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<thead>
<tr>
<th>Title</th>
<th>Generating a Bergson-Samuelson Social Welfare Ordering From Partial Welfare Judgements</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Suzumura, Kotaro; Xu, Yongsheng</td>
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<td>Citation</td>
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Generating a Bergson-Samuelson Social Welfare Ordering
From Partial Welfare Judgements

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From Partial Welfare Judgements

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Abstract

In this paper, we examine the possibility of generating a Pareto-compatible Bergson-Samuelson social welfare ordering from the collection of reflexive, consistent and incomplete extensions of the Pareto quasi-ordering through set-theoretic union. A necessary and sufficient condition for the possibility of this generativity property will be identified. 

Keywords: Bergson-Samuelson Social Welfare Ordering, Pareto Quasi-Ordering, Ordering Extension, New Welfare Economics

[JEL Classification Numbers: C60, D60, D70]
1 Introduction

It was none other than Paul Samuelson (1981, p.225) who identified two versions of the New Welfare Economics. The first is “[t]he narrow version that emphasized and stopped short at ‘compensation payments’ made by gainers to losers ... .” The second is “Bergson’s synthesis of the Old Welfare Economics of the additive-hedonistic type with the more general notion of a Social Welfare Function that introduces, from outside positivistic economic science, ethical norming of alternative states of the world.” According to his sweeping verdict, “Bergson not only clarified the relationship of the general New Welfare Economics to the previous Old Welfare Economics; but, as well, his 1938 analysis enabled scholars to understand how the narrow ‘New Welfare Economics’ of Hicks, Kaldor, Scitovsky, and dozen others reached only the state of necessary conditions rather than that of necessary-and-sufficient conditions.” We are thus left with an impression that, after Bergson’s synthesis in terms of the Social Welfare Function, there is no independent room for the compensationist school of the New Welfare Economics. This paper represents a modest attempt to find a somewhat different niche for the compensationist school à la “Hicks, Kaldor, Scitovsky, and dozen others.”

Rcollect that Samuelson (1947/1983, p.221) does not show any interest whatsoever in the way how the Bergson Social Welfare Function is generated from the profile of individual preference orderings on the set of alternative states of the world:

Without inquiring into its origins, we take as a starting point for our discussion a function of all the economic magnitudes of a system which is supposed to characterize some ethical belief — that of a benevolent despot, or a complete egoist, or “all men of good will,” a misanthrope, the state, race, or group mind, God, etc. Any possible opinion is admissible ... . We only require that the belief be such as to admit of an unequivocal answer to whether one configuration of the economic system is “better” or “worse” than any other or “indifferent,” and that these relationships are transitive ... .

In sharp contrast with Samuelson’s complete lack of interest in the process or rule

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through which the Bergson Social Welfare Function is generated, the central focus of social choice theory à la Kenneth Arrow (1951/1963) and Amartya Sen (1970/1979) is precisely the existence of a democratic and informationally efficient social choice process or rule for generating the Bergson Social Welfare Function. In between these polar extremes lies the approach of the New Welfare Economics of the compensationist school which tried to extend the applicability of the Pareto unanimity principle to the situation of interpersonal conflict through the medium of compensatory payments between gainers and losers. In this context, a logical question of crucial importance naturally suggests itself. Starting from the Pareto quasi-ordering, can we generate a complete ordering by taking union of the incomplete extensions of the Pareto quasi-ordering? This seems to be one possible way of interpreting the analytical scenario of the New Welfare Economics of the compensationist school. The purpose of this paper boils down to examining the logical coherence of this scenario.

Apart from this Introduction, this paper consists of three sections. In Section 2, the basic concepts, notations, and definitions are introduced. In Section 3, we formalize the problem of generating a Bergson-Samuelson social welfare ordering, and present the basic possibility theorems and their interpretations. Section 4 concludes this paper with several remarks.

2 Basic Concepts, Notations, and Definitions

The set of individuals in the society is $N = \{1, 2, \cdots, n\}$, where $2 \leq n < +\infty$. $X$ is the universal set of social alternatives, where $3 \leq |X| < +\infty$. Each individual $i \in N$ has a preference ordering $R_i$ over $X$. For any $x, y \in X$, $(x, y) \in R_i$ means that $x$ is judged by $i$ to be at least as good as $y$. $P(R_i)$ and $I(R_i)$ denote, respectively, the asymmetric part and the symmetric part of $R_i$.

Given a profile $R^N = (R_1, R_2, \ldots, R_n)$ of individual preference orderings, let the Pareto quasi-ordering, to be denoted by $\rho(R^N)$, be defined by

(1) $\rho(R^N) = \cap_{i \in N} R_i$.

Given a binary relation $R$ on $X$, we say that a binary relation $R^*$ on $X$ is an extension of $R$ if and only if $R \subset R^*$ and $P(R) \subset P(R^*)$ hold. If an extension $R^*$ of $R$ is an ordering on $X$, $R^*$ is called an ordering extension of $R$. When $R^*$ is an extension of $R$, $R$ is said to be a compatible sub-relation of $R^*$. A binary relation $R$ on $X$ is consistent if and only if there exists no finite subset $\{x^1, x^2, \cdots, x^t\}$ of $X$,
where $2 \leq t$, such that $(x^1, x^2) \in P(R), (x^2, x^3) \in R, \ldots$, and $(x^t, x^1) \in R$. Note that a transitive binary relation is consistent, but a consistent binary relation need not be transitive.

The following result is due to Suzumura (1976; 1983, Theorem A(5)), which is a generalization of Szpilrajn’s (1930) classic ordering extension theorem.

**Lemma 2.1.** A binary relation $R$ has an ordering extension if and only if it is consistent.

### 3 Generativity Theorems

The task of this paper is to examine the possibility of extending the Pareto quasi-ordering $\rho(R^N)$ into a complete ordering on $X$. For mnemonic convenience, such an ordering is called a Pareto-compatible Bergson-Samuelson social welfare ordering.

To facilitate this analysis, given any profile $R^N$, let $\sigma(R^N)$ be a generic collection of reflexive, consistent, and incomplete binary relations on $X$, each one of which strictly extends the Pareto quasi-ordering $\rho(R^N)$. Let $\Sigma(R^N)$ be the set of all such collections. Given a profile $R^N$ and a collection $\sigma(R^N) \in \mathcal{P}(R^N)$, let us define the set-theoretic union of all sets in $\sigma(R^N)$:

$$(2) \quad T_\sigma(R^N) := \bigcup_{Q \in \sigma(R^N)} Q.$$

Under what condition(s) on $\sigma(R^N) \in \Sigma(R^N)$, can we assure that $T_\sigma(R^N)$ qualifies as a Pareto-compatible Bergson-Samuelson social welfare ordering?

To confirm that this question is non-vacuous, consider the following example.

**Example 3.1.** Let $X = \{x, y, z\}$ and $\rho(R^N) = \Delta(X) \cup \{(x, y)\}$, where $\Delta(X) := \{(a,a) | a \in X\}$. Then, the set of all logically possible reflexive, consistent and incomplete binary relations on $X$ that strictly extend $\rho(R^N)$ is given by $\{Q_1, \ldots, Q_6\}$, where

- $Q_1 = \{(y,z)\} \cup \rho(R^N)$;
- $Q_2 = \{(z,y)\} \cup \rho(R^N)$;
- $Q_3 = \{(z,y), (y,z)\} \cup \rho(R^N)$;
- $Q_4 = \{(x,z)\} \cup \rho(R^N)$;
- $Q_5 = \{(z,x)\} \cup \rho(R^N)$;
- $Q_6 = \{(x,z), (z,x)\} \cup \rho(R^N)$.

It is easy, if tedious, to check that $\Sigma(R^N)$ consists of 63 collections of binary relations, viz., $\Sigma(R^N) = \{\sigma_1(R^N), \ldots, \sigma_{63}(R^N)\}$, where, for example,

$$\sigma_t(R^N) = \{Q_t\} \ (t = 1, \ldots, 6), \ \sigma_7(R^N) = \{Q_1, Q_4\}.$$
\[ \sigma_8(\mathcal{R}^N) = \{Q_1, Q_5\}, \sigma_9(\mathcal{R}^N) = \{Q_2, Q_4\}. \]

It can be checked that \( T_{\sigma_7}(\mathcal{R}^N) = Q_1 \cup Q_4 \) is an ordering, whereas \( T_{\sigma_8}(\mathcal{R}^N) = Q_1 \cup Q_5 \) is cyclic. \( \blacksquare \)

This example illustrates that \( T_{\sigma}(\mathcal{R}^N) \) can be an ordering only under some appropriate restriction(s) on \( \sigma(\mathcal{R}^N) \). Such a restriction can be expressed in terms of the set \( \Sigma(\mathcal{R}^N) \) as follows.

**Theorem 3.1.** Given a profile \( \mathcal{R}^N \), there exists a \( \sigma^*(\mathcal{R}^N) \in \Sigma(\mathcal{R}^N) \) such that \( T_{\sigma^*}(\mathcal{R}^N) := \bigcup_{Q \in \sigma^*(\mathcal{R}^N)} Q \) is an ordering if and only if \( \Sigma(\mathcal{R}^N) \neq \emptyset \).

Thus, \( T_{\sigma^*}(\mathcal{R}^N) \) is an ordering if and only if there are at least two distinct pairs of alternatives that cannot be ranked by the weak Pareto principle. This is a rather weak requirement. Note that the condition specified in this theorem is non-vacuous. To see this, we have only to consider the following example.

**Example 3.2.** Let \( X = \{x, y, z\} \) and \( \rho(\mathcal{R}^N) = \Delta(X) \cup \{(x, y), (x, z)\} \). Then, there exists no reflexive, consistent and incomplete binary relation on \( X \) which strictly extends the Pareto quasi-ordering \( \rho(\mathcal{R}^N) \). Therefore, \( \Sigma(\mathcal{R}^N) = \emptyset \) in this case. \( \blacksquare \)

**Proof of Theorem 3.1.** The “only if” part being trivially true, we have only to prove the “if” part. Suppose that the profile \( \mathcal{R}^N \) is such that \( \Sigma(\mathcal{R}^N) \neq \emptyset \). Then,

\[ (3) \quad \text{There exist } a, b, c, d \in X \text{ such that } \{(a, b), (b, a), (c, d), (d, c)\} \cap \rho(\mathcal{R}^N) = \emptyset, \text{ and } \{a, b\} \text{ and } \{c, d\} \text{ are two distinct pairs.} \]

By virtue of Lemma 2.1, \( \rho(\mathcal{R}^N) \), being transitive, has an ordering extension \( R \). For all distinct \( x, y \in X \) such that \( \{(x, y), (y, x)\} \cap \rho(\mathcal{R}^N) = \emptyset \), let

\[ Q_{x,y} = R - \{(x, y), (y, x)\}. \]

By construction, \( Q_{x,y} \) is incomplete. Since \( R \) is an ordering, \( Q_{x,y} \) must be reflexive and consistent. \( R \) being an ordering extension of \( \rho(\mathcal{R}^N) \), \( Q_{x,y} \) extends \( \rho(\mathcal{R}^N) \). Note also that (3) holds. Therefore, \( Q_{x,y} \neq \rho(\mathcal{R}^N) \). Thus, we have shown that \( Q_{x,y} \) is reflexive, consistent, incomplete, and strictly extends \( \rho(\mathcal{R}^N) \).

Let \( \sigma^*(\mathcal{R}^N) \) be defined by

\[ \sigma^*(\mathcal{R}^N) = \{Q_{x,y} \mid \forall x, y \in X : \{(x, y), (y, x)\} \cap \rho(\mathcal{R}^N) = \emptyset\}. \]

From this definition, it is clear that \( T_{\sigma^*}(\mathcal{R}^N) = \bigcup_{Q \in \sigma^*(\mathcal{R}^N)} Q = R \). \( R \) being an ordering extension of \( \rho(\mathcal{R}^N) \), the proof is complete. \( \blacksquare \)
Although Theorem 3.1 guarantees the existence of $\sigma^*(R^N) \in \Sigma(R^N)$ which can generate an ordering $T_{\sigma^*}(R^N) := \cup_{Q \in \sigma^*(R^N)} Q$, it does not say anything about the structure of $\sigma^*(R^N)$. In view of this arguably unsatisfactory nature of Theorem 3.1, an alternative approach to the problem of generating a Pareto-compatible Bergson-Samuelson social welfare ordering through set-theoretical union may be in order.

Given a profile $R^N$ of individual preference orderings, a $\sigma^M(R^N) \in \Sigma(R^N)$ is called the maximal collection of consistent extensions of $\rho(R^N)$ if, for any incomplete and strict extension $Q' \notin \sigma^M(R^N)$ of $\rho(R^N)$, $T_{\sigma^M}(R^N) \cup Q'$ is not consistent, where $T_{\sigma^M}(R^N) := \cup_{Q \in \sigma^M(R^N)} Q$.

We are now ready to present the following:

**Theorem 3.2.** The maximal collection $\sigma^M(R^N)$ of consistent extensions of $\rho(R^N)$ can generate a Pareto-compatible ordering $T_{\sigma^M}(R^N) := \cup_{Q \in \sigma^M(R^N)} Q$ if and only if $\Sigma(R^N) \neq \emptyset$ and $T_{\sigma^M}(R^N)$ is consistent.

Before we present the proof of Theorem 3.2, two remarks may be in order. In the first place, as was shown by Example 3.2, $\Sigma(R^N)$ may be empty for some profile $R^N$. Hence the condition that $\Sigma(R^N) \neq \emptyset$ is non-vacuous. In the second place, $T_{\sigma^M}(R^N)$ can be inconsistent for some maximal collection $\sigma^M(R^N)$ of consistent extensions of $\rho(R^N)$, so that the condition for $T_{\sigma^M}(R^N)$ being consistent is also a non-vacuous requirement for generating a Pareto-compatible ordering through set-theoretic union.

To verify this fact, let us present the following:

**Example 3.3.** Let $X = \{x, y, z\}$ and $\rho(R^N) = \Delta(X)$. Consider

$$Q_1 = \{(x, y)\} \cup \rho(R^N); Q_2 = \{(y, z)\} \cup \rho(R^N); Q_3 = \{(z, x)\} \cup \rho(R^N),$$

and let $\sigma^M(R^N) := \{Q_1, Q_2, Q_3\}$. It can be checked that $\sigma^M(R^N)$ is a maximal collection of consistent extensions of $\rho(R^N)$, and yet $T_{\sigma^M}(R^N) := Q_1 \cup Q_2 \cup Q_3$ is cyclic. □

**Proof of Theorem 3.2.** We first observe that if $T_{\sigma^M}(R^N)$ is an ordering, $\sigma^M(R^N) \neq \emptyset$ so that $\Sigma(R^N) \neq \emptyset$ follows immediately. $T_{\sigma^M}(R^N)$ being transitive, consistency of $T_{\sigma^M}(R^N)$ follows immediately.

Next, we show the converse. Let $\Sigma(R^N) \neq \emptyset$. We note that, given $\Sigma(R^N) \neq \emptyset$, we can always construct a maximal collection $\sigma^M(R^N)$ of consistent extensions of $\rho(R^N)$. Let $T_{\sigma^M}(R^N) := \cup_{Q \in \sigma^M(R^N)} Q$ be consistent. Our task is to show that $T_{\sigma^M}(R^N)$ is an
ordering. Note that each and every \( Q \in \sigma^{M}(R^N) \) is reflexive, consistent and incomplete. Each \( Q \in \sigma^{M}(R^N) \) being a strict extension of \( \rho(R^N) \) and being incomplete, the following must be true:

(4) There exist two distinct pairs \( \{a, b\} \) and \( \{c, d\} \) such that \( \{(a, b), (b, a), (c, d), (d, c)\} \cap \rho(R^N) = \emptyset. \)

Suppose that \( T_{\sigma^{M}}(R^N) \) is not an ordering. Since \( T_{\sigma^{M}}(R^N) \) is reflexive by definition and consistent by assumption, there must exist \( x, y \in X \) such that \( \{(x, y), (y, x)\} \cap T_{\sigma^{M}}(R^N) = \emptyset \). Let \( R \) be an ordering extension of \( T_{\sigma^{M}}(R^N) \), which exists by virtue of Lemma 2.1, and let \( (x \circ y) \) be the relationship between \( x \) and \( y \) according to \( R \). Consider \( Q = \{(x \circ y)\} \cup \rho(R^N) \). By (4), \( Q \) is a strict extension of \( \rho(R^N) \), which is not an ordering. Note that \( T_{\sigma^{M}}(R^N) \cup Q \) is consistent. Since \( \sigma^{M}(R^N) \) is a maximal collection of consistent extensions of \( \rho(R^N) \), it must be true that \( Q \in \sigma^{M}(R^N) \). That is to say, \( (x \circ y) \in T_{\sigma^{M}}(R^N) \), which is a contradiction with \( \{(x, y), (y, x)\} \cap T_{\sigma^{M}}(R^N) = \emptyset \). Thus, \( T_{\sigma^{M}}(R^N) \) must be an ordering.

\[ \square \]

4 Concluding Remarks

This paper explored one possible method of interpreting the analytical scenario of the New Welfare Economics of the compensationist school. According to this scenario, the purpose of the New Welfare Economics was to expand the applicability of the Pareto unanimity rule to the situation of interpersonal conflict through the medium of compensatory payments between gainers and losers. The condition under which the Bergson Social Welfare Ordering could be generated in terms of set-theoretical union of the incomplete and consistent extensions of the Pareto quasi-ordering was identified.

In concluding this paper, it may be worthwhile to recollect that Kotaro Suzumura (1999) explored the logical connection between the two versions of the New Welfare Economics along a related but distinct avenue. It took the Pareto-compatible Bergson-Samuelson Social Welfare Ordering as given from outside, and explored the conditions under which the social choice according to this Social Welfare Ordering could be recovered by means of the social choices according to the Pareto-compatible subrelations thereof.\(^2\) It is hoped that the two approaches to the New Welfare Eco-

\(^2\) See Suzumura and Xu (2003) for some related results.
nomics of the compensationalist school would play complementary role in clarifying its logical relationship with the New Welfare Economics of the Social Welfare Function school.
References


