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AN ARBITRAGE APPROACH TO THE PRICING OF CATASTROPHE OPTIONS INVOLVING THE COX PROCESS*

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Abstract

We investigate the valuation and hedging of catastrophe options, whose claim arrival process is modeled by the Cox process or a doubly stochastic Poisson process. Employing the non-arbitrage principle we obtain closed form formula for the pricing of the option. Various hedging parameters are also computed.

Keywords: catastrophe options, Cox process, pricing
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I *Introduction*

The high level of catastrophic events, such as major typhoons and/or disastrous earthquakes etc., have had a heavy impact on the reinsurance market. These events are usually infrequent, hard to predict reliably, and difficult to effectively prevent. In order for insurance companies to hedge themselves against the resulting risks and also to stabilize insurance markets, from the 1990s, catastrophe futures and options written on these futures have been introduced into trading as alternative reinsurance products [e.g., Cummins et al. (2004), Froot (2001), Jaffee and Russell (1997)]. As the first successful example of a catastrophe bond underwritten by a non-financial firm, we recall the contract issued in 1999 by Oriental Land Company Ltd., which is intended to cover the losses of Tokyo Disneyland caused by a possible earthquake in the Tokyo area. In recent years, these types of catastrophe (CAT) financial products have established themselves as a special financial category.

One reason to consider such products in the markets is that although the insured catastrophe risks can be very large, its relative size to other risks may be comparable. For instance, the total loss of the Hanshin-Awaji earthquake having occured in 1994 is estimated to be 10 trillion yen (Disaster Reduction and Human Renovation Institution (http: // www.dri.ne.jp)); which is roughly of the same order of a 2 percent change in the value of the Tokyo Stock Exchange, which is not such an infrequent event.

Typical examples of CAT contracts include the CatEPut, which is the first catastrophe equity put option and a registered trademark of Aon Limited. The payoff function of the CatEPut at maturity T with exercise value of K has the form

$$\text{Payoff} = 1_{\{X_T > L\}}\max\{K - S_T,\ 0\} = \begin{cases} K - S_T & \text{if } S_T < K \text{ and } X_T > L \\ 0 & \text{if } S_T \geq K \text{ or } X_T \leq L, \end{cases} \tag{1}$$

where $S = S_t$ denotes the market price and $X = X_t$ the stochastic event. The parameter L is the trigger level. Here $1_{\{.\}}$ stands for the indicator function; namely, $1_{\{X_T > L\}} = 1$ if $X_T > L$ and $= 0$ if $X_T \leq L$. Thus the option can be exercised, in other words, becomes in-the-money, only when claims X_T exceed L; the owner of the option can hedge against catastrophic losses exceeding the existing coverage limit during the life of the option. We note that the CatEPut, in this sense, is classified among the so-called double trigger options.

The pricing of these structured risk management products should be concerned with incomplete markets [Vaugirard (2003)]. As to the CatEPut, Cox et al. (2004) made the first step via a non-arbitrage approach when the loss process X follows a Poisson process and obtained a closed form solution. Later Jaimungal and Wang (2006) extended the results of Cox et al. (2004) so that X is assumed to follow a compound Poisson process and the risk-free interest rate is governed by a mean-reverting stochastic process. Furthermore the hedging parameters Delta, Gamma and Rho are also calculated.

In this note we deal with the pricing of a double trigger CAT option when the claim arrival process X follows the Cox process or a doubly stochastic Poisson process. Since the Cox process allows the intensity to be stochastic, it is suitable for the modeling of catastrophic events. Indeed Dassios and Jang (2003) employ the Cox process with shot noise intensity (see (2) in §2 below) on this ground, and establish a pricing formulae for a stop-loss reinsurance contract for catastrophic events and catastrophe insurance derivatives. Their technique is based on utilizing an equivalent martingale probability measure; however, since the shot noise process possesses many parameters, the calculation becomes rather cumbersome. We additionally remark that the model of Dassios and Jang (2003) does not contain the market price process. Here, on the other hand, we adopt an elementary method for both the modeling as well as the computation and derive a closed form pricing formula by directly clarifying the equivalent martingale measure. Several generalizations will be indicated together.

Concerning the valuation of other products in this family, we recall, for instance, that catastrophe insurance futures and call spreads are treated in Cummins and Geman (1995); a floating retention insurance is also discussed in Cox et al. (2004), whereas the pricing of a reinsurance contract with catastrophe bonds is developed in Lee and Yu (2007). We also refer to the references cited therein.

The paper is organized as follows. In §2 we recall the basic properties of the Cox process and present our basic model. Based on the non-arbitrage technique, we exhibit a pricing formula in §3. §4 is devoted to calculating various hedging parameters, namely, Greeks. Generalizations and discussions of our findings are examined a little further in §5.

II *Preliminary*

Firstly, we recall the definition and properties of our basic process; namely, the so-called Cox process, or a doubly stochastic Poisson process [e.g., Lando (1998)]. As a general reference we refer to a comprehensive monograph of Rolski et al. (1998).

It is well observed that the Poisson process has been used as a claim arrival process in insurance modeling. However, it is also recognized, at the same time, that the homogeneous Poisson process is inappropriate for modeling the resulting claims for catastrophic events, because the process has deterministic intensity. Cox processes or doubly stochastic Poisson processes, on the other hand, provide flexibility in modeling, since the intensity process is allowed to be stochastic.

Let $\Lambda := \{\lambda_t\}_{t\geq 0}$ be an intensity process; namely, a nonnegative, measurable, and locally integrable stochastic process. A counting process $\{N(t;\Lambda)\}_{t\geq 0}$ is called a Cox process or a doubly stochastic Poisson process with intensity Λ if for each sequence $\{k_i\}_{i=1,2,\cdots,n}$ of nonnegative integers, and for $0<t_1\leq s_1\leq t_2\leq s_2\leq\cdots\leq t_{n-1}\leq s_{n-1}\leq t_n\leq s_n$, there holds

$$P\Big(\bigcap_{i=1}^{n}\{N(s_i;\Lambda)-N(t_i;\Lambda)=k_i\}\Big)=\prod_{i=1}^{n}E\Big[\frac{1}{k_i!}\Big(\int_{t_i}^{s_i}\lambda_u d_u\Big)^{k_i}\exp\Big(-\int_{t_i}^{s_i}\lambda_u d_u\Big)\Big].$$

One typical example, which is favorably used to measure the effect of catastrophic events, is the shot noise process [e.g., Cox and Isham (1986), Dassios and Jang (2003), Kluppelberg and Mikosch (1995)]. Suppose $0<s_1<s_2<\cdots$ are the points of a Poisson process with the intensity ρ and $\{y_i\}_{i=1,2,\cdots}$ are independently and identically distributed nonnegative random variables with $E(y_i)<\infty$, then

$$\lambda_t=\lambda_0 e^{-\delta t}+\sum_{i\,:\,s_i\leq t}y_i e^{-\delta(t-s_i)}, \tag{2}$$

where λ_0 is the initial value and δ denote the rate of exponential decay. In Dassios and Jang (2003) it is generalized that parameters ρ, δ are allowed to be time-dependent. We clearly imagine that s_i corresponds to the time at which catastrophe i occurs and y_i supplies the jump size of the catastrophe i.

Now we specify our model, which is represented by the next system.

$$S_t = S_0 \exp\Big\{ -\alpha N(t;\Lambda) + \Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma W_t \Big\} \tag{3}$$
$$B_t = e^{rt},$$

where $W_t (t \geq 0)$ denotes the standard Brownian motion which is independent of the Cox process $N(t;\Lambda)$, and α, μ, σ are assigned positive constants. The interest rate r is free of risk and is kept constant. It is easy to recognize that if a large claim occurs at time t, then the price changes abruptly from $S_{t\_}$ to $S_t = e^{-\alpha(N(t;\Lambda)-N(t\_;\Lambda))} S_{t\_}$. If no large claims are made during the interval, then S_t follows a typical geometric Brownian motion over the same interval, with the drift μ and the volatility σ.

Under the process (3) we intend to obtain a pricing of the double trigger catastrophe options where $X_t = N(t;\Lambda)$. We note once again that our formulation deals with an incomplete market; several choices of equivalent martingale measure will exist for the valuation. Our current business does not involve deciding what is the suitable one to use; we merely proceed as simply as possible.

III *Pricing Formula*

Let $C(t)$ denote the value of the option at time t $(< T)$ whose payoff function at the maturity T is prescribed by

$$C(T) = 1_{\{X_T > L\}} \max\{S_T - K, 0\}. \tag{4}$$

We assume that $X_t = N(t;\Lambda)$ with $X_0 = 0$ and the stock price S_t is described by (3). In this section we want to estimate an arbitrage-free value of $C(t)$. It is easy to see that the price $P(t)$ of the CatEPut (1) under the same X_t is proceeded similarly.

We begin by showing the next lemma, which is a straight extension of Lemma 1 in Cox et al. (2004) and gives a key tool to the valuation.

Lemma 3.1 *Let* $\{N(t;\Lambda)\}_{t \geq 0}$ *be a Cox process with intensity* $\Lambda = \{\lambda_t\}_{t \geq 0}$. Then

$$\exp\{-\alpha N(t;\Lambda) + \log(M_{\Lambda_t}(k))\}_{t \geq 0}$$

is a martingale, where $k := 1 - e^{-\alpha}$ *and* $M_{\Lambda_t}(\cdot)$ *denotes the moment generating function of the aggregated process* $\Lambda_t := \int_0^t \lambda_s ds$.

Proof. Firstly, we learn that

$$E[e^{-\alpha N(t;\Lambda)}] = E[E[e^{-\alpha N(t;\Lambda)} \mid \mathscr{F}_t^\Lambda]] = E[\sum_{l=0}^{\infty} e^{-\alpha l} P(N(t;\Lambda) = l))]$$

$$= E\left[\sum_{l=0}^{\infty} e^{-\alpha l}\frac{\Lambda_t^l}{l!}e^{-\Lambda_t}\right] = E\left[\exp\{\Lambda_t(e^{-\alpha}-1)\}\right],$$

where $\mathscr{F}_t^{\Lambda} := \sigma\{\lambda_s \mid 0 \le s \le t\}$.

We compute for $0 \le s \le t \le T$

$$\begin{aligned}
&E[\exp\{-\alpha N(t;\Lambda)+\log(M_{\Lambda_t}(k))\} \mid \mathscr{F}_s^{\Lambda}]\\
&\quad = E[\exp\{-\alpha(N(t;\Lambda)-N(s;\Lambda))+\log(M_{\Lambda_t}(k))-\log(M_{\Lambda_s}(k))\} \mid \mathscr{F}_s^{\Lambda}]\cdot\\
&\qquad\quad \cdot E[\exp\{-\alpha N(s;\Lambda)+\log(M_{\Lambda_s}(k))\} \mid \mathscr{F}_s^{\Lambda}]\\
&\quad = E[\exp\{-\alpha(N(t;\Lambda)-N(s;\Lambda))+\log(M_{(\Lambda_t-\Lambda_s)}(k))\} \mid \mathscr{F}_s^{\Lambda}]\cdot\\
&\qquad\quad \cdot \exp\{-\alpha N(s;\Lambda)+\log(M_{\Lambda_s}(k))\}\\
&\quad = E[\exp\{-\alpha(N(t;\Lambda)-N(s;\Lambda)) \mid \mathscr{F}_s^{\Lambda}]M_{(\Lambda_t-\Lambda_s)}(k)\cdot\\
&\qquad\quad \cdot \exp\{-\alpha N(s;\Lambda)+\log(M_{\Lambda_s}(k))\}\\
&\quad = M_{(\Lambda_t-\Lambda_s)}(-k)M_{(\Lambda_t-\Lambda_s)}(k)\exp\{-\alpha N(s;\Lambda)+\log(M_{\Lambda_s}(k))\}\\
&\quad = \exp\{-\alpha N(s;\Lambda)+\log(M_{\Lambda_s}(k))\},
\end{aligned}$$

which completes the proof of Lemma.

For the expression of $C(t)$ we understand that it is the expected discounted value under the risk-neutral measure Q. Precisely stated, the process $\{e^{-rt}S_t\}_{t\ge 0}$ is a Q-martingale and we have

$$C(t) = E_t^Q[e^{-r(T-t)}1_{\{N(T;\Lambda)>L\}}\max\{S_T-K,\,0\}].$$

To obtain the measure Q we apply the change of measure. Indicating the original variable by P like $W_t^P := W_t$, we define

$$W_t^Q := W_t^P + \frac{(\mu-r)t+\log(M_{\Lambda_t}(k))}{\sigma},$$

and the Radon-Nikodym derivative process as follows.

$$\frac{dQ}{dP} = \exp\left\{-\frac{1}{2}\int_0^t\left(\frac{\mu-r+\gamma_k(s)}{\sigma}\right)^2 ds - \int_0^t \frac{\mu-r+\gamma_k(s)}{\sigma}dW_s^P\right\},$$

where we have put $\gamma_k(t) := E[\lambda_t\exp\{k\Lambda_t\}]/M(k)$. Taking Lemma 3.1 into account and observing that

$$e^{-rt}S_t = S_0\exp\{-\alpha N(t;\Lambda)+\log(M_{\Lambda_t}(k))\}\exp\left\{\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right\},$$

we conclude that $\{e^{-rt}S_t\}$ is a Q-martingale. To be explicit, we have for $0 \le s \le t$

$$E_s^Q[e^{-rt}S_t] = e^{-rs}S_s.$$

Now we deduce the closed form pricing formula.

Theorem 3.2 *The price $C(t)$ of the double trigger option whose payoff function at the maturity T is (4) is given by*

$$C(t) = \sum_{l=L+1-N(t;\Lambda)}^{\infty} (S_t e^{-\alpha l + \log(M_{(\Lambda_T - \Lambda_t)}(k))} \Phi(d_l + \sigma\sqrt{T-t}) - Ke^{-r(T-t)} \Phi(d_l)) \cdot$$
$$\cdot E\left[\frac{(\Lambda_T - \Lambda_t)^l}{l!} e^{-(\Lambda_T - \Lambda_t)]}\right]$$

where $k := 1 - e^{-\alpha}$ *and*

$$d_l := \frac{\log(S_t/K) + r(T-t) - \alpha l + \log(M_{(\Lambda_T - \Lambda_t)}(k))}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}. \tag{5}$$

Here $\Phi(x)$ *denotes the cumulative distribution function for the standardized normal distribution*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy =: \int_{-\infty}^{x} \varphi(y) dy.$$

It is easy to see that the quantity $S_t e^{-\alpha l + \log(M_{(\Lambda_T - \Lambda_t)}(k))} \Phi(d_l + \sigma\sqrt{T-t}) - K \cdot \cdot e^{-r(T-t)} \Phi(d_l)$ represents the Black-Scholes pricing formula of vanilla call option whose underlying process is given by $\{S_t e^{-\alpha l} + \log(M_{(\Lambda_T - \Lambda_t)}(k))\}$ and the exercise value K; thus our formula may be understood as an extension of that obtained by Merton (1976, (16)).

Proof. We just calculate the relevant quantity, which is carried out as follows:

We write

$$S_T = e^{r(T-t)} S_t \exp\{-\alpha(N(T;\Lambda) - N(t;\Lambda)) + \log(M_{(\Lambda_T - \Lambda_t)}(k))\} \cdot$$
$$\cdot \exp\left\{\sigma(W_T^Q - W_t^Q) - \frac{1}{2}\sigma^2(T-t)\right\}.$$

Taking account of the Black-Scholes formula and calculating conditionally on the number of claims $N(T;\Lambda)$, we infer that

$$C(t) = E[1_{\{N(T;\Lambda)>L\}} (S_t e^{-\alpha(N(T;\Lambda) - N(t;\Lambda)) + \log(M_{(\Lambda_T - \Lambda_t)}(k))} \cdot$$
$$\cdot \Phi(d_{N(T;\Lambda) - N(t;\Lambda)} + \sigma\sqrt{T-t}) - Ke^{-r(T-t)} \Phi(d_{N(T;\Lambda) - N(t;\Lambda)}))]$$
$$= \sum_{l=L+1-N(t;\Lambda)}^{\infty} (S_t e^{-\alpha l + \log(M_{(\Lambda_T - \Lambda_t)}(k))} \Phi(d_l + \sigma\sqrt{T-t}) - Ke^{-r(T-t)} \Phi(d_l)) \cdot$$
$$\cdot E\left[\frac{(\Lambda_T - \Lambda_t)^l}{l!} e^{-(\Lambda_T - \Lambda_t)}\right].$$

This finishes the proof.

IV *Hedging Parameters*

Here we consider the hedging strategy. In particular we compute various Greeks. The calculation is rather straightforward. We summarize them as the next proposition, whose proof will be safely omitted.

Proposition 4.1 *Let $C(t)$ denote the price of the double trigger option whose payoff function at the maturity T is given by (4). Subsequently, the Delta, Gamma, Rho, and Vega of $C(t)$ are computed as the next expressions, respectively.*

$$\Delta(t) = \frac{\partial C(t)}{\partial S} = \sum_{l=L+1-N(t;\Lambda)}^{\infty} e^{-\alpha l + \log(M_{(\Lambda_T-\Lambda_t)}(k))} \Phi(d_l + \sigma\sqrt{T-t}) \cdot \\ \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!} e^{-(\Lambda_T-\Lambda_t)}\right]$$

$$\Gamma(t) = \frac{\partial^2 C(t)}{\partial S^2} = \sum_{l=L+1-N(t;\Lambda)}^{\infty} \frac{e^{-\alpha l + \log(M_{(\Lambda_T-\Lambda_t)}(k))}}{S\sigma\sqrt{T-t}} \varphi(d_l + \sigma\sqrt{T-t}) \cdot \\ \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!} e^{-(\Lambda_T-\Lambda_t)}\right]$$

$$\rho(t) = \frac{\partial C(t)}{\partial r} = \sum_{l=L+1-N(t;\Lambda)}^{\infty} K(T-t)\Phi(d_l) \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!} e^{-(\Lambda_T-\Lambda_t)}\right]$$

$$\nu(t) = \frac{\partial C(t)}{\partial \sigma} = \sum_{l=L+1-N(t;\Lambda)}^{\infty} S\sqrt{T-t}\varphi(d_l + \sigma\sqrt{T-t}) \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!} e^{-(\Lambda_T-\Lambda_t)}\right],$$

where d_l is defined by (5) and $\varphi(x) = \Phi'(x)$.

V *Discussions*

We have established a pricing formula of double trigger catastrophe options through the non-arbitrage principle. Catastrophe financial products have been introduced, partly due to the need to provide alternative hedging tools and thus stabilize insurance markets. For the relevant claim arrival process, we employ the Cox process or a doubly stochastic Poisson process, which is customarily used as a model of catastrophic events. The nature of the problem means we are forced to treat the incomplete market. Invoking the change of measure we evaluate the non-arbitrage price of options, while hedging parameters are also computed.

Slight extensions of our present results are possible; for example, the risk-free interest rate r is permitted to be stochastic, say, a mean-reverting process. Confer the related work of Jaimungal and Wang (2006). We may also specialize the Cox process as, say, the shot noise process (2), although the parameters would be involved. Observe the elaborate study of Dassios and Jang (2003). These are accomplished rather directly.

However, several challenging issues remain to be further pursued. For example, there may be different means availbale to handle the incompleteness [e.g., Bjork (2004), Fujita (2002)]. One promising trick in economics theory involves appealing to the optimality of a certain utility function. It is interesting to find the appropriate rule to select the relevant type of utility functions. In any case we wish to know or ensure how the benchmark price should be. Next, but no less important, a relatively significant research theme involves developing a fast and accurate algorithm to numerically compute the catastrophe insurance products. Although certain methods have been introduced, considerable room remains for innovation or improvement. We will return to these topics in our future investigation.

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