a system with positive profits whenever the exploitation of labor exists. This observation was pointed out by Brody (1970), Bowles and Gintis (1981), Samuelson (1982), and was named “Generalized Commodity Exploitation Theorem (GCET)” by Roemer (1982).

\(^2\)After the seminal work by Morishima (1973), there were many generalizations and discussions of FMT. While the original FMT is discussed in the simple Leontief economy with homogeneous labor, the generalization of FMT to the Leontief economy with heterogeneous labor was made by Fujimori (1982), Krause (1982), etc. The problem of generalizing FMT to the von Neumann economy was discussed by Steedman (1977) and one solution was proposed by Morishima (1974). Furthermore, Roemer (1980) generalized the theorem to a convex cone economy. These arguments may reflect the robustness of FMT.
but choose to belong to the *working class*: there is no other available option for the least wealthy agents. Thus, the existence of labor-exploiters and labor-exploited reflects unequal opportunity of life options.\(^3\)

In this paper, we first reexamine two definitions of Marxian exploitation of labor which have been discussed in Marxian exploitation theory: one is Morishima’s (1974) type and the other is Roemer’s type (1982). We show that in general convex cone economies, CECP no longer holds true, whichever of the Roemer and the Morishima types of exploitation we assume. Second, we propose two new definitions of Marxian exploitation of labor, each of which is given as the difference between one unit of labor supplied by an agent per day and the amount of direct labor socially necessary to provide the agent with his *income* per day. In contrast to the two traditional definitions, CECP can be shown to hold true in general convex cone economies under the two new definitions. We could also resolve, under the two new definitions, most of the difficulties that Marxian economic theory has faced. That is, the difficulty of FMT in the general convex cone economy, that Petri (1980) and Roemer (1980) pointed out, is resolved.

Note that Roemer (1982) argued that the epistemological role of CECP in our understanding of the capitalist economy is as an axiom, although the formal version of it emerges as a theorem. So, if we wish to verify CECP, we must seek an appropriate model which will preserve this principle as a theorem. By this reason, Roemer (1982) insisted that the Roemer (1982) definition of labor exploitation is superior to the Morishima (1974) one. Based upon this argument, which he made by himself, however, the Roemer (1982) type of labor exploitation will also be shown to be unjustified, since CECP fails to hold even in the model with the Roemer (1982) exploitation. Hence, we propose a plausible axiom that every formulation of labor exploitation should meet to be considered ‘Marxian.’ By using this axiom, we discuss what is the necessary and sufficient condition for ‘Marxian’ formulations of labor exploitation to preserve CECP as a theorem in general convex cone economies. Based upon this characterization result, we discuss that the above

\(^3\)This argument was criticized by some Marxian theorists, such as Bowles and Gintis (1990) and Devine and Dymski (1991, 1992), since it assumed a standard neoclassical labor market, which was regarded as not a *real* but an *ideal* model of the capitalist economy by these critics. However, as Yoshihara (1998) showed, those theorems essentially hold true even if the neoclassical labor market is replaced by a non-neoclassical labor market with efficiency wage contracts, which was interpreted as a more realistic aspect of the capitalist economy by those same critics.
mentioned new definitions are the most plausible formulations for Marxian exploitation of labor.

In the following paper, section 2 defines a basic economic model with convex cone production technology, and also introduces the equilibrium notion in this paper and alternative formulations, including our new definitions, for Marxian exploitation of labor. Section 3 discusses the robustness of CECP under the various definitions of labor exploitation in general convex cone economies. Section 4 discusses the performance of the new definition in terms of FMT. Finally, section 5 provides some concluding remarks.

2 The Basic Model

2.1 Production

Let \( P \) be the production set, which is assumed to be a convex cone. \( P \) has elements of the form \( \alpha = (-\alpha_0, -\alpha, \overline{\alpha}) \) where \( \alpha_0 \in \mathbb{R}_+ \), \( \alpha \in \mathbb{R}^m_+ \), and \( \overline{\alpha} \in \mathbb{R}_+^m \). Thus, elements of \( P \) are vectors in \( \mathbb{R}^{2m+1} \). The first component, \(-\alpha_0\), is the direct labor input of the process \( \alpha \); and the next \( m \) components, \(-\alpha\), are the inputs of goods used in the process; and the last \( m \) components, \( \overline{\alpha} \), are the outputs of the \( m \) goods from the process. We denote the net output vector arising from \( \alpha \) as \( \mathbf{b}^{\alpha} \equiv \alpha - \alpha \). We assume that \( P \) is a closed convex cone containing the origin in \( \mathbb{R}^{2m+1} \). Moreover, it is assumed that:

A 1. \( \forall \alpha \in P \) s.t. \( \alpha_0 \geq 0 \) and \( \underline{\alpha} \geq 0 \), \( \overline{\alpha} \geq 0 \Rightarrow \alpha_0 > 0 \); and

A 2. \( \forall \) commodity \( m \) vector \( c \in \mathbb{R}_+^m \), \( \exists \alpha \in P \) s.t. \( \hat{\alpha} \geq c \).

Given such \( P \), we will sometimes use the notations like \( P(\alpha_0 = 1) \) and \( \hat{P}(\alpha_0 = 1) \), where

\[
P(\alpha_0 = 1) \equiv \{(-\alpha_0, -\alpha, \overline{\alpha}) \in P \mid \alpha_0 = 1\}
\]

and

\[
\hat{P}(\alpha_0 = 1) \equiv \{\hat{\alpha} \in \mathbb{R}^m \mid \exists \alpha = (-1, -\alpha, \overline{\alpha}) \in P \ s.t. \overline{\alpha} - \alpha \geq \hat{\alpha}\}.
\]

Given a market economy, any price system is denoted by \( p \in \mathbb{R}_+^m \), which is a price vector of \( m \) commodities. Moreover, a subsistent vector of commodities \( b \in \mathbb{R}_+^m \) is also necessary in order to supply one unit of labor per day. We assume that the nominal wage rate is normalized to unity when it purchases the subsistent consumption vector only. By assumption, \( p b = 1 \) holds.
2.2 A Model of Accumulation and Marxian Equilibrium Notion

For the sake of simplicity, we follow the same setting as that in Roemer (1982; Chapter 5). That is, our schematic model of a capitalist economy is that all agents are accumulators who seek to expand the value of their endowments as rapidly as possible. Let us denote the set of agents by $N$ with generic element $\nu$. All agents have access to the same technology $P$, but they differ in their bundles of endowments. An agent $\nu \in N$ can engage in three types of economic activity: he can sell his labor power $\gamma_0^\nu$, he can hire the labor powers of others to operate $\beta_0^\nu = \alpha_0^\nu - \beta_0^\nu$, or he can work for himself to operate $\alpha_0^\nu = (\alpha_0^\nu, \beta_0^\nu, \gamma_0^\nu) \in P$. His constraint is that he must be able to afford to lay out the operating costs in advance for the activities he chooses to operate, either with his own labor or hired labor, funded by the value of his endowment. He can choose the activity level of each of $\alpha_0^\nu$, $\beta_0^\nu$, and $\gamma_0^\nu$ under the constraints of his capital and labor endowments. Thus, given $(p, w)$, where $w$ is a nominal wage rate, his program is:

$$\max_{(\alpha^\nu; \beta^\nu; \gamma_0^\nu)} \left[ p \left( \alpha_0^\nu - \alpha^\nu \right) \right] + \left[ p \left( \beta_0^\nu - \beta^\nu \right) - w \beta_0^\nu \right] + \left[ w \gamma_0^\nu \right]$$

such that

$$p \alpha_0^\nu + p \beta_0^\nu \leq p \omega^\nu \equiv W^\nu,$$

$$\alpha_0^\nu + \gamma_0^\nu \leq 1.$$

Given $(p, w)$, let $A^\nu (p, w)$ be the set of actions $(\alpha^\nu; \beta^\nu; \gamma_0^\nu) \in P \times P \times \mathbb{R}_+$ which solve $\nu$'s program at prices $(p, w)$.

The equilibrium notion of this model is given as follows:

**Definition 1** [Roemer (1982; Chapter 5)]: A **reproducible solution** (RS) for the economy specified above is a pair $((p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N})$, where $p \in \mathbb{R}^m_+, w \geq pb = 1$, and $(\alpha^\nu; \beta^\nu; \gamma_0^\nu) \in P \times P \times \mathbb{R}_+$, such that:

(a) $\forall \nu \in N$, $(\alpha^\nu; \beta^\nu; \gamma_0^\nu) \in A^\nu (p, w)$ (revenue maximization);

(b) $\underline{\alpha} + \underline{\beta} \leq \omega$ (social feasibility),

where $\underline{\alpha} \equiv \sum_{\nu \in N} \alpha^\nu$, $\underline{\beta} \equiv \sum_{\nu \in N} \beta^\nu$, and $\omega \equiv \sum_{\nu \in N} \omega^\nu$;

(c) $\beta_0^\nu \leq \gamma_0^\nu$ (labor market equilibrium).
where $\beta_0 \equiv \sum_{\nu \in N} \beta'_0$ and $\gamma_0 \equiv \sum_{\nu \in N} \gamma'_0$; and
\[ (d) \hat{\alpha} + \hat{\beta} \geq \alpha_0 b + \beta_0 b \text{ (reproducibility),} \]
where $\hat{\alpha} \equiv \sum_{\nu \in N}(\alpha' - \hat{\alpha}'\nu)$, $\hat{\beta} \equiv \sum_{\nu \in N}(\beta' - \hat{\beta}'\nu)$, and $\alpha_0 \equiv \sum_{\nu \in N} \alpha'_0$.

The three parts except (a) need some comments. Part (d) says that net outputs should at least replace employed workers’ total consumption. This is equivalent to requiring that the vector of social endowments does not decrease in terms of components, because (d) is equivalent to $\omega - (\hat{\alpha} + \alpha_0 b) + \hat{\beta} \geq \omega$, where the right hand side is the social stocks at the beginning of this period, the left hand side is the stocks at the beginning of the next period. Part (b) says that intermediate inputs must be available from current stocks. Here, we assume that wage goods are dispensed at the end of each production period, therefore stocks need not be sufficient to accommodate them as well. Finally, (c) is the condition of labor market equilibrium. This condition allows strict inequality between labor demand $\beta_0$ and labor supply $\gamma_0$. If it holds in strict inequality, then the nominal wage rate is driven down to the subsistence wage $w = pb = 1$. If it holds in equality, then it might hold that $w \geq pb = 1$.

The existence of the reproducible solution in this definition is guaranteed. Let $P(\omega) \equiv \{\alpha = (-\alpha_0, -\alpha, \underline{\alpha}) \in P \mid \alpha \leq \omega\}$ and $\alpha_0(\omega) \equiv \max\{\alpha_0 \mid \exists \alpha = (-\alpha_0, -\alpha, \underline{\alpha}) \in P(\omega)\}$. Then:

**Proposition 1:** Let $b \in \mathbb{R}^m_+$ and $\alpha_0(\omega) < |N|$. Under $A1$, $A2$, and stationary expectation of prices, a reproducible solution (RS) of Definition 1 exists for the economy specified above.

**Proof.** It follows from Theorem 2.5 of Roemer (1980; 1981) that for any non-negative values $(W'\nu)_{\nu \in N}$, a quasi-reproducible solution (QRS) $(p, (\alpha'; \beta'; \gamma'_0)_{\nu \in N})$ [Roemer (1980; 1981)] exists. (Note that QRS is defined by Definition 1(a), 1(c), and 1(d).) Thus, it suffices to show that for each $(\omega'\nu)_{\nu \in N}$, there exists a QRS $(p, (\alpha'; \beta'; \gamma'_0)_{\nu \in N})$ which constitutes an RS under the initial endowments $(\omega'\nu)_{\nu \in N}$. Take any $(\omega'\nu)_{\nu \in N}$ such that for any $\nu \in N$, $\omega' \in \mathbb{R}^m_+$. Let $S \equiv \{p \in \mathbb{R}^m_+ \mid pb = 1\}$. Given $(\omega'\nu)_{\nu \in N}$, let $\mathbb{W} : S \rightarrow \mathbb{R}^m_+$ such that for any $p \in S$, $\mathbb{W}(p) = (p\omega'\nu)_{\nu \in N}$. Let $\varphi : \mathbb{R}^n_+ \rightarrow S$ be a correspondence such that for any $\mathbb{W} = (W'\nu)_{\nu \in N} \in \mathbb{R}^n_+$, $p \in \varphi(\mathbb{W})$ implies that there exists $(\alpha'; \beta'; \gamma'_0)_{\nu \in N} \in P^n$ such that $(p, (\alpha'; \beta'; \gamma'_0)_{\nu \in N})$ is a QRS under $\mathbb{W}$. Then, define $\Psi \equiv \varphi \circ \mathbb{W}$. Thus, $\Psi$ is a correspondence from $S$ into itself.

We can show that this $\Psi$ is upper hemi-continuous with non-empty, convex-compact valued. First, it is obvious that $\Psi$ is non-empty compact-
valued. Since $W$ is a continuous function, so that it suffices to show $\varphi$ is upper semi-continuous. Let $W^\mu \to W$ as $\mu \to \infty$, $p^\mu \in \varphi (W^\mu)$ for each $\mu$, and $p^\mu \to p$. Suppose $p \notin \varphi (W)$. Then, by definition of QRS, it implies that for any $(\alpha, \beta)$ with $\alpha \equiv \sum_{\nu \in N} \alpha^\nu$ and $\beta \equiv \sum_{\nu \in N} \beta^\nu$ such that $(\alpha^\nu; \beta^\nu; \gamma_0^\nu) \in \mathcal{A}^\nu (p, 1)$ for any $\nu \in N$, $\hat{\alpha} + \hat{\beta} \notin (a_0 + \beta_0) b$ holds. Then, for large enough $\mu$, $p^\mu$ has only $(\alpha^\nu; \beta^\nu) \in \mathcal{A} (p^\mu, 1) \equiv \sum_{\nu \in N} \mathcal{A}^\nu (p^\mu, 1)$ with $\alpha^\mu + \beta^\mu \notin (a_0^\mu + \beta_0^\mu) b$. This is a contradiction, since $p^\mu \in \varphi (W^\mu)$. Thus, $p \in \varphi (W)$. Finally, we can show that $\varphi (W)$ is convex-valued for any $W \in \mathbb{R}_+^n$. Let $p, p' \in \varphi (W)$ such that $(p, (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N})$ and $(p', (\alpha'^\nu; \beta'^\nu; \gamma_0'^\nu)_{\nu \in N})$ are QRSs under $W$. Note that $p$ is a supporting price for the production plan $\alpha + \beta$, whereas $p'$ is a supporting price for $\alpha' + \beta'$. Take any $p'' \equiv tp + (1 - t) p'$, where $t \in (0, 1)$. Then, since $P$ is convex-cone, there exists $\alpha'' + \beta''$ such that $\alpha'' + \beta'' = s (\alpha + \beta) + (1 - s) (\alpha' + \beta')$ for some $s \in [0, 1]$, and $p''$ is a supporting price for $\alpha'' + \beta''$. Thus, there exists $(\alpha''^\nu; \beta''^\nu; \gamma_0''^\nu)_{\nu \in N} \in \mathcal{X}_{\nu \in N} \mathcal{A}^\nu (p'', 1)$ such that $\sum_{\nu \in N} \alpha''^\nu + \sum_{\nu \in N} \beta''^\nu = \alpha'' + \beta''$. Since $\alpha'' + \beta''$ is a convex combination between $\alpha + \beta$ and $\alpha' + \beta'$, $\alpha''^\nu + \beta_0''^\nu < |N|$ holds true, so that $\beta_0''^\nu < \gamma_0''^\nu = |N| - \alpha_0''$. This condition is consistent with the wage rate $w = 1$.

Furthermore, $\hat{\alpha}'' + \hat{\beta}'' \geq (\alpha_0'' + \beta_0''^\nu) b$ follows from $\hat{\alpha} + \hat{\beta} \geq (a_0 + \beta_0) b$ and $\alpha'' + \beta'' \geq (\alpha_0'' + \beta_0''^\nu) b$. Thus, $(p'', (\alpha''^\nu; \beta''^\nu; \gamma_0''^\nu)_{\nu \in N})$ is a QRS under $W$. This implies $p'' \in \varphi (W)$.

Hence, by the Kakutani’s fixed point theorem, there exists $p^* \in S$ such that $p^* \in \Psi (p^*)$. By the construction of $\Psi$, this $p^*$ has its corresponding $(\alpha^*; \beta^*; \gamma_0^*_{\nu \in N})$ such that $(p^*, (\alpha^*; \beta^*; \gamma_0^*_{\nu \in N})_{\nu \in N})$ constitutes an RS under $(\omega^*_{\nu \in N})$. ■

Given an RS $((p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N})$, let $\alpha^{p,w} \equiv \sum_{\nu \in N} \alpha^\nu + \sum_{\nu \in N} \beta^\nu$, which is the aggregate production activity actually accessed in this RS. Thus, the pair $((p, w), \alpha^{p,w})$ is the summary information of this RS. In the following, we sometimes use $((p, w), \alpha^{p,w})$ or only $(p, w)$ for the representation of the RS $((p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N})$. Also, in the following, we assume an RS with full employment (that is, Definition 1(c) holds in equality) for the sake of simplicity. Then, under any such RS $((p, w), \alpha^{p,w})$, every agent $\nu \in N$ gets a revenue $\Pi^\nu (p, w) \equiv \pi_{\text{max}} (p, w) p \omega^\nu + w$. 

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2.3 Various Formulations for Marxian Exploitation of Labor

In economic models with joint production such as the von Neumann model and the convex cone model, the issue of what is a plausible formulation for Marxian exploitation of labor is controversial. There are two formulations for Marxian exploitation of labor, which are well-known in the literature of mathematical Marxian economics; one is Morishima (1974), and the other is Roemer (1982). In addition to these, there are various potential formulations that are plausible definitions for Marxian exploitation of labor. In this subsection, we discuss a general condition that every formulation for labor exploitation has to satisfy to be considered Marxian. Then, by this condition, the class of plausible formulations for Marxian exploitation of labor is identified. We show that both the Morishima (1974) and the Roemer (1982) definitions meet these conditions. We also introduce three alternative definitions for Marxian exploitation of labor, which also meet the condition.

Given any economy \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \), and any RS \((p, w), \alpha^{p,w}\), let \( N^\text{ter} \subseteq N \), \( N^\text{ted} \subseteq N \), and \( N^\text{ter} \cap N^\text{ted} = \emptyset \). Also, let \( B(p, \Pi^\nu (p, w)) \equiv \{ f^\nu \in \mathbb{R}^m_+ \mid pf^\nu = \Pi^\nu (p, w) \} \), \( B_+ (p, \Pi^\nu (p, w)) \equiv \{ f^\nu \in \mathbb{R}^m_+ \mid pf^\nu \geq \Pi^\nu (p, w) \} \), and \( B_- (p, \Pi^\nu (p, w)) \equiv \{ f^\nu \in \mathbb{R}^m_+ \mid pf^\nu \leq \Pi^\nu (p, w) \} \). Let \( c \in \mathbb{R}^m_+ \) be a vector of produced commodities. Let

\[
\phi (c) \equiv \{ \alpha \in \mathcal{P} \mid \alpha_0 = 1 \} \cap \mathbb{R}^m_+ \text{ such that } \bar{p} \geq pc_0 \text{ and for any } \nu \in N,
\]

\[
\nu \in N^\text{ter} \iff \exists c' \in B_-(p, \Pi^\nu (p, w)) \text{ s.t. } c' \geq \bar{c} \text{ and } \exists \alpha \in \phi (c') \text{ with } \alpha_0 > 1;
\]

\[
\nu \in N^\text{ted} \iff \exists c'' \in B_+(p, \Pi^\nu (p, w)) \text{ s.t. } c'' \leq c \text{ and } \exists \alpha \in \phi (c'') \text{ with } \alpha_0 < 1.
\]

The axiom \textbf{LE} requires choosing two commodity vectors \( \bar{c}, \underline{c} \in \mathbb{R}^m_+ \), each of which can be produced as a net output by supplying one unit of labor. These \( \bar{c} \) and \( \underline{c} \) are considered as reference consumption bundles to identify the income range of non-exploited non-exploiting agents: any agent \( \nu \in N \) with income \( pc \leq \Pi^\nu (p, w) \leq p\bar{c}, \) who supplies one unit of labor, is regarded as
neither exploited nor exploiting, since the amount of socially necessary labor that he can receive from any consumption through his income is exactly one unit. Thus, if an agent $\nu \in N$ supplies one unit of labor and receives $\Pi^\nu(p, w) < p_c$, then he has a consumption bundle $c^\nu \in B^-(p, \Pi^\nu(p, w))$ with $c^\nu \leq c$ such that $c^\nu$ is produced as a net output with less than one unit of labor. Then, LE requires that such an agent should be defined as ‘exploited.’

The parallel argument can be also applied to the case of ‘exploiter.’ We think all potential formulations for Marxian notion of labor exploitation should have this property.

We can see that both the Morishima (1974) and the Roemer (1982) definitions of labor exploitation, which we will provide below, satisfy this axiom.

First:

**Definition 2:** The Morishima (1974) labor value of commodity vector $c$, $l.v. (c)$, is given by

$$l.v. (c) \equiv \min \{ \alpha_0 \mid \alpha = (-\alpha_0, -\alpha_1, \alpha_2) \in \phi (c) \} .$$

It is easy to see that $\phi (c)$ is non-empty by A2. Also,

$$\{ \alpha_0 \mid \alpha = (-\alpha_0; -\alpha_1; \alpha_2) \in \phi (c) \}$$

is bounded from below by 0, by the assumption $0 \in P$ and A1. Thus, $l.v. (c)$ is well-defined since $P$ is compact. Moreover, by A1, $l.v. (c)$ is positive whenever $c \neq 0$, so that $e (c)$ is well-defined.

Then:

**Definition 3:** A producer $\nu \in N$ is exploited in the Morishima (1974) sense if and only if:

$$\max_{f^\nu \in B(p, \Pi^\nu(p, w))} l.v. (f^\nu) < 1,$$

and he is an exploiter in the Morishima (1974) sense if and only if:

$$\min_{f^\nu \in B(p, \Pi^\nu(p, w))} l.v. (f^\nu) > 1.$$

To see Definition 3 satisfies LE, let us define:

$$\zeta \equiv \{ c \in \mathbb{R}^m_+ \mid \exists \alpha \in \phi (c) : \alpha_0 = 1 \& \alpha_0 \text{ is minimized over } \phi (c) \} .$$

That is, $\zeta = \partial \tilde{P} (\alpha_0 = 1) \cap \mathbb{R}^m_+$. Then, given an RS $(p, w)$, let $c \in \zeta$ be such that $p_c c \geq pc$ for all $c \in \zeta$. Also, let $c \in \zeta$ be such that $p_c \leq pc$ for all $c \in \zeta$. 9
We can check that \( \nu \) is an exploiter in the Morishima (1974) sense if and only if \( \Pi' (p, w) > p \bar{c} \). Also, \( \nu \) is exploited in the Morishima (1974) sense if and only if \( \Pi' (p, w) < p \bar{c} \). This argument implies that Definition 3 satisfies LE.

In contrast to the Morishima (1974) labor value, the definition of labor value in Roemer (1982) depends, in part, on the particular equilibrium the economy is in. Given a price system \((p, w)\), let

\[
\pi_{\text{max}} (p, w) \equiv \max \left\{ \frac{p \alpha - (p \alpha + w \alpha_0)}{p \alpha} \mid \alpha = (-\alpha_0, -\alpha, \overline{\alpha}) \in P \right\}
\]

and

\[
\mathcal{P} (p, w) \equiv \left\{ \alpha = (-\alpha_0, -\alpha, \overline{\alpha}) \in P \mid \frac{p \alpha - (p \alpha + w \alpha_0)}{p \alpha} = \pi_{\text{max}} (p, w) \right\}.
\]

Then, let

\[
\phi (c; p, w) \equiv \left\{ \alpha \in \mathcal{P} (p, w) \mid \tilde{\alpha} \geq c \right\},
\]

which is the set of those profit-rate-maximizing actions which produce, as net output vectors, at least \(c\). Then:

**Definition 4:** The Roemer (1982) labor value of commodity vector \(c\), \( l.v. (c; p, w) \), is given by

\[
l.v. (c; p, w) \equiv \min \{ \alpha_0 \mid \alpha = (-\alpha_0, -\alpha, \overline{\alpha}) \in \phi (c; p, w) \}.
\]

Then:

**Definition 5:** Let \((p, w)\) be a price of RS. A producer \(\nu \in N\) is exploited in the Roemer (1982) sense if and only if:

\[
\max_{f' \in B(p, \Pi' (p, w))} l.v. (f'; p, w) < 1,
\]

and he is an exploiter in the Roemer (1982) sense if and only if:

\[
\min_{f' \in B(p, \Pi' (p, w))} l.v. (f'; p, w) > 1.
\]

It is easy to verify that \(l.v. (c; p)\) is well-defined, and has a positive value whenever \(c \neq 0\). Also, \(l.v. (c; p) \geq l.v. (c)\) holds.

To see that Definition 5 satisfies LE, let us define for any \((p, w)\),

\[
\theta (p, w) \equiv \left\{ c \in \mathbb{R}_+^m \mid \exists \alpha \in \phi (c; (p, w)) : \alpha_0 = 1 \text{ & } \alpha_0 \text{ is minimized over } \phi (c; (p, w)) \right\}.
\]
Then, given an RS \((p, w)\), let \(\tau \in \theta_{(p,w)}\) be such that \(p\tau \geq pc\) for all \(c \in \theta_{(p,w)}\). Also, let \(\zeta \in \theta_{(p,w)}\) be such that \(p\zeta \leq pc\) for all \(c \in \theta_{(p,w)}\). We can check that \(\nu\) is an exploiter in the Roemer (1982) sense if and only if \(\Pi^\nu(p, w) > p\tau\). Also, \(\nu\) is exploited in the Roemer (1982) sense if and only if \(\Pi^\nu(p, w) < p\zeta\).

This argument implies that Definition 5 satisfies LE.

In addition to the above two definitions of labor exploitation, we also propose two new definitions. Following Roemer (1982), we still adopt the definition of labor value of commodities as in Definition 4. However, we refine the definition of labor exploitation from Roemer’s (1982). The first new definition is given as follows:

**Definition 6:** Let \(((p, w), \alpha_{p,w})\) be an RS. A producer \(\nu \in N\) is exploited if and only if:

\[
\min_{f^\nu \in B(p, \Pi^\nu(p, w))} l.v.\left(f^\nu; p, w\right) < 1,
\]

and he is an exploiter if and only if:

\[
\min_{f^\nu \in B(p, \Pi^\nu(p, w))} l.v.\left(f^\nu; p, w\right) > 1.
\]

We can see that Definition 6 satisfies LE by choosing \(\tau \in \theta_{(p,w)}\) as \(p\tau \geq pc\) for all \(c \in \theta_{(p,w)}\), and \(\zeta = \tau\).

Note that \(\min_{f^\nu \in B(p, \Pi^\nu(p, w))} l.v.\left(f^\nu; p, w\right)\) in Definition 6 can be regarded as the indirect labor value of \(\nu\)’s income. This implies that the labor value in Definition 6 is concerned not with an agent’s consumption vector, but rather with an agent’s income earned. Thus, this new definition implies the following: Suppose an economy is under a reproducible solution \(((p, w), \alpha_{p,w})\). Then, if the minimal amount of labor socially necessary to provide each agent \(\nu\) with income \(\Pi^\nu(p, w)\) under the RS \(((p, w), \alpha_{p,w})\) is less (resp. more) than unity, then \(\nu\) is exploited (resp. exploiter).

The second new definition is given as follows. Given any RS \(((p, w), \alpha_{p,w})\), let \(\hat{\alpha}^N_{p,w} = \frac{\alpha_{p,w}}{\alpha^0}\). Moreover, for any \(\nu \in N\), let \(t^\nu \geq 0\) be such that \(pt^\nu\hat{\alpha}^N_{p,w} = \Pi^\nu(p, w)\). Then:

**Definition 7:** Let \(((p, w), \alpha_{p,w})\) be an RS. A producer \(\nu \in N\) is exploited if and only if:

\[
l.v.\left(t^\nu\hat{\alpha}^N_{p,w}; p, w\right) < 1,
\]

and he is an exploiter if and only if:

\[
l.v.\left(t^\nu\hat{\alpha}^N_{p,w}; p, w\right) > 1.
\]
We can see that Definition 7 satisfies LE by choosing $\tau = \hat{\alpha}_{p,w}^N$ and $\zeta = \hat{\alpha}_{p,w}^N$.

The labor value in Definition 7 is also concerned not with an agent’s consumption vector, but rather with an agent’s income earned. The difference of Definition 7 from Definition 6 is that the amount of labor socially necessary to provide each agent $\nu$ with income $\Pi^\nu(p, w)$ is given by examining the actually accessed social production path $\alpha_{p,w}$, rather than the minimizer over $\mathcal{T}(p, w)$. Under this definition, the following relationship holds:

$$\text{total labor employed} = \text{labor value of national income} = \text{net product}.$$ 

This macroeconomic identity has been required as a basic property of labor value in Marxian economic theory.\(^4\)

We may also consider a more subjective notion of labor exploitation. Suppose that there is a representative agent of this economy, and introduce this agent’s welfare function $U : \mathbb{R}_m^+ \to \mathbb{R}$. This $U$ is continuous and strictly monotonic on $\mathbb{R}_m^+$, and it should have the following property: for any RS $((p, w), \alpha_{p,w})$, $\hat{\alpha}_{p,w}^N$ is the maximizer of $U(c)$ over $B(p, p\hat{\alpha}_{p,w}^N)$. Given this welfare function $U$, let $c_{U}^{\text{max}} \in \mathbb{R}_m^+$ be the maximizer of $U(c)$ over $\zeta$. Then:

**Definition 8:** Let $((p, w), \alpha_{p,w})$ be an RS. A producer $\nu \in N$ is exploited if and only if:

$$\Pi^\nu(p, w) < p c_{U}^{\text{max}},$$

and he is an exploiter if and only if:

$$\Pi^\nu(p, w) > p c_{U}^{\text{max}}.$$

We can see that Definition 8 satisfies LE by choosing $\tau = c_{U}^{\text{max}}$ and $\zeta = c_{U}^{\text{max}}$.

This definition is extended from Matsuo (2006), although Matsuo provides only the definition of exploited agents in order to discuss FMT.

### 3 CECP in Accumulation Economies

In the following discussion, we will examine the viability of the above five definitions of labor exploitation respectively by checking whether CECP [Roemer

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\(^4\) The macroeconomic identity is also satisfied by the labor value formulation of Flaschel (1983), although it has an extremely different form from the other formulations: the labor value formulation of Flaschel (1983) is given by an *additive sum of individual labor values*, whereas the Morishima (1974) and the Roemer (1982) formulations of labor value are respectively given by an *optimal value* of labor input. Note that the labor value formulation in Definition 7 is also given by the Roemer (1982) type.
(1982; Chapter 5) holds true under each of these definitions. We will show that among these definitions, only Definitions 6 and 7 preserve CECP as a theorem even in general convex cone economies.

Following Roemer (1982; Chapter 5), let us define possible classes. At every RS in the model of section 2.2, different producers relate differently to the means of production. An individually optimal solution for an agent \( \nu \) at the RS consists of three vectors \((\alpha^\nu; \beta^\nu; \gamma^\nu_0)\). According to whether these vectors are either zero or nonzero at the RS, all producers are classified into the following four types: that is, \((+,+,0)\), \((+,0,0)\), \((+,0,+), \text{ and } (0,0,+), \) where “+” means a nonzero vector in the appropriate place. Here, the notation \((+,+,0)\) implies, for instance, that an agent works for his own ‘shop’ and hires others’ labor powers; while \((+,0,+)\) implies that an agent works for his own ‘shop’ and also sells his own labor power to others, etc.

Let us define four disjoint classes as follows:

\[
C^H = \{ \nu \in N \mid \mathcal{A}^\nu (p, w) \text{ has a solution of the form } (+,+,0) \setminus (+,0,0) \},
\]

\[
C^{PB} = \{ \nu \in N \mid \mathcal{A}^\nu (p, w) \text{ has a solution of the form } (+,0,0) \},
\]

\[
C^S = \{ \nu \in N \mid \mathcal{A}^\nu (p, w) \text{ has a solution of the form } (+,0,+) \setminus (+,0,0) \},
\]

\[
C^P = \{ \nu \in N \mid \mathcal{A}^\nu (p, w) \text{ has a solution of the form } (0,0,+) \}.
\]

We can see that the set of producers \( N \) can be divided into these four classes at any RS.

Then:

**Proposition 2** [Roemer (1982; Chapter 5)]: Let \((p, w)\) be an RS with \( \pi^{\max} (p, w) > 0 \). Then,

\[
\begin{align*}
\nu & \in C^H \iff W^\nu > \max_{\alpha \in P(p,w)} \left[ \frac{p\alpha}{\alpha_0} \right], \\
\nu & \in C^{PB} \iff \min_{\alpha \in P(p,w)} \left[ \frac{p\alpha}{\alpha_0} \right] \leq W^\nu \leq \max_{\alpha \in P(p,w)} \left[ \frac{p\alpha}{\alpha_0} \right], \\
\nu & \in C^S \iff 0 < W^\nu < \min_{\alpha \in P(p,w)} \left[ \frac{p\alpha}{\alpha_0} \right], \\
\nu & \in C^P \iff W^\nu = 0.
\end{align*}
\]

Now, CECP, which is a principle we would like to verify, is introduced as follows:
Class Exploitation Correspondence Principle (CECP) [Roemer (1982)]:
For any economy defined as in section 2, and any reproducible solution with a positive profit rate, it holds that:
(A) every member of $C^H$ is an exploiter.
(B) every member of $C^S \cup C^P$ is exploited.

First, we mention that under any of Definitions 3, 5, 6, 7, and 8, CECP holds true if the production possibility set is given by the Leontief technology. Let $A$ be an $m \times m$ non-negative square matrix with input-output coefficient $a_{ij} \geq 0$ for any $i, j = 1, \ldots, m$, and $L$ be a positive $1 \times m$ vector with labor input coefficient $L_j > 0$ for any $j = 1, \ldots, m$. Then, let $P_{(A,L)} \equiv \{(Lx, -Ax, x) | x \in \mathbb{R}_+^m \}$. Then:

**Proposition 3:** Under $A1$, $A2$, and stationary expectation of prices, let $(p, w)$ be an RS with $\pi_{\text{max}} (p, w) > 0$ for an economy $(N; (P_{(A,L)}), b; (\omega^\nu)_{\nu \in \mathbb{N}})$. Then, CECP holds under any of Definitions 3, 5, 6, 7, and 8.

Note that in the economy with Leontief technology, Definitions 3 and 5 are equivalent. In this case, CECP holds as Roemer (1982; Chapter 4) discussed.

Insert Figure 1 around here.

Figure 1 illustrates that CECP holds under Definitions 3 and 5 in a two-goods economy with Leontief technology. The complete proof of this proposition will be given after Theorem 3 is discussed.

Next, we show that in general convex cone economies, CECP holds true under Definition 6:

**Theorem 1:** Under $A1$, $A2$, and stationary expectation of prices, let $(p, w)$ be a RS with $\pi_{\text{max}} (p, w) > 0$. Then, it holds that:
(A) every member of $C^H$ is an exploiter in the sense of Definition 6.
(B) every member of $C^S \cup C^P$ is exploited in the sense of Definition 6.

**Proof.** The part (A) is already shown by Roemer (1982). Thus, we only show the part (B). Given an RS $(p, w)$, let $\bar{c} \in \theta_{(p,w)}$ be such that $p\bar{c} \geq pc$ for all $c \in \theta_{(p,w)}$. Also, let $\bar{c} \in \theta_{(p,w)}$ be such that $p\bar{c} \leq pc$ for all $c \in \theta_{(p,w)}$. Let $\theta^*_{(p,w)} \equiv \{ \alpha \in \overline{P} (p, w) | \exists c \in \mathbb{R}_+^m : l.v. (c; p, w) = 1, \alpha \geq c, \& \alpha_0 = 1 \}$.
To verify $C^S \cup C^P$ consisted of exploited agents in the sense of Definition 6, we have to show

$$\frac{p\tilde{c} - w}{\pi_{\text{max}}(p, w)} \geq \min_{\alpha \in \Gamma(p, w)} \left[ \frac{p\alpha}{\alpha_0} \right]. \quad (1)$$

Since $\overline{P}(p, w)$ is a cone, we can normalize the right hand side of the inequality (1) by taking $\alpha \in P(p, w)$ for which $\alpha_0 = 1$. Thus, by introducing a notation $\Gamma(p, w) \equiv \{ \alpha \in P(p, w) \mid \alpha_0 = 1 \}$, (1) can be reduced to

$$\frac{p\tilde{c} - w}{\pi_{\text{max}}(p, w)} \geq \min_{\alpha \in \Gamma(p, w)} p\alpha. \quad (2)$$

Note that

$$\min_{\alpha \in \Gamma(p, w)} p\alpha \leq \min_{\alpha \in \theta^*(p, w)} p\alpha \quad (3),$$

since $\Gamma(p, w) \supseteq \theta^*(p, w)$. Then, it is sufficient to show:

$$\frac{p\tilde{c} - w}{\pi_{\text{max}}(p, w)} \geq \min_{\alpha \in \theta^*(p, w)} p\alpha. \quad (4)$$

Taking $\pi_{\text{max}}(p, w) p\alpha \equiv p\tilde{c} - w$ for any $\alpha \in \Gamma(p, w)$, the inequality (4) is equivalent to:

$$p\tilde{c} - w \geq \min_{\alpha \in \theta^*(p, w)} p\tilde{\alpha} - w. \quad (5)$$

Note that $\tilde{\alpha} \in \theta^*(p, w)$ for any $\alpha \in \theta^*(p, w)$. Thus, for any $\tilde{\alpha} \in \theta^*(p, w)$ such that $\alpha \in \theta^*(p, w)$, $p\tilde{c} \geq p\tilde{\alpha}$, which implies (5) holds true. This implies the part (B) is shown. 

Insert Figure 2 around here.

Second, we discuss that except for Definitions 6 and 7, the other aforementioned three definitions cannot preserve CECP as a theorem. For any set $S \subseteq \mathbb{R}^m_+$, let $co \{ S \}$ denote the convex hull of $S$, and $\text{comp} \{ S \}$ denote the comprehensive hull of $S$. Given any economy $\langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$, and any RS $((p, w), \alpha^{p,w})$, note that $\pi_{\text{max}}(p, w) = \frac{p\tilde{\alpha}^{p,w} - w\alpha_0^{p,w}}{p\alpha_0^{p,w}}$ follows from the definition of RS. Thus, there exists $\alpha^{p,w*} \in \Gamma(p, w)$ such that for some $t > 0$, $t\alpha^{p,w*} = \alpha^{p,w}$. Moreover, there exists $c^{p,w} \in \zeta$ such that $pc^{p,w} \geq pc$ for any $c \in \zeta$. Since $\tilde{\alpha}^{p,w*} \in \zeta$ by Definition 1(d), we have $pc^{p,w} \geq p\tilde{\alpha}^{p,w*}$. Then:
Lemma 1: Under $A1$, $A2$, there exists an economy $\langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$ which has an RS $\langle (p, 1), (\alpha^p) \rangle$ such that $pc^p > p\alpha$ for any $\alpha \in \Gamma(p, 1)$.

Proof. Let us consider the following von Neumann system:

$$B = \begin{bmatrix} 5 & 3 & 9.8 & 0 \\ 25.5 & 4.5 & 0 & 5.25 \end{bmatrix}, A = \begin{bmatrix} 3.5 & 2 & 8 & 0 \\ 4.5 & 3 & 0 & 3.5 \end{bmatrix}, L = \begin{bmatrix} 0.75 & 1 & 0.6 & 1 \end{bmatrix}. $$

Define a production possibility set $P(B,A,L)$ by

$$P(B,A,L) \equiv \{ (-Lx, -Ax, Bx) \in \mathbb{R}_- \times \mathbb{R}_-^2 \times \mathbb{R}_+^2 \mid x \in \mathbb{R}_+^4 \}. $$

This $P(B,A,L)$ is a closed convex cone in $\mathbb{R}_- \times \mathbb{R}_-^2 \times \mathbb{R}_+^m$ with $0 \in P(B,A,L)$. Moreover, $P(B,A,L)$ is shown to satisfy $A1$ and $A2$.

Let $e_j \in \mathbb{R}_+^m$ be a unit column vector with 1 in the $j$-th component and 0 in any other component. Then, $\alpha^1 \equiv (-L \omega_1, -A \omega_1, B_1)$, $\alpha^2 \equiv (-L \omega_2, -A \omega_2, B_2)$, $\alpha^3 \equiv (-L \omega_3, -A \omega_3, B_3)$, and $\alpha^4 \equiv (-L \omega_4, -A \omega_4, B_4)$. Moreover,

$$\hat{\alpha}^1 \equiv (B - A) \begin{bmatrix} 1.5 \\ 0.75 \end{bmatrix}, \hat{\alpha}^2 \equiv (B - A) \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, $$

$$\hat{\alpha}^3 \equiv (B - A) \begin{bmatrix} 1.8 \\ 0 \end{bmatrix}, \hat{\alpha}^4 \equiv (B - A) \begin{bmatrix} 0 \\ 1.75 \end{bmatrix}. $$

Also, let $\hat{\alpha}(\alpha_0 = 1) = co \{(2, 1), (1, 1.5), (3, 0), (0, 1.75), 0\}$. Let $b = (1, 1)$, and the social endowment of capital is given by $\omega = (2|N|, 3|N|)$. Then, for any economy $\langle N; (P(B,A,L), b); (\omega^\nu)_{\nu \in N} \rangle$ with $\sum_{\nu \in N} \omega^\nu = \omega$, a pair $((p, 1), |N| \alpha^2)$ with $p = (0.5, 0.5)$ constitutes an RS. Note that

$$\frac{[p(B - A) - L] e_1}{pA e_1} = \frac{3}{32}, \frac{[p(B - A) - L] e_2}{pA e_2} = \frac{1}{10}, $$

$$\frac{[p(B - A) - L] e_3}{pA e_3} = \frac{3}{40}, \frac{[p(B - A) - L] e_4}{pA e_4} = -1 \frac{1}{14}. $$

This implies that $\Gamma(p, 1) = \{ \alpha^2 \} = \theta^p_{(p, 1)}$ and $\theta_{(p, 1)} = comp \{ \hat{\alpha}^2 \}$. Thus,

$$\min_{\alpha \in \Gamma(p, 1)} \frac{[p \alpha]}{[\alpha_0]} = \min_{\alpha \in \Gamma(p, 1)} \frac{p \alpha}{\alpha_0} = \max_{\alpha \in \Gamma(p, 1)} \frac{p \alpha}{\alpha_0} = \max_{\alpha \in \Gamma(p, 1)} \frac{[p \alpha]}{[\alpha_0]} = p \alpha^2. $$

Let $H_{+}(p, \alpha^2) \equiv \{ c \in \mathbb{R}_+^2 \mid pc = p\alpha^2 \}$, $H_{+}(p, \alpha^2) \equiv \{ c \in \mathbb{R}_+^2 \mid pc > p\alpha^2 \}$, and $H_{-}(p, \alpha^2) \equiv \{ c \in \mathbb{R}_+^2 \mid pc < p\alpha^2 \}$. Moreover, $\zeta_{+} \equiv \zeta \cap H_{+}(p, \alpha^2)$, $\zeta_{-} \equiv \zeta \cap H_{-}(p, \alpha^2)$. Thus,

$$\zeta_{+} = \zeta_{-} = \zeta. $$

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\( \zeta \cap H_-(p, \hat{\omega}^2) \). Note that \( \zeta_+ = \text{co} \{(1,1.5), (2,1)\} \cup \text{co} \{(2,1), (3,0)\} \setminus \{(1,1.5)\}, \)
\( \zeta_- = \text{co} \{(0,1.75), (1,1.5)\} \setminus \{(1,1.5)\}, \) and \( \hat{\omega}^2 = (1,1.5) \). Since \( \omega_{\nu_1} \in \zeta \) implies \( \omega_{\nu_1} = (2,1) \), we have \( \omega_{\nu_1} > \hat{\omega}^2 \). Thus, we obtain a desired result.

Insert Figure 3 around here.

Let \( \hat{\Gamma}(p, 1) \equiv \{ \hat{\alpha} \in \mathbb{R}_+^n \mid \alpha \in \Gamma(p, 1) \} \). Then, the following theorem gives us a necessary condition for formulations of labor exploitation satisfying \( \textbf{LE} \) in order to preserve \( \textbf{CECP} \) as a theorem:

**Theorem 2:** Under A1, A2, let \( \langle N; (P,b); (\omega_{\nu})_{\nu \in \mathbb{N}} \rangle \) be an economy with an RS \((p_1, \alpha^{n_1})\). Then, for any definition of labor exploitation satisfying \( \textbf{LE} \), if \( \textbf{CECP} \) holds under this definition, its corresponding \( \tau, \xi \in \zeta \) imply \( \tau, \xi \in \hat{\Gamma}(p, 1) \).

**Proof.** Let \( \langle N; (P,b); (\omega_{\nu})_{\nu \in \mathbb{N}} \rangle \) be an economy with an RS \((p_1, \alpha^{n_1})\) such that \( \omega_{\nu} > p\hat{\alpha} \) for any \( \alpha \in \hat{\Gamma}(p, 1) \). Let

\[
\alpha_{\max}(p,1) \equiv \arg \max_{\alpha \in \Gamma(p,1)} \omega_{\nu};\text{ and } \alpha_{\min}(p,1) \equiv \arg \min_{\alpha \in \Gamma(p,1)} \omega_{\nu}.
\]

Then, by Proposition 2, we have

\[
C^H = \{ \nu \in N \mid \Pi^\nu (p,1) > \max (p_1,\omega_{\nu}) + 1 \};
\]
\[
C^{PB} = \{ \nu \in N \mid \max (p_1,\omega_{\nu}) + 1 \leq \Pi^\nu (p,1) \leq \max (p_1,\omega_{\nu}) + 1 \};
\]
\[
C^S = \{ \nu \in N \mid 1 < \Pi^\nu (p,1) < \max (p_1,\omega_{\nu}) + 1 \};
\]
\[
C^P = \{ \nu \in N \mid \Pi^\nu (p,1) = 1 \}.
\]

Insert Figures 4 and 5 around here.

Let \( H(p, \hat{\alpha}) \equiv \{ c \in \mathbb{R}_+^m \mid pc = p\hat{\alpha} \}, \)
\( H_+(p, \hat{\alpha}) \equiv \{ c \in \mathbb{R}_+^m \mid pc > p\hat{\alpha} \}, \) and
\( H_-(p, \hat{\alpha}) \equiv \{ c \in \mathbb{R}_+^m \mid pc < p\hat{\alpha} \}. \) Moreover, \( \zeta_+ \equiv \zeta \cap H_+(p, \hat{\alpha}_{\max}(p,1)), \)
\( \zeta_- \equiv \zeta \cap H_-(p, \hat{\alpha}_{\min}(p,1)). \) Then, \( \zeta = \zeta_+ \cup \hat{\Gamma}(p,1) \cup \zeta_- \).

**Case 1:** Consider any definition of labor exploitation satisfying \( \textbf{LE} \), and for this definition, its corresponding \( \tau, \xi \in \zeta \) have the property that \( \tau \in \zeta_+ \).
Thus, \( \omega_{\nu} > p\hat{\alpha}_{\max}(p,1) \). Since \( p\hat{\alpha}_{\max}(p,1) = \max (p_1,\omega_{\nu}) + 1 \), we can
construct an economy \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) with \( \sum_{\nu \in N} \omega^\nu = \omega \), such that for some \( \nu \in N \), \( p^\nu > \pi^\max (p, 1) \omega^\nu + 1 > p^\nu \alpha^\max(p,1) \) holds. Thus, this agent \( \nu \) belongs to \( C^H \), as per Proposition 2. However, \( p^\nu > \pi^\max (p, 1) \omega^\nu + 1 = \Pi^\nu (p, 1) \) implies that \( \nu \) is not an exploiter.

**Case 2):** Consider any definition of labor exploitation satisfying LE, and for this definition, its corresponding \( \tau, c \in \zeta \) have the property that \( \tau \in \zeta \). Then, since \( p^\nu \geq p^c \) by LE, \( c \in \zeta \). Thus, \( p^c < p^\alpha \min(p,1) \). Then, we can construct an economy \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) with \( \sum_{\nu \in N} \omega^\nu = \omega \), such that for some \( \nu \in N \), \( p^\nu < \pi^\max (p, 1) \omega^\nu + 1 < p^\alpha \min(p,1) \) holds. Note that this agent \( \nu \) belongs to \( C^S \), as per Proposition 2. However, \( p^c < \pi^\max (p, 1) \omega^\nu + 1 = \Pi^\nu (p, 1) \) implies that \( \nu \) is not exploited.

**Case 3):** Finally, consider any definition of labor exploitation satisfying LE, and for this definition, its corresponding \( \tau, c \in \zeta \) have the property that \( \tau \in \Gamma(p, 1) \) and \( p^\nu > p^c \). If \( p^c < p^\alpha \min(p,1) \), then the argument of **Case 2** can be applied.

In summary, the arguments of the above three cases imply that if a definition of labor exploitation satisfying LE preserves \( \text{CECP} \) as a theorem, then its corresponding \( \tau, c \in \zeta \) imply \( \tau, c \in \Gamma(p, 1) \).

By the above Theorem 2, we can show that both the Morishima (1974) and the Roemer (1982) formulations for Marxian labor exploitation cannot preserve \( \text{CECP} \) as a theorem:

**Corollary 1:** Under A1, A2, \( \text{CECP} \) cannot hold under Definition 3.

**Proof.** Let \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) be the economy constructed in Lemma 1. Then, this economy has an RS \( (\alpha^{p,1}, \omega^\nu) \) such that \( p^\nu \alpha^{p,1} > p^\alpha \) for any \( \alpha \in \Gamma(p, 1) \). In this economy, if the Morishima (1974) formulation of labor exploitation (Definition 3) is applied, then \( c = \alpha^4 \) and

\[
\tau \in \left\{ c \in \mathbb{R}_+^2 \mid \exists t \in [0, 1] : c = t (2, 1) + (1 - t) (3, 0) \right\}.
\]

Insert Figure 6 around here.

Note \( \Gamma(p, 1) = \{ \alpha^2 \} \). Then, by Theorem 2, \( \text{CECP} \) violates under Definition 3. ■

**Corollary 2:** Under A1, A2, \( \text{CECP} \) cannot hold under Definition 5.
Proof. Let \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) be the economy constructed in Lemma 1 as in the proof of Corollary 1. In this economy, if the Roemer (1982) formulation of labor exploitation (Definition 6) is applied, then \( c = (1, 0) \) and \( \sigma = \alpha^2 \).

Insert Figure 7 around here.

Then, since \( \widehat{\Gamma}(p, 1) = \{ \alpha^2 \} \), by Theorem 2, CECP violates under Definition 5.

We can also show that even Definition 8 cannot preserve CECP as a theorem.

**Corollary 3:** Under A1, A2, CECP cannot hold under Definition 8.

Proof. Let \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) be the economy constructed in Lemma 1 as in the proof of Corollary 1. In this economy, if Definition 8 is applied as a formulation of labor exploitation, and the welfare function \( U \) of the representative agent has the following properties: \( \widehat{\alpha}^N_{(p,1)} = \alpha^2 \) and \( c^\text{max}_U = (2, 1) \). Thus, \( c = (2, 1) \) and \( \sigma = (2, 1) \).

Insert Figure 8 around here.

Then, since \( \widehat{\Gamma}(p, 1) = \{ \alpha^2 \} \), by Theorem 2, CECP violates under Definition 8.

The necessary condition for formulations of labor exploitation satisfying LE to preserve CECP as a theorem also constitutes a sufficient condition, as the following theorem shows:

**Theorem 3:** Under A1, A2, and stationary expectation of prices, if a definition of labor exploitation satisfying LE has the property that \( \sigma, c \in \widehat{\Gamma}(p, 1) \), then CECP holds true under this definition.

Proof. Let \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) be any economy with an RS \((p, 1), \alpha^p, 1\). Remember that, by Proposition 2, we have

\[
\begin{align*}
C^H &= \{ \nu \in N \mid \Pi^\nu(p, 1) > \pi^\text{max}(p, 1) \rho p^\text{max}(p, 1) + 1 \} ; \\
C^{PB} &= \{ \nu \in N \mid \pi^\text{max}(p, 1) \rho p^\text{min}(p, 1) + 1 \leq \Pi^\nu(p, 1) \leq \pi^\text{max}(p, 1) \rho p^\text{max}(p, 1) + 1 \} ; \\
C^S &= \{ \nu \in N \mid 1 < \Pi^\nu(p, 1) < \pi^\text{max}(p, 1) \rho p^\text{min}(p, 1) + 1 \} ; \\
C^P &= \{ \nu \in N \mid \Pi^\nu(p, 1) = 1 \} .
\end{align*}
\]
Since the definition of labor exploitation satisfies LE, there are \( \vec{c}, \underline{c} \in \zeta \) such that \( p\vec{c} \geq p\underline{c} \) under the RS \(((p, 1), \alpha, p)\). Note that if \( \vec{c}, \underline{c} \in \hat{\Gamma}(p, 1) \) under this definition of labor exploitation, then
\[
\pi^{\max}(p, 1) p\alpha^{\min(p, 1)} + 1 \leq p\underline{c} \leq p\vec{c} \leq \pi^{\max}(p, 1) p\alpha^{\max(p, 1)} + 1.
\]
By LE, any agent \( \nu \in N \) with \( W^\nu \) under this RS such that \( \Pi^\nu(p, 1) < p\underline{c} \) is exploited, whereas any agent \( \nu \in N \) with \( W^\nu \) under this RS such that \( \Pi^\nu(p, 1) > p\vec{c} \) is an exploiter. Thus, any \( \nu \in C^H \) becomes an exploiter, whereas any \( \nu \in C^S \cup C^P \) is exploited in this economy. Thus, CECP holds under this definition of labor exploitation.

**Proof of Proposition 3:** In the economy with Leontief technology, \( \hat{\Gamma}(p, 1) = \zeta \) holds, so the equivalence between Definitions 3 and 5 follows. Then, it is shown that CECP holds as in Roemer (1982; Chapter 4). Thus, it suffices to show that CECP holds in the case of Definition 8. Note that by definition, \( c_{U}^{\max} \in \zeta \) so that \( c_{U}^{\max} \in \hat{\Gamma}(p, 1) \) in the economy with Leontief technology. Since \( \vec{c} = c_{U}^{\max} = \underline{c} \) by Definition 8, it follows from Theorem 3 that CECP holds under Definition 8.

**Corollary 4:** Under A1, A2, CECP holds true under Definition 7.

**Proof.** Note that in Definition 7, \( \vec{c} = \hat{\alpha}^{p,w} \) and \( \underline{c} = \hat{\alpha}^{p,w} \). Since \( \hat{\alpha}^{p,w} \in \hat{\Gamma}(p, 1) \) by definition, the desired result follows from Theorem 3.

Note that Definition 6 also satisfies the condition \( \vec{c}, \underline{c} \in \hat{\Gamma}(p, 1) \).

There may potentially be another formulation of labor exploitation which satisfies LE and the condition \( \vec{c}, \underline{c} \in \hat{\Gamma}(p, 1) \). However, at least to the best of my knowledge, except for our Definitions 6 and 7, there is no other explicit formulation of labor exploitation in the current literature of mathematical Marxian economics, which satisfies LE and the condition \( \vec{c}, \underline{c} \in \hat{\Gamma}(p, 1) \). Thus, I believe that each of Definitions 6 and 7 could represent one of the most plausible formulations for Marxian exploitation of labor.
4 FMT in general convex cone economies

In this section, we discuss that the new formulations of labor exploitation given by Definitions 6 and 7 resolve the well-known difficulty in FMT under joint production economies. Let us consider an economy $\langle N; (P, b) ; (\omega^\nu)_{\nu \in N} \rangle$ in which there is a partition $N_1$ and $N_2$ of the society $N$. That is, $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. Let us assume that for any $\nu \in N_1$, $\omega^\nu \in \mathbb{R}_m^{m+}$, and for any $\nu \in N_2$, $\omega^\nu = 0$. Furthermore, every agent $\nu \in N_1$ is assumed to engage solely in operating $\beta^\nu \in P$ so as to maximize his profit revenue, whereas every agent $\nu \in N_2$ is assumed to engage solely in selling $\gamma^\nu \in [0, 1]$ so as to maximize his wage revenue.

In such a framework, Morishima (1974) showed that if the economy is under the von Neumann balanced growth equilibrium, then the warranted profit rate is positive if and only if the Morishima (1974) labor exploitation is positive (that is, $l.v. (b) < 1$). However, Petri (1980) and Roemer (1980) showed that if the economy is under the RS, then FMT cannot hold: there is a case that the maximal profit rate is positive under no exploitation in the sense of Morishima (1974) (that is, $l.v. (b) = 1$). Furthermore, Roemer (1980; 1981) showed that the following assumption is the necessary and sufficient condition for FMT to hold true under the RS and the Morishima (1974) labor exploitation:

A 3. (Independence of Production) $\forall (-\alpha_0, -\underline{a}, \overline{a}) \in P$, $\forall \alpha \geq 0$, and $\forall 0 \leq c \leq \overline{a}$, $\exists (-\alpha_0', -\underline{a}', \overline{a}') \in P$ s.t. $\overline{a}' - \underline{a}' \geq c$ and $\alpha_0' < \alpha_0$.

This assumption is rather strong, since every production set having inferior production processes is eliminated by this assumption. Moreover, just excluding such production sets is not the real resolution since the failure of FMT occurs in production sets with inferior production processes.

However, if the Morishima (1974) labor exploitation is replaced by our Definitions 6 and 7, then, without A3, FMT can hold true even under RS. The following theorems illustrate this:

Theorem 4: Under A1, A2, and stationary expectation of prices, let $((p, 1), \alpha^{p,1})$ be a reproducible solution (RS). Then, the RS yields positive total profits if and only if every worker in $N_2$ is exploited in the sense of Definition 6.
Proof. ($\Rightarrow$): Let $((p, 1), \alpha^{p, 1})$ be an RS with a positive total profit. Thus,

$$p \cdot \left( \sum_{\nu \in N_1} (\beta' - \beta^\nu) \right) - \sum_{\nu \in N_1} \beta^\nu = p \cdot \left( \sum_{\nu \in N_1} (\beta' - \beta^\nu) - \sum_{\nu \in N_1} \beta^\nu b \right) = p \cdot (\hat{\alpha}^{p, 1} - \alpha^{p, 1}_0 b) > 0.$$ 

Since $p \in \mathbb{R}_+^m$ and $\hat{\alpha}^{p, 1} \geq \alpha^{p, 1}_0 b$ by Definition 1(d), the last strict inequality implies $\hat{\alpha}^{p, 1} \geq \alpha^{p, 1}_0 b$ and $\hat{\alpha}^{p, 1} \neq \alpha^{p, 1}_0 b$. Let $f \in \mathbb{R}_+^m$ be such that $pf = pb$ and $\alpha^{p, 1}_0 f = t\hat{\alpha}^{p, 1}$ for some $0 < t < 1$. Then, by the convex cone property of the production set, $l.v.(\alpha^{p, 1}_0 f; p, 1) \leq l.v.(\alpha^{p, 1}_0 b; p, 1)$, and $l.v.(\alpha^{p, 1}_0 f; p, 1) < l.v.(\alpha^{p, 1}_0 b; p, 1)$ holds whenever $f \neq b$. Thus, $l.v.(\alpha^{p, 1}_0 f; p, 1) < \alpha^{p, 1}_0$. By linearity, $l.v.(f; p, 1) < 1$, which implies $\min_{f \in B(p, 1)} l.v.(f; p, 1) < 1$, so that every worker is exploited in the sense of Definition 6.

($\Leftarrow$): Since there is no RS with a negative total profit, it suffices to discuss only the case of zero profit. Let $(p, \alpha^{p, 1})$ be an RS with a zero total profit. Thus, $p \cdot (\hat{\alpha}^{p, 1} - \alpha^{p, 1}_0 b) = 0$. By Definition 1(d), $\hat{\alpha}^{p, 1} \geq \alpha^{p, 1}_0 b$. Let $f \in \mathbb{R}_+^m$ be such that $pf = pb$ and $\alpha^{p, 1}_0 f = t\hat{\alpha}^{p, 1}$ for some $0 < t \leq 1$. Then, $p \cdot (\hat{\alpha}^{p, 1} - \alpha^{p, 1}_0 f) = 0$ and $\alpha^{p, 1}_0 f = t\hat{\alpha}^{p, 1}$ imply that $t = 1$. Thus, $\hat{\alpha}^{p, 1} = \alpha^{p, 1}_0 b$ holds whenever $p > 0$. Note for this RS $(p, \alpha^{p, 1})$, any profit-maximizing production points $\alpha' \in \overline{P}(p, 1) \cap \partial P(\alpha_0 = 1)$ has the property that $p\hat{\alpha}' = 1$ by $\pi^{\max}(p, 1) = 0$. Thus, for any $\alpha' \in \overline{P}(p, 1) \cap \partial P(\alpha_0 = 1)$, $p\hat{\alpha}' = \frac{p\hat{\alpha}^{p, 1}}{\alpha^{p, 1}_0} = pb$.

This implies for any $f \in \mathbb{R}_+^m$ such that $pf = pb$, $l.v.(f; p, 1) \geq 1$ holds. Hence, $\min_{f \in B(p, 1)} l.v.(f; p, 1) = 1$, so that no worker is exploited in the sense of Definition 6.

If $p \geq 0$, it may be the case that $\hat{\alpha}^{p, 1} \geq \alpha^{p, 1}_0 b$ and $\hat{\alpha}^{p, 1} \neq \alpha^{p, 1}_0 b$. However, as $p \cdot (\hat{\alpha}^{p, 1} - \alpha^{p, 1}_0 b) = 0$ and $\{\beta^\nu\}_{\nu \in N_1}$ constitutes a profit-maximizing production plan at $p$, $b \in \partial \hat{P}(\alpha_0 = 1)$ holds. By the same argument as above, for any $f \in \mathbb{R}_+^m$ such that $pf = pb$, $l.v.(f; p, 1) \geq 1$ holds. Thus, $\min_{f \in B(p, 1)} l.v.(f; p, 1) = 1$, so that no worker is exploited in the sense of Definition 6.

**Theorem 5:** Under $A1$, $A2$, and stationary expectation of prices, let $((p, 1), \alpha^{p, 1})$ be a reproducible solution (RS). Then, the RS yields positive total profits if and only if every worker in $N_2$ is exploited in the sense of Definition 7.
The proof of Theorem 5 is similar to that of Theorem 4, and we therefore omit it.

Note that FMT cannot hold under RS if the labor exploitation is given by the Roemer (1982) type (Definition 5 in this paper). This difficulty cannot be resolved even if A3 is imposed. The proof is easily obtained by considering the economy that we constructed in the proof of Lemma 1. (See Figure 4.) In that economy, we can see that \( l.v. (b; p, 1) = 1 \), that implies every worker is not exploited in the sense of Roemer (1982), though the maximal profit rate is positive under the RS of that economy. Since that economy satisfies A3, we obtain the proof of the above statement.

Note also that FMT cannot hold true in general convex cone economies with heterogeneous consumption demand of workers if the definition of labor exploitation is either the Morishima (1974) type or the Roemer (1982) type. We can see that this difficulty is also resolved under Definitions 6 and 7. The detail discussion for this issue is given by Yoshihara (2006).

5 Concluding Remarks

As shown in the theorems in this paper, the proposed new definitions of labor exploitation performs well in terms of both FMT and CECP. However, the new definitions have exclusively distinct characteristics in comparison with the previous definitions such as Morishima (1974) and Roemer (1982), which may give us new insights on the Marxian theory of labor exploitation and the theory of labor value.

First, Roemer (1982) claimed that prices should emerge logically prior to labor values so as to preserve CECP as a theorem in general convex cone economies. According to all the theorems in this paper, we would also follow the above claim of Roemer (1982) in order to verify not only CECP, but also FMT in general convex cone economies.

Second, in the orthodox Marxian argument, labor exploitation was explained by using the concept of the labor value of labor power. The labor value of labor power could be defined in the Morishima (1974) framework as the minimal amount of direct labor necessary to produce the subsistent consumption vector as a net output. This could be accepted by the orthodox Marxist as the formulation of the socially necessary labor time to reproduce labor power. In such an argument, the subsistent consumption vector plays a crucial role in the formulation of the labor value of labor power. In De-
finition 6 of this paper, however, the labor value of labor power might be defined as the minimal amount of direct labor socially necessary to provide workers with the income by which they can respectively purchase at least the subsistent consumption vector. Also, in Definition 7, the labor value of labor power might be defined as the amount of direct labor socially necessary to provide workers with the income, which is evaluated via the actually used social production path. In both of these formulations, the subsistent consumption vector is used, at most indirectly, to define the labor value of labor power. Thus, even the labor value of labor power no longer emerges logically prior to the price of labor power (wage income). Hence, the concepts of labor value in these new definitions are completely irrelevant to theories of exchange values of commodities and labor power.

In spite of such a significant difference of these new definitions from the orthodox Marxian notion of labor exploitation, they would be justified, according to the scenario Roemer (1982) offered, since both FMT and CECP hold true for these new definitions. Note that we still need a further conceptual argument about which of Definitions 6 and 7 is more appropriate formulation for Marxian notion of labor exploitation. We leave this for future occasions.

6 References


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\[ v \in C^p \cup C^s \Rightarrow v \text{ is exploited,} \]
\[ v \in C^H \Rightarrow v \text{ is an exploiter.} \]

**Figure 1**
Class-Exploitation Correspondence Principle holds under Definitions 3 and 5 in economies with Leontief technology.
\[
\begin{align*}
\max_{\alpha \in \Gamma(p,1)} \pi p \bar{\alpha} + 1 &= \min_{\alpha \in \Gamma(p,1)} \pi p \bar{\alpha} + 1 \\
\arg\min_{c \in B[p,1]} l.v(c, (p,1)) &= \bar{\alpha}^* \\
\end{align*}
\]

**Figure 2:** Class-Exploitation Correspondence Principle in a general convex cone economy when the formulation of exploitation is given by Definition 6.
\[
\hat{\alpha}^4 = (0, 1.75) \\
\hat{\alpha}^2 = (1, 1.5) \\
\hat{\alpha}^3 = (2, 1) \\
\hat{\alpha}^3 = (3, 0)
\]

\[p = (0.5, 0.5)\]

Figure 3
\[ \hat{\alpha}^4 = (0, 1.75) \]
\[ \hat{\alpha}^2 = (1, 1.5) \]
\[ \hat{\alpha} = (2, 1) \]
\[ \partial \hat{P}(\alpha_0 = 1) \]

Figure 4
Figure 5
\[ \hat{\alpha}^4 = \zeta \]

\[ \bar{P}(p,1) \]

\[ \partial \hat{P}(\alpha_0 = 1) = \zeta \]

\[ \theta_{(p,1)} \]

\[ \hat{\alpha}^3 \]

\[ (3,0) \]

\[ \hat{\alpha}^2 \]

\[ (2,1) \]

\[ \bar{c} \]

\[ P = (0.5, 0.5) \]

**Figure 6:** The Morishima (1974) definition for Marxian Labor Exploitation meets LE, but violates CECP.
Figure 7: The Roemer (1982) definition for Marxian Labor Exploitation meets LE, but violates CECP.
Figure 8: Definition 8 for Marxian Labor Exploitation meets LE, but violates CECP.
Figure 9: Definition 7 for Marxian Labor Exploitation meets LE, and preserves CECP as a theorem.