Lexicographic Compositions of Two Criteria for Decision Making*

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Abstract

This paper considers two distinct procedures to lexicographically compose two criteria for social or individual decision making. The first procedure composes two binary relations into one, and then selects its maximal elements. The second procedure first selects the set of maximal elements of the first binary relation, and then within that set, chooses the maximal elements of the second binary relation. We show several distinct sets of conditions for the choice functions representing these two procedures to satisfy non-emptiness and choice-consistency conditions such as contraction consistency (Chernoff, 1954) and path independence (Arrow, 1963). We also examine the relationships between the outcomes of the two procedures. Then, we investigate under what conditions the outcomes of each procedure is independent of the order of lexicographic application of two criteria. Examples for applications of the results in the economic environments are also presented.

1 Introduction

In the process of social decision making, people often advocate multiple criteria on which the desirability of alternatives should be judged. A typical example is the equity-efficiency trade-off. People say that economic growth is desirable because the welfare of most individuals increases, while at the same time they insist that an equitable distribution is essential for social stability. As often argued, however, economic growth may give rise to an inequitable distribution of income and wealth.

Even a single individual's decision may be based upon multiple criteria. As Sen (1985) argues, an individual has not only material preferences over his own consumptions but also has value judgments based on, for instance, the sense of obligation, which may contradict his material preferences. A family's decision also typically involves multiple criteria. Parents' interest often conflicts with children's interest on, for example, video games.

When two criteria, each regarded as reasonable for itself, are in contradiction with each other, one resolution would be to give priority to one criterion over the other. For such lexicographic applications of multiple criteria, however, we can consider two distinct procedures of choice, which are described in the following. Let us first postulate that each criterion is expressed by a binary relation on the set X of all alternatives. In the first procedure, which we call procedure α, we first compose lexicographically two binary relations R^1 and R^2 into one binary relation $P(R^1, R^2)$ in the following way: an alternative x is better than an alternative y for $P(R^1, R^2)$ if and only if (i) x is superior to y for R^1 or (ii) y is not superior to x for R^1 and x is better than y for R^2. Then, for each subset S of alternatives, we select the set $C_{P(R^1,R^2)}(S)$ of maximal elements for $P(R^1, R^2)$.

By contrast, in the second procedure, which we call procedure β, for each subset S of alternatives, we first choose the set of maximal elements in S for the first criterion R^1, and then select within that set, its subset of maximal elements for the second criterion R^2.

Procedure α has been introduced and examined by Tadenuma (2002, 2005), while procedure β has been introduced by Aizerman (1985) and Aizerman and Aleskerov

(1995), and studied more recently by Manzini and Mariotti (2005), Tadenuma (2005) and Houy (2007).

When a decision-maker has multiple criteria, his behavior becomes different from a simple maximizer of a single binary relation. It is more difficult to have consistent choices under multiple criteria than under a single criterion. In this paper, we study under what conditions the choice correspondence derived from each procedure to lexicographically compose two criteria satisfy non-emptiness and various properties of choice-consistency such as contraction consistency (Chernoff, 1954) and path independence (Arrow, 1963). We also examine relationships between the outcomes of procedures α and β.

Another interesting question would be whether the final outcome depends on the order of application of the multiple criteria. When we evaluate allocations, which criterion should we apply first, the efficiency criterion or the equity criterion? Such a question is important if the order of application of the two criteria affects the final outcome. But if it is irrelevant, then we do not have to be concerned about which criterion we should take first. We investigate under what conditions the outcomes of the choice correspondence of each procedure are independent of the order of lexicographic application of the two criteria.

All the results in this paper are derived without specific restrictions on the set of alternatives, but we present applications of the results in the classical division problem of infinitely divisible commodities.

There are many examples in which two criteria, each of which seems reasonable for itself, contradict each other. In economics and social choice theory, the social preference relation that has been most widely accepted is the Pareto domination. However, the Pareto criterion is silent about the distributional equity of allocations but concerns only efficient use of resources. On the other hand, several interesting concepts of distributional equity have been introduced and extensively studied in economics. Two of them are central: *no-envy* and *egalitarian-equivalence*.[1] It was Feldman and

[1] The concept of no-envy was introduced by Foley (1967) and Kolm (1972), and that of egalitarian-equivalence by Pazner and Schmeidler (1978).

Kirman (1974) and Kolm (1972) who pointed out that there is a fundamental conflict between the Pareto criterion and the equity-as-no-envy criterion: there often exist two allocations x and y such that x Pareto dominates y whereas x is not envy-free but y is. The same kind of conflict also arises between the Pareto criterion and the equity-as-egalitarian-equivalence criterion.

Social choice theory on abstract domains has also been extended to take account of intersituational comparisons of individuals.[2] In this "extended sympathy" approach, Suzumura (1981a, b) studied choice-consistency of social choice functions satisfying some conditions concerning Pareto efficiency and equity-as-no-envy in the framework of abstract social choice. Tadenuma (2002, 2005) introduced various lexicographic compositions of the Pareto criterion and the no-envy criterion, and of Pareto and egalitarian-equivalence, respectively, in the classical division problem, and examined rationality of the social preference relations. Tadenuma (2005) also showed that the set of allocations selected by procedure α with the Pareto criterion and the egalitarian-equivalence criterion from the set of all feasible allocations is independent of the order of lexicographic application of the two criteria, and that the essential reason for this independence is because the set of allocation selected by procedure β is also independent of the order of application.

The present paper generalizes the results in these works by showing general conditions for non-emptiness and path independence of choice functions representing procedures α and β, clarifying their relationships, and deriving conditions for independence of the order of application of multiple criteria.

2 Basic Definitions and Notation

Let X be a (finite or infinite) set of alternatives, and $\mathcal{X}$ the set of all finite subsets of X. A binary relation on X is a set $R \subseteq X \times X$. The set of all binary relations on X is denoted $\mathcal{R}$. Given $R \in \mathcal{R}$, define $P(R) \in \mathcal{R}$ by $(x, y) \in P(R) \Leftrightarrow [(x, y) \in R$ and $(y, x) \notin$

[2]Notable earlier contributions in this line of research are Harsanyi (1955), Suppes (1966), Pattanaik (1968), Sen (1970), Hammond (1976) and Arrow (1977).

R], and $I(R) \in \mathcal{R}$ by $(x, y) \in I(R) \Leftrightarrow [(x, y) \in R$ and $(y, x) \in R]$. Given $R \in \mathcal{R}$, a sequence $(x^1, \ldots, x^K) \subseteq X$ is a *cycle* for R if $(x^1, x^2), (x^2, x^3), \ldots, (x^{K-1}, x^K), (x^K, x^1) \in R$. A binary relation $R \in \mathcal{R}$ is

- *complete* if for all $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$;
- *transitive* if for all $x, y, z \in X$, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$;
- *quasi-transitive* if for all $x, y, z \in X$, $(x, y) \in P(R)$ and $(y, z) \in P(R)$ imply $(x, z) \in P(R)$;
- *asymmetric* if for all $x, y \in X$, $(x, y) \in R$ implies $(y, x) \notin R$;
- *acyclic* if there exist no cycle for R.

Note that acyclicity implies asymmetry by the above definitions.

A choice function is a function $C : \mathcal{X} \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \mathcal{X}$. Given $R \in \mathcal{R}$, we define the choice function $C_{P(R)}$ as the one selecting the set of maximal elements for every $S \in \mathcal{X}$, that is,

$$\forall S \in \mathcal{X},\ C_{P(R)}(S) = \{x \in X \mid \forall y \in X,\ (y, x) \notin P(R)\}.$$

We say that a choice function C is *rationalizable by a binary relation* $R \in \mathcal{R}$ if $C = C_{P(R)}$.

Given two choice function C_A and C_B, the choice function $C_B C_A$ is defined by

$$C_B C_A(S) = C_B(C_A(S))$$

for every $S \in \mathcal{X}$.

In the following, we often consider the classical division problem with n agents and m infinitely divisible commodities defined as follows. Let $N = \{1, \ldots, n\}$ be the set of agents. The consumption set of each agent is $\mathbb{R}^m_+$. Let $\mathcal{R}_E$ be the set of complete, transitive and strictly monotonic[3] relations on $\mathbb{R}^m_+$. Each agent $i \in N$ is endowed

[3] A preference relation $\succsim$ is strictly monotonic if for all $a, b \in \mathbb{R}^m_+$, $a > b$ implies $a \succ b$, where $a > b$ is defined as $a \geq b$ and $a \neq b$.

with a preference relation $\succsim_i \in \mathcal{R}_E$. The associated strict preference relation and the indifference relation are defined as above, and denoted $\succ_i$ and $\sim_i$, respectively. An allocation is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^{mn}_+$ where each $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}^m_+$ is the consumption bundle of agent $i \in N$. The set of alternatives in this division problem is defined as $X = \mathbb{R}^{mn}_+$.

3 Choice-Consistency Properties

In this section, we introduce some desirable properties of choice functions. A very basic requirement is that at least one alternative should be chosen from any set.

Non-Emptiness: For every $S \in \mathcal{X}$, $C(S) \neq \emptyset$.

Our next three properties require "consistency" of choices in related situations. The first choice-consistency property means that if the set of available alternatives "shrinks" but previously chosen alternatives are still available, then those alternatives should remain chosen. This is a fundamental requirement of choice-consistency, and it is satisfied by any choice function that is rationalizable by some binary relation.

Contraction Consistency (Chernoff, 1954): For all $S, T \in \mathcal{X}$ with $T \subseteq S$, $T \cap C(S) \subseteq C(T)$.

The second property requires "the independence of the final choice from the path to it" (Arrow, 1951, p.120). In real choice situations, we often divide the set of alternatives into several parts in the first round, and make final choices from the alternatives that have survived in the first round. This property requires that the final choices should not depend on the way we divide the set of alternatives in the first round.

Path Independence: For all $S, T \in \mathcal{X}$, $C(C(S) \cup C(T)) = C(S \cup T)$.

The third choice-consistency property says that if an alternative is chosen from every pair containing it in the set S, then it should be chosen from S.

Condorcet Consistency: For every $S \in \mathcal{X}$ and every $x \in S$, if $x \in C(\{x, y\})$ for every $y \in S$, then $x \in C(S)$.

The choice-consistency properties are related with rationalizability of the choice functions. The following results are well-known.[4]

Proposition 1 (Blair et al., 1976) *A choice function C satisfies Non-Emptiness, Contraction Consistency and Condorcet Consistency if and only if it is rationalizable by a binary relation R such that $P(R)$ is acyclic.*

Given a binary relation $R \in \mathcal{R}$, $C_{P(R)}$ satisfies Condorcet Consistency by definition. As we have noted, any choice function that is rationalizable by a binary relation satisfies Contraction Consistency. Hence, we have the following corollary.

Corollary 1 *Let $R \in \mathcal{R}$ be given. The choice function $C_{P(R)}$ satisfies Non-Emptiness if and only if $P(R)$ is acyclic.*

Similar relations hold for Path Independence and rationalizability by a quasi-transitive binary relation.

Proposition 2 (Plott, 1973) *A choice function C satisfies Non-Emptiness, Path Independence and Condorcet Consistency if and only if it is rationalizable by a quasi-transitive binary relation.*

Corollary 2 *Let $R \in \mathcal{R}$ be given. The choice function $C_{P(R)}$ satisfies Non-Emptiness and Path Independence if and only if R is quasi-transitive.*

4 Lexicographic Composition of Two Binary Relations

As we have mentioned in the introduction, we consider two distinct procedures to compose two criteria for decision making. This section focuses on procedure α in which we first compose two binary relations R^1 and R^2 into one, and then choose its maximal elements. Formally, for all $R^1, R^2 \in \mathcal{R}$, we define $P(R^1, R^2) \in \mathcal{R}$ by

$$P(R^1, R^2) = \{(x, y) \in X \times X \mid (x, y) \in P(R^1) \text{ or } [(y, x) \notin P(R^1) \text{ and } (x, y) \in P(R^2)]\}.$$

[4] A good reference for these results is Suzumura (1983, Ch.2).

We call $P(R^1, R^2)$ the *lexicographic composition of* R^1 *and* R^2. Notice that $P(R^1, R^2)$ is asymmetric and hence $P(P(R^1, R^2)) = P(R^1, R^2)$.

We examine under what conditions the choice function $C_{P(R^1,R^2)}$ satisfies Non-Emptiness and Path Independence. By Corollaries 1 and 2, our examination reduces to checking acyclicity and quasi-transitivity of $P(R^1, R^2)$. We also present examples for applications of the results in economic environments.

Our first result gives a necessary and sufficient condition for $P(R^1, R^2)$ to be acyclic, and equivalently, for $C_{P(R^1,R^2)}$ to be non-empty.

Proposition 3 *Let $R^1, R^2 \in \mathcal{R}$. The lexicographic composition $P(R^1, R^2)$ is acyclic if and only if for every cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^1) \cup P(R^2)$,there exists $k \in \{1, \ldots, K-1\}$ such that $(x^{k+1}, x^k) \in P(R^1)$ or $(x^1, x^K) \in P(R^1)$.*

Proof. The "only if" part: Suppose that there exists a cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^1) \cup P(R^2)$ such that for every $k \in \{1, \ldots, K-1\}$, $(x^{k+1}, x^k) \notin P(R^1)$ and $(x^1, x^K) \notin P(R^1)$. We will show that $P(R^1, R^2)$ has a cycle. For every $k \in \{1, \ldots, K-1\}$, since $(x^k, x^{k+1}) \in P(R^1) \cup P(R^2)$ and $(x^{k+1}, x^k) \notin P(R^1)$ it follows that either $(x^k, x^{k+1}) \in P(R^1)$ or $[(x^k, x^{k+1}) \in P(R^2)$ and $(x^{k+1}, x^k) \notin P(R^1)]$. By definition, $(x^k, x^{k+1}) \in P(R^1, R^2)$ for every $k \in \{1, \ldots, K-1\}$. Similarly, we have $(x^K, x^1) \in P(R^1, R^2)$. Thus, $(x^1, \ldots, x^K) \subseteq X$ is a cycle for $P(R^1, R^2)$.

The "if" part: Suppose that $P(R^1, R^2)$ has a cycle $(x^1, \ldots, x^K) \subseteq X$. If $(x^{k+1}, x^k) \in P(R^1)$ for some $k \in \{1, \ldots, K-1\}$, then by definition, $(x^k, x^{k+1}) \notin P(R^1, R^2)$, which is a contradiction. Thus, for every $k \in \{1, \ldots, K-1\}$, $(x^{k+1}, x^k) \notin P(R^1)$, and hence either $(x^k, x^{k+1}) \in P(R^1)$ or $(x^k, x^{k+1}) \notin P(R^1)$. In the latter case, since $(x^k, x^{k+1}) \in P(R^1, R^2)$, we have $(x^k, x^{k+1}) \in P(R^2)$. Therefore, $(x^k, x^{k+1}) \in P(R^1) \cup P(R^2)$ for every $k \in \{1, \ldots, K-1\}$. Similarly, we have $(x^1, x^K) \notin P(R^1)$ and $(x^K, x^1) \in P(R^1) \cup P(R^2)$. Thus, $(x^1, \ldots, x^K) \subseteq X$ is a cycle for $P(R^1) \cup P(R^2)$ and $(x^{k+1}, x^k) \notin P(R^1)$ for every $k \in \{1, \ldots, K-1\}$, and $(x^1, x^K) \notin P(R^1)$. ■

Notice that Proposition 3 implies that acyclicity of $P(R^1)$ is a necessary condition for $P(R^1, R^2)$ to be acyclic.

Although Proposition 3 gives a necessary and sufficient condition for $P(R^1, R^2)$ to be acyclic, it does not tell what properties of the primitive binary relations, R^1 and R^2, guarantee acyclicity of $P(R^1, R^2)$. Next we provide several sufficient conditions for $P(R^1, R^2)$ to be acyclic or quasi-transitive, which may be useful in many contexts.

Proposition 4 *Let $R^1, R^2 \in \mathcal{R}$. If R^1 is complete and transitive, and R^2 is quasi-transitive, then $P(R^1, R^2)$ is quasi-transitive.*

Proof. Let $x, y, z \in X$. Assume that $(x, y) \in P(R^1, R^2)$ and $(y, z) \in P(R^1, R^2)$. From $(x, y) \in P(R^1, R^2)$, we have (1) $(x, y) \in P(R^1)$ or (2) $(x, y) \notin P(R^1)$, $(y, x) \notin P(R^1)$ and $(x, y) \in P(R^2)$. By completeness of R^1, $(x, y) \notin P(R^1)$ and $(y, x) \notin P(R^1)$ if and only if $(x, y) \in I(R^1)$. Similarly, it follows from $(y, z) \in P(R^1, R^2)$ that (3) $(y, z) \in P(R^1)$ or (4) $(y, z) \in I(R^1)$ and $(y, z) \in P(R^2)$. If (1) and [(3) or (4)] hold true, then by transitivity of R^1, we have $(x, z) \in P(R^1)$, and hence $(x, z) \in P(R^1, R^2)$. Similarly, (2) and (3) together imply $(x, z) \in P(R^1)$ and $(x, z) \in P(R^1, R^2)$. Finally, if (2) and (4) hold, then $(x, z) \in I(R^1)$ follows from transitivity of R^1, and $(x, z) \in P(R^2)$ from quasi-transitivity of R^2. Hence, we have $(x, z) \in P(R^1, R^2)$. ■

Example 1 Envy-free allocations. An allocation $x \in \mathbb{R}^{mn}_+$ is *envy-free* if for all $i, j \in N$, $(x_i, x_j) \in \succsim_i$. Let $F \subset \mathbb{R}^{mn}_+$ be the set of envy-free allocations. Define $R^F \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R_F$ if and only if $x \in F$ or $y \notin F$.

Define $R^P \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^P$ if and only if for all $i \in N$, $(x_i, y_i) \in \succsim_i$. The social preference relation R^P is called the *weak Pareto domination*, and the associated strict social preference relation $P(R^P)$ the *Pareto domination*.

Since R^F is complete and transitive, and R^P is quasi-transitive, it follows from Proposition 4 that $P(R^F, R^P)$ is quasi-transitive. Hence, the choice function $C_{P(R^F, R^P)}$ is not empty and satisfies the Path Independence condition.

Example 2 Ranking by the number of envy instances. For each $x \in \mathbb{R}^{mn}_+$, define the set $H(x) \subset N \times N$ by

$$H(x) = \{(i, j) \in N \times N \mid (x_j, x_i) \in \succ_i\}.$$

The set $H(x)$ is the set of all *instances of envy* at x. Following Feldman and Kirman (1974), define $R^H \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^H$ if and only if $\#H(x) \leq \#H(y)$.

Then, R^H is complete and transitive. By Proposition 4, $P(R^H, R^P)$ is quasi-transitive (Tadenuma, 2002).

Example 3 Egalitarian-equivalent allocations. An allocation $x \in \mathbb{R}^{mn}_+$ is *egalitarian-equivalent* if there exists $a \in \mathbb{R}^m_+$ such that for all $i \in N$, $(x_i, a) \in \sim_i$. Let $E \subset \mathbb{R}^{mn}_+$ be the set of egalitarian-equivalent allocations. Define $R^E \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^E$ if and only if $x \in E$ or $y \notin E$.

Then, R^E is complete and transitive. By Proposition 4, $P(R^E, R^P)$ is quasi-transitive (Tadenuma, 2005).

Proposition 5 *Let $R^1, R^2 \in \mathcal{R}$. If R^1 is complete and transitive, and $P(R^2)$ is acyclic, then $P(R^1, R^2)$ is acyclic.*

Proof. Assume that R^1 is complete and transitive, and $P(R^2)$ is acyclic. Suppose, on the contrary, that there exists a cycle $(x^1, \ldots, x^K) \in X$ for $P(R^1, R^2)$. Because R^1 is complete, for all $x, y \in X$, $(x, y) \in P(R^1, R^2)$ implies that $(x, y) \in R^1$. Hence, we have $(x^k, x^{k+1}) \in R^1$ for all $k \in \{1, \ldots, K-1\}$ and $(x^K, x^1) \in R^1$. Therefore, by transitivity of R^1, for all $k, k' \in \{1, \ldots, K\}$, $(x^k, x^{k'}) \in I(R^1)$. Then, since $(x^1, \ldots, x^K) \in X$ is a cycle for $P(R^1, R^2)$, we must have $(x^k, x^{k+1}) \in P(R^2)$ for all $k \in \{1, \ldots, K-1\}$ and $(x^K, x^1) \in P(R^2)$. This contradicts acyclicity of $P(R^2)$. ■

If $R^1 \in \mathcal{R}$ is only quasi-transitive, then even if R^1 is complete and $R^2 \in \mathcal{R}$ is complete and transitive, $P(R^1, R^2)$ may have a cycle.

Example 4 Define $R^{\hat{P}} \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^{\hat{P}}$ if and only if $(y, x) \notin P(R^P)$. Sen (1970) called $R^{\hat{P}}$ the *Pareto extension rule*. Notice that $P(R^{\hat{P}}) = P(R^P)$, and $R^{\hat{P}}$ is complete and quasi-transitive. As we noted, R^F, R^H, and R^E are all complete and transitive. However, none of $P(R^{\hat{P}}, R^F)$, $P(R^{\hat{P}}, R^H)$ and $P(R^{\hat{P}}, R^E)$ is acyclic (Tadenuma, 2002, 2005). Notice that, since $P(R^{\hat{P}}) = P(R^P)$, none of $P(R^P, R^F)$, $P(R^P, R^H)$ and $P(R^P, R^E)$ is acyclic either.

However, some cases where $R^1 \in \mathcal{R}$ is only quasi-transitive, $R^2 \in \mathcal{R}$ is complete and transitive and still $P(R^1, R^2)$ shows no cycle, will be of special interest. For simplicity of expression, we define the following binary relation: for all $x, y \in X$,

$$(x, y) \in \Gamma \Leftrightarrow [(x, y) \notin P(R^1), (y, x) \notin P(R^1) \text{ and } (x, y) \in P(R^2)].$$

Proposition 6 *Let $R^1, R^2 \in \mathcal{R}$. Suppose that R^1 is quasi-transitive, and that R^2 is complete and transitive. Suppose further that the following two conditions hold:*
(A) *for all $x, y, z \in X$, if $(x, y) \in \Gamma$, $(y, z) \in P(R^1)$ and $(z, y) \in P(R^2)$, then $(x, z) \in P(R^1)$.*
(B) *for all $x, y, z \in X$, if $(x, y) \in \Gamma$ and $(y, z) \in \Gamma$, then $(z, x) \notin P(R^1)$.*
Then, the lexicographic composition $P(R^1, R^2)$ is acyclic.

Proof. Assume that R^1 is quasi-transitive, that R^2 is complete and transitive, and that conditions (A) and (B) are satisfied. Suppose, on the contrary, that that $P(R^1, R^2)$ has a cycle. Let $(x^1, ..., x^k)$ be a cycle of the smallest cardinality for $P(R^1, R^2)$. Since $P(R^1, R^2)$ is asymmetric, $k \geq 3$.

Assume that $(x^1, x^2), (x^2, x^3) \in P(R^1)$. Then, by quasi-transitivity of R^1, $(x^1, x^3) \in P(R^1)$ and then $(x^1, x^3, ..., x^k)$ is a cycle for $P(R^1, R^2)$ which contradicts the fact that $(x^1, ..., x^k)$ is a cycle of the smallest cardinality for $P(R^1, R^2)$.

Assume that $(x^1, x^2), (x^2, x^3) \in \Gamma$. Then, by definition, $(x^1, x^2), (x^2, x^3) \in P(R^2)$ and by transitivity of R^2, $(x^1, x^3) \in P(R^2)$. Moreover, by condition (B), $(x^3, x^1) \notin P(R^1)$ which implies that $(x^1, x^3) \in P(R^1, R^2)$. Then $(x^1, x^3, ..., x^k)$ is a cycle for $P(R^1, R^2)$ which contradicts the fact that $(x^1, ..., x^k)$ is a cycle of the smallest cardinality for $P(R^1, R^2)$.

Let $(x^1, ..., x^k)$ be one of the smallest cycles for $P(R^1, R^2)$ with $k \geq 3$. From what we have shown above, with no loss of generality, we can set $(x^1, x^2) \in \Gamma$ and $(x^2, x^3) \in P(R^1)$. We distinguish two cases.
(1) If $(x^3, x^2) \in P(R^2)$, then by condition (A), $(x^3, x^1) \in P(R^1)$. By quasi-transitivity of R^1, $(x^2, x^3), (x^3, x^1) \in P(R^1)$ implies $(x^2, x^1) \in P(R^1)$ which contradicts $(x^1, x^2) \in \Gamma$.
(2) If $(x^3, x^2) \notin P(R^2)$, then by completeness of R^2, $(x^2, x^3) \in R^2$. Together with

$(x^1, x^2) \in \Gamma$ and transitivity of R^2, we have $(x^1, x^3) \in P(R^2)$. If $(x^3, x^1) \in P(R^1)$, then by quasi-transitivity of R^1, $(x^2, x^1) \in P(R^1)$, which contradicts $(x^1, x^2) \in \Gamma$. Hence, $(x^3, x^1) \notin P(R^1)$. But then, $(x^1, x^3) \in P(R^1, R^2)$, and $(x^1, x^3, ..., x^k)$ is a cycle for $P(R^1, R^2)$, which contradicts the fact that $(x^1, ..., x^k)$ is a cycle of the smallest cardinality for $P(R^1, R^2)$. ■

Example 5 For each $i \in N$, let $a_i \in \mathbb{R}^m_+$ be the *reference bundle for agent* i. (Examples of reference bundles are (i) the equal division bundle for all agents under a social resource constraint, (ii) initial endowment bundles in a private ownership economy, (iii) minimum bundles to meet some basic functionings.) Define $R^B \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^B$ if and only if $\#\{i \in N \mid (x_i, a_i) \in \succsim_i\} \geq \#\{i \in N \mid (y_i, a_i) \in \succsim_i\}$. Clearly, R^B is complete and transitive. Notice that if $(x, y) \in P(R^P)$ where R^P is the weak Pareto domination defined above, then it never occurs that $(y, x) \in P(R^B)$. Hence, condition (A) in Proposition 6 is vacuously satisfied. Furthermore, if $(x, y) \in \Gamma$ and $(y, z) \in \Gamma$, then $(x, z) \in P(R^B)$ by transitivity of R^B, and hence $(z, x) \notin P(R^P)$. Therefore, condition (B) in Proposition 6 is also met. We can conclude that the the lexicographic composition $P(R^P, R^B)$ is acyclic. Of course, the same result holds for $P(R^{\hat{P}}, R^B)$

Often an equity criterion dichotomizes allocations into equitable and non-equitable ones. In such a case, we can define a complete and transitive binary relation R^2 as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^2$ if and only if x is equitable or y is not equitable. Note that from this definition, $(x, y) \in P(R^2)$ if and only if x is equitable *and* y is not equitable. Moreover, in this case, R^2 has at most two indifference classes. Hence, the condition (B) in Proposition 6 is irrelevant because for all $x, y, z \in \mathbb{R}^{mn}_+$, $(x, y) \in \Gamma$ and $(y, z) \in \Gamma$ cannot occur together. Therefore, we have the following corollary.

Corollary 3 *Let $R^1, R^2 \in \mathcal{R}$. Suppose that R^1 is quasi-transitive, and that R^2 is complete and transitive, and has at most two indifference classes. Suppose further that for all $x, y, z \in X$, if $(x, y) \in \Gamma$, $(y, z) \in P(R^1)$ and $(z, y) \in P(R_2)$, then $(x, z) \in P(R^1)$. Then, the lexicographic composition $P(R^1, R^2)$ is acyclic.*

Let $A \subset \mathbb{R}^{mn}_+$ be such that for all $a, b \in A$, $a \geq b$ or $b \geq a$. An allocation $x \in \mathbb{R}^{mn}_+$ is *A-egalitarian-equivalent* if there exists $a \in A$ such that $(x_i, a) \in \sim_i$ for all $i \in N$. Let $E_A \subset \mathbb{R}^{mn}_+$ be the set of A-egalitarian-equivalent allocations. Define R^{E_A} as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^{E_A}$ if and only if $x \in E_A$ or $y \notin E_A$. Notice that $(x, y) \in P(R^{E_A})$ if and only if $x \in E_A$ and $y \notin E_A$. One can check that R^{E_A} is complete and transitive.

Proposition 7 *Let R^{E_A} be defined as above. Let R^P be the weak Pareto domination. Then, the lexicographic composition $P(R^P, R^{E_A})$ is acyclic.*

Proof. As noted above, R^{E_A} is complete and transitive, and R^P is quasi-transitive. In view of Corollary 3, it is enough to show that for all $x, y, z \in \mathbb{R}^{mn}_+$, if $(x, y) \in \Gamma$, $(y, z) \in P(R^P)$ and $(z, y) \in P(R_{E_A})$, then $(x, z) \in P(R^P)$.

Suppose that $(x, y) \in \Gamma$, $(y, z) \in P(R^P)$ and $(z, y) \in P(R_{E_A})$. Because $(x, y) \in P(R^{E_A})$, we have $x \in E_A$ and $y \notin E_A$. Thus, there exists $a \in A$ such that $(x_i, a) \in \sim_i$ for all $i \in N$. Since $(z, y) \in P(R_{E_A})$, we have $z \in E_A$ and $y \notin E_A$. Hence, there exists $b \in A$ such that $(z_i, b) \in \sim_i$ for all $i \in N$. If $(y_i, x_i) \in \sim_i$ for all $i \in N$, then $(y_i, a) \in \sim_i$ for all $i \in N$, which contradicts $y \notin E_A$. Therefore, $(y, x) \notin P(R^P)$ holds only if there exists $i^* \in N$ such that $(x_{i^*}, y_{i^*}) \in \succ_{i^*}$. Since $(y, z) \in P(R^P)$, we have $(y_i, z_i) \in \succsim_i$ for all $i \in N$, and in particular, for agent i^*. Hence, $(x_{i^*}, z_{i^*}) \in \succ_{i^*}$. We also have $(a, x_{i^*}) \in \sim_{i^*}$ and $(z_{i^*}, b) \in \sim_{i^*}$. By transitivity of $\succsim_{i^*}$, $(a, b) \in \succ_{i^*}$. Since $a, b \in S$, either $a > b$ or $b > a$. By strict monotonicity of $\succsim_{i^*}$, we have $a > b$. Then, for all $i \in N$, $(x_i, a) \in \sim_i$, $(a, b) \in \succ_i$ and $(b, z_i) \in \sim_i$. It follows from transitivity of $\succsim_i$ that $(x_i, z_i) \in \succ_i$. Thus, we have $(x, z) \in P(R^P)$.■

5 Lexicographic Composition of Two Choice Functions

In this section, we study the procedure β to compose two criteria, namely, we first choose the set of maximal elements for the first binary relation R^1, and then from this set we select its subset of maximal elements for the second binary relation R^2. Formally, the procedure is represented by the choice function $C_{P(R^2)}C_{P(R^1)}$.

It is clear from the definition that $C_{P(R^2)}C_{P(R^1)}(S) = C_{P(R^2)}(C_{P(R^1)}(S)) \neq \emptyset$ for every $S \in \mathcal{X}$ if both R^1 and R^2 are acyclic. However, even if there exists a cycle S for $P(R^2)$, $C_{P(R^2)}C_{P(R^1)}(S) \neq \emptyset$ holds as long as $P(R^1)$ is acyclic and $P(R^1)$ ranks at least one pair in S. Our next result provides a necessary and sufficient condition for $C_{P(R^2)}C_{P(R^1)}$ to satisfy non-emptiness.

Proposition 8 *Let $R^1, R^2 \in \mathcal{R}$. The choice function $C_{P(R^2)}C_{P(R^1)}$ satisfies non-emptiness if and only if $P(R^1)$ is acyclic and for every cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^2)$, there exist $k, \ell \in \{1, \ldots, K\}$ such that $(x^k, x^\ell) \in P(R^1)$.*

Proof. The "only if" part: It is clear that if $P(R^1)$ has a cycle $T = (x^1, \ldots, x^K) \subseteq X$, then $C_{P(R^1)}(T) = \emptyset$ and $C_{P(R^2)}C_{P(R^1)}(T) = C_{P(R^2)}(C_{P(R^1)}(T)) = C_{P(R^2)}(\emptyset) = \emptyset$. Thus, $C_{P(R^2)}C_{P(R^1)}$ does not satisfy non-emptiness. Suppose that there exists a cycle $T = (x^1, \ldots, x^K) \subseteq X$ for $P(R^2)$ such that for all $k, \ell \in \{1, \ldots, K\}$, $(x^k, x^\ell) \notin P(R^1)$. Then, by definition of $C_{P(R^1)}$, $C_{P(R^1)}(T) = T$. Then, $C_{P(R^2)}C_{P(R^1)}(T) = C_{P(R^2)}(C_{P(R^1)}(T)) = C_{P(R^2)}(T) = \emptyset$. Therefore, $C_{P(R^2)}C_{P(R^1)}$ violates non-emptiness.

The "if" part: Suppose that $P(R^1)$ is acyclic and for every cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^2)$, there exist $k, \ell \in \{1, \ldots, K\}$ such that $(x^k, x^\ell) \in P(R^1)$. Let $S \in \mathcal{X}$. Since R^1 is acyclic, $C_{P(R^1)}(S) \neq \emptyset$. Moreover, if $P(R^2)$ has a cycle $(x^1, \ldots, x^K) \subseteq C_{P(R^1)}(S)$, then there exist $k, \ell \in \{1, ..., K\}$ such that $(x_k, x_\ell) \in P(R^1)$. Then, $x_\ell \notin C_{P(R^1)}(S)$, which is a contradiction. Thus, $P(R^2)$ does not have a cycle in $C_{P(R^1)}(S)$, and hence $C_{P(R^2)}C_{P(R^1)}(S) \neq \emptyset$. Since this holds true for every $S \in \mathcal{X}$, $C_{P(R^2)}C_{P(R^1)}$ satisfies non-emptiness.■

Comparing Propositions 3 and 8, one can see that if $C_{P(R^1,R^2)}$ satisfies non-emptiness (or equivalently, $P(R^1, R^2)$ is acyclic), then $C_{P(R^2)}C_{P(R^1)}$ satisfies non-emptiness as well. Then, if one has to compose lexicographically two criteria for decision making, non-emptiness of the composition of the choice functions is more easily obtained than the non-emptiness of the composition of the binary relations.

Corollary 4 *Let $R^1, R^2 \in \mathcal{R}$. If the choice function $C_{P(R^1,R^2)}$ satisfies non-emptiness, then $C_{P(R^2)}C_{P(R^1)}$ also satisfies non-emptiness.*

The following example shows that the converse of Corollary 4 does not hold true.

Example 6 Let R^P be the weak Pareto domination, and let R^F be defined as in Example 1. As noted above, R^P is quasi-transitive and R^F is transitive. Hence, for every finite set $S \in \mathbb{R}^m_+$, $C_{P(R^P)}(S) \neq \emptyset$, and $C_{P(R^F)}(C_{P(R^P)}(S)) \neq \emptyset$. However, there exists a cycle for $P(R^P, R^F)$ (Tadenuma, 2002), and hence $C_{P(R^P,R^F)}$ does not satisfy non-emptiness. The same result holds for the other criteria given in Example 4, namely $C_{P(R^H)}C_{P(R^P)}$ and $C_{P(R^E)}C_{P(R^P)}$ satisfy non-emptiness whereas $C_{P(R^H,R^P)}$ and $C_{P(R^E,R^P)}$ do not.

Corollary 4 also follows from the next proposition.

Proposition 9 *For every $S \in \mathcal{X}$, $C_{P(R^1)}(S) \cap C_{P(R^2)}(S) \subseteq C_{P(R^1,R^2)}(S) \subseteq C_{P(R^2)}C_{P(R^1)}(S)$.*

Proof. To show $C_{P(R^1)}(S) \cap C_{P(R^2)}(S) \subseteq C_{P(R^1,R^2)}(S)$, let $S \in \mathcal{X}$ and $x \in C_{P(R^1)}(S) \cap C_{P(R^2)}(S)$. Then, there exists no $y \in S$ with $(y,x) \in P(R^1)$, nor does $y \in S$ with $(y,x) \in P(R^2)$. By definition of $P(R^1, R^2)$, there exists no $y \in S$ such that $(y,x) \in P(R^1, R^2)$. Hence, $x \in C_{P(R^1,R^2)}(S)$.

To prove $C_{P(R^1,R^2)}(S) \subseteq C_{P(R^2)}C_{P(R^1)}(S)$, let $S \in \mathcal{X}$ and $S \ni x \notin C_{P(R^2)}C_{P(R^1)}(S)$. By definition, $x \notin C_{P(R^2)}C_{P(R^1)}(S)$ implies 1) $\exists y \in S$ with $(y,x) \in P(R^1)$ or 2) $\exists y \in S$ such that $\forall z \in S, (z,y) \notin P(R^1)$ and $(y,x) \in P(R^2)$. If 1) is satisfied, then $(y,x) \in P(R^1)$ which implies that $(y,x) \in P(R^1, R^2)$. Hence, $x \notin C_{P(R^1,R^2)}(S)$. If 2) is satisfied, then, $(x,y) \notin P(R^1)$ and $(y,x) \in P(R^2)$. Then, by definition, $(y,x) \in P(R^1, R^2)$. Hence, $x \notin C_{P(R^1,R^2)}(S)$.■

We now examine the choice consistency properties of the lexicographic composition of choice functions.

Proposition 10 *Assume that $C_{P(R^2)}C_{P(R^1)}$ satisfies Non-Emptiness. Then, $C_{P(R^2)}C_{P(R^1)}$ satisfies Contraction Consistency if and only if $\forall x, y, z \in X$, $[(x,y) \in$*

$P(R^1)$ and $(y,z) \in \Gamma]$ implies $(x,z) \in P(R^1,R^2)$. Moreover, if $C_{P(R^2)}C_{P(R^1)}$ satisfies Non-Emptiness and Contraction Consistency, then, $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1,R^2)}$.

Proof. The "if" part: Assume that $\forall x, y, z \in X$, $[(x,y) \in P(R^1)$ and $(y,z) \in \Gamma]$ implies $(x,z) \in P(R^1,R^2)$, and that $C_{P(R^2)}C_{P(R^1)}$ does not satisfy Non-Emptiness. First, let us show that $P(R^1,R^2)$ is acyclic. Suppose, on the contrary, that $P(R^1,R^2)$ has a cycle. Let $B = (x_1, ..., x_n)$ be one of the cycles with the smallest cardinality. Since $P(R^1,R^2)$ is asymmetric, the cardinality of B is strictly greater than 2. By Proposition 8, it cannot be the case that B is a cycle for $P(R^1)$ or for Γ. Then, we necessarily have one of the following three cases:
(1) exists $i \in \{1, ..., n-2\}$ such that $(x_i, x_{i+1}) \in P(R^1)$ and $(x_{i+1}, x_{i+2}) \in \Gamma$.
(2) $(x_{n-1}, x_n) \in P(R^1)$ and $(x_n, x_1) \in \Gamma$.
(3) $(x_n, x_1) \in P(R^1)$ and $(x_1, x_2) \in \Gamma$.
In case (1), by assumption, we have, $(x_i, x_{i+2}) \in P(R^1,R^2)$ and then $(x_1, ..., x_i, x_{i+2}, ..., x_n)$ is a cycle for $P(R^1,R^2)$ which contradicts the fact that B is one of the cycles with the smallest cardinality. In case (2), we have, $(x_{n-1}, x_1) \in P(R^1,R^2)$ and then $(x_1, ..., x_{n-1})$ is a cycle for $P(R^1,R^2)$ which contradicts the fact that B is one of the cycles with the smallest cardinality. In case (3), we have, $(x_n, x_2) \in P(R^1,R^2)$ and then $(x_2, ..., x_n)$ is a cycle for $P(R^1,R^2)$ which contradicts the fact that B is one of the cycles with the smallest cardinality. Hence, $P(R^1,R^2)$ is acyclic.

Assume now that $\forall x, y, z \in X$, $[(x,y) \in P(R^1)$ and $(y,z) \in \Gamma]$ implies $(x,z) \in P(R^1,R^2)$. Let $x \in S \in \mathcal{X}$ be such that $x \notin C_{P(R^2)}C_{P(R^1)}(S)$. Let $T \in \mathcal{X}$ be such that $S \subseteq T$. Let us show that $x \notin C_{P(R^2)}C_{P(R^1)}(T)$. If $\exists y \in S, (y,x) \in P(R^1)$, then by definition, $x \notin C_{P(R^2)}C_{P(R^1)}(T)$ which completes the proof. If $\nexists y \in S, (y,x) \in P(R^1,R^2)$, then $x \in C_{P(R^2)}C_{P(R^1)}(S)$ which is a contradiction with the assumptions. Then, $\exists y \in S, (y,x) \in \Gamma$. If $y \in C_{P(R^1)}(T)$, then, by definition, $x \notin C_{P(R^2)}C_{P(R^1)}(T)$, which completes the proof. Assume $\exists z \in T, (z,y) \in P(R^1)$. By acyclicity of $P(R^1,R^2)$ shown above, $\exists n \in \mathbb{N}, \exists (z_0, ..., z_n) \in T$ mutually different and different from x and y such that $\forall i \in \{1, ..., n\}$, $(z_i, z_{i-1}) \in P(R^1)$, $z_0 = z$ and $\forall z' \in T, (z', z_n) \notin P(R^1)$. By definition, $z_n \in C_{P(R^1)}(T)$. Moreover, we have, $(z,y) \in P(R^1)$ and $(y,x) \in \Gamma$, then by assumption, $(z,x) \in P(R^1,R^2)$. If $(z,x) \in P(R^1)$, then, by definition, $x \notin C_{P(R^2)}C_{P(R^1)}(T)$, which completes the proof. If $(z,x) \in \Gamma$, then, $(z_1, z) \in P(R^1)$ and $(z,x) \in \Gamma$, therefore, by assumption, $(z_1, x) \in P(R^1, R^2)$. If $(z_1, x) \in P(R^1)$, then, by definition, $x \notin C_{P(R^2)}C_{P(R^1)}(T)$, which completes the proof. If $(z_1, x) \in \Gamma$, then, $(z_2, z_1) \in P(R^1)$ and $(z_1, x) \in \Gamma$, therefore, by assumption, $(z_2, z_1) \in P(R^1, R^2)$. Iterating until $(z_n, x) \in \Gamma$ completes the proof since, we have shown that $z_n \in C_{P(R^1)}(T)$ and then, $x \notin C_{P(R^2)}C_{P(R^1)}(T)$.

The "only if"part: Assume that $C_{P(R^2)}C_{P(R^1)}$ satisfies Non-Emptiness and Contraction Consistency. Suppose, on the contrary, that $\exists x, y, z \in X$ such that $(x, y) \in P(R^1)$, $(y, z) \in \Gamma$ and $(x, z) \notin P(R^1, R^2)$. Then, by definition, $z \in C_{P(R^2)}C_{P(R^1)}(\{x, y, z\})$. However, $\{y\} = C_{P(R^2)}C_{P(R^1)}(\{y, z\})$, which is a contradiction with Contraction Consistency of $C_{P(R^2)}C_{P(R^1)}$.

Now, let us show that if $C_{P(R^2)}C_{P(R^1)}$ satisfies Non-Emptiness and Contraction Consistency, then, $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1,R^2)}$. Let $x \in S \in \mathcal{X}$. If $\exists y \in S$ such that $(y, x) \in P(R^1, R^2)$, then by definition, $x \notin C_{P(R^2)}C_{P(R^1)}(\{x, y\})$, and by Contraction Consistency of $C_{P(R^2)}C_{P(R^1)}$, we have $x \notin C_{P(R^2)}C_{P(R^1)}(S)$. If $\forall y \in S, (y, x) \notin P(R^1, R^2)$, then by definition, $x \in C_{P(R^2)}C_{P(R^1)}(S)$. ■

The following example shows that even if $C_{P(R^1,R^2)}$ satisfies Non-Emptiness and Contraction Consistency (or equivalently, $P(R^1, R^2)$ is acyclic), it is possible that $C_{P(R^2)}C_{P(R^1)} \neq C_{P(R^1,R^2)}$ and $C_{P(R^2)}C_{P(R^1)}$ violates Contraction Consistency.

Example 7 Let $S = \{x, y, z\}$. Assume that $P(R^1) = \{(y, x)\}$ and $P(R^2) = \{(z, y)\}$. Then, $P(R^1, R^2)$ is acyclic and $C_{P(R^1,R^2)}$ satisfies Non-Emptiness and Contraction Consistency. However, $C_{P(R^2)}C_{P(R^1)}(S) = C_{P(R^2)}(\{x, z\}) = \{x, z\}$ and $C_{P(R^1,R^2)}(S) = \{z\}$. Hence, $C_{P(R^2)}C_{P(R^1)} \neq C_{P(R^1,R^2)}$.

Let $T = \{x, y\} \subset S$. Then, although $x \in T \cap C_{P(R^2)}C_{P(R^1)}(S)$, $C_{P(R^2)}C_{P(R^1)}(T) = \{y\}$. This is a violation of Contraction Consistency.

Corollary 5 *Let $R^1, R^2 \in \mathcal{R}$ be given. If $C_{P(R^2)}C_{P(R^1)}$ satisfies Non-Emptiness and Contraction Consistency, then $C_{P(R^1,R^2)}$ satisfies Non-Emptiness and Contraction Consistency. However, the converse does not hold true.*

In contrast to the above result, requiring Non-Emptiness and Path Independence for $C_{P(R^2)}C_{P(R^1)}$ is equivalent to requiring the same conditions for $C_{P(R^1,R^2)}$ as the following proposition shows.

Proposition 11 *The following three statements are equivalent.*

1. *$P(R^1, R^2)$ is quasi-transitive.*

2. *$C_{P(R^1,R^2)}$ satisfies Non-Emptiness and Path Independence.*

3. *$C_{P(R^2)}C_{P(R^1)}$ satisfies Non-Emptiness and Path Independence.*

Proof. The equivalence of 1 with 2 is already shown in Corollary 2. We prove the equivalence of 1 with 3.

1 $\Rightarrow$ 3: First, let us prove that if $P(R^1, R^2)$ is quasi-transitive, then $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1,R^2)}$. Let $S \in \mathcal{X}$. From Proposition 9, $C_{P(R^1,R^2)}(S) \subseteq C_{P(R^2)}C_{P(R^1)}(S)$. To show that $C_{P(R^2)}C_{P(R^1)}(S) \subseteq C_{P(R^1,R^2)}(S)$, let $x \notin C_{P(R^1,R^2)}(S)$ and $x \in S$. Then, by definition, $\exists y \in S$ such that $(y, x) \in P(R^1, R^2)$. Since $P(R^1, R^2)$ is quasi-transitive and asymmetric, it is acyclic. Then, there exists an integer n such that there exists a sequence $(y^0, ..., y^n)$ such that $\forall k \in \{1, ..., n\}, (y^k, y^{k-1}) \in P(R^1, R^2)$, $y^0 = x$ and $\forall z \in S, (z, y^n) \notin P(R^1, R^2)$. Then, $y^n \in C_{P(R^1)}(S)$. By quasi-transitivity of $P(R^1, R^2)$, $(y^n, x) \in P(R^1, R^2)$. If $(y^n, x) \in P(R^1)$, $x \notin C_{P(R^2)}C_{P(R^1)}(S)$ follows by definition. If $(y^n, x) \notin P(R^1)$ and $(y^n, x) \in P(R^2)$, $x \notin C_{P(R^2)}C_{P(R^1)}(S)$ follows by $y^n \in C_{P(R^1)}(S)$.

Thus, $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1,R^2)}$. However, $C_{P(R^1,R^2)}$ satisfies Path Independence since $P(R^1, R^2)$ is quasi-transitive. Hence, $C_{P(R^2)}C_{P(R^1)}$ also satisfies Path Independence.

3 $\Rightarrow$ 1: Assume that $C_{P(R^2)}C_{P(R^1)}$ satisfies Non-Emptiness and Path Independence. Since Path Independence implies Contraction Consistency, it follows from Proposition 10 that $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1,R^2)}$. Then, by Corollary 2, $P(R^1, R^2)$ is quasi-transitive. ■

To illustrate this result, let us go back to Examples 1, 2 and 3. We have shown that $C_{P(R^F,R^P)}$, $C_{P(R^H,R^P)}$ and $C_{P(R^E,R^P)}$ are non-empty and path independent. Then, by

Proposition 11, we can conclude that $C_{P(R^P)}C_{P(R^F)}$, $C_{P(R^P)}C_{P(R^H)}$ and $C_{P(R^P)}C_{P(R^E)}$ are also non-empty and path independent.

6 Order Independence of Lexicographic Compositions

In this section, we investigate under what conditions the outcomes of each choice procedure are independent of the order of lexicographic applications of two criteria.

The next result, which is based on Proposition 9, shows that if procedure α satisfies order independence, then it always chooses the intersection of the set of maximal elements of the first criterion with that of the second criterion, irrespective of the order of application of the two criteria.

Proposition 12 *For every $S \in \mathcal{X}$, if $C_{P(R^1,R^2)}(S) = C_{P(R^2,R^1)}(S)$, then $C_{P(R^1,R^2)}(S) = C_{P(R^2,R^1)}(S) = C_{P(R^1)}(S) \cap C_{P(R^2)}(S)$.*

Proof. Let $S \in \mathcal{X}$. Let $x \in C_{P(R^1,R^2)}(S)$. By Proposition 9, $x \in C_{P(R^1)}(S)$. Hence, $C_{P(R^1,R^2)}(S) \subseteq C_{P(R^1)}(S)$. Similarly, $C_{P(R^2,R^1)}(S) \subseteq C_{P(R^2)}(S)$. Then, $C_{P(R^1,R^2)}(S) = C_{P(R^2,R^1)}(S) \subseteq C_{P(R^1)}(S) \cap C_{P(R^2)}(S)$. On the other hand, by Proposition 9, $C_{P(R^1)}(S) \cap C_{P(R^2)}(S) \subseteq C_{P(R^1,R^2)}(S) = C_{P(R^2,R^1)}(S)$. Thus, $C_{P(R^1,R^2)}(S) = C_{P(R^2,R^1)}(S) = C_{P(R^1)}(S) \cap C_{P(R^2)}(S)$. ■

The following results suggest that order independence is quite a strong requirement.

Lemma 1 *If $P(R^1) \cup P(R^2)$ is asymmetric, then $C_{P(R^1,R^2)} = C_{P(R^2,R^1)} = C_{P(R^1)\cup P(R^2)} = C_{P(R^1)} \cap C_{P(R^2)}$.*

Proof. Suppose that $P(R^1) \cup P(R^2)$ is asymmetric. Then, if $(x,y) \in P(R^1)$, then $(y,x) \notin P(R^2)$, and if $(x,y) \in P(R^2)$, then $(y,x) \notin P(R^1)$. Hence, we have $P(R_1,R_2) = P(R^1) \cup P(R^2) = P(R_2,R_1)$ and thus $C_{P(R^1,R^2)} = C_{P(R^2,R^1)} = C_{P(R^1)\cup P(R^2)}$.

To prove $C_{P(R^1)\cup P(R^2)} \subseteq C_{P(R^1)} \cap C_{P(R^2)}$, let us consider $S \in \mathcal{X}$ and $S \ni x \notin C_{P(R^1)} \cap C_{P(R^2)}(S)$. Then, $\exists y \in S$ such that $(y,x) \in P(R^2)$ or $(y,x) \in P(R^1)$. Hence, $(y,x) \in P(R^1) \cup P(R^2)$ and by asymmetry of $P(R^1) \cup P(R^2)$, $(x,y) \notin P(R^1) \cup P(R^2)$. Then $(y,x) \in P(P(R^1) \cup P(R^2))$. Hence, $x \notin C_{P(R^1)\cup P(R^2)}(S)$.

To prove $C_{P(R^1)} \cap C_{P(R^2)} \subseteq C_{P(R^1)\cup P(R^2)}$, let us consider $S \in \mathcal{X}$ and $S \ni x \notin C_{P(R^1)\cup P(R^2)}$. Then, by definition, $\exists y \in S$ such that $(y,x) \in P(R^1) \cup P(R^2)$. Then, $x \notin C_{P(R^1)}$ or $x \notin C_{P(R^2)}$.■

We now give a necessary and sufficient condition for the lexicographic composition of two binary relations to be non-empty and order independent.

Corollary 6 *$C_{P(R^1,R^2)} = C_{P(R^2,R^1)}$ and $C_{P(R^1,R^2)}$ satisfies non-emptiness if and only if $P(R^1) \cup P(R^2)$ is acyclic.*

Proof. The "if" part follows directly from Lemma 1 and Corollary 1.

To prove the "only if" part, let $C_{P(R^1,R^2)} = C_{P(R^2,R^1)}$, $C_{P(R^1,R^2)}$ satisfy non-emptiness and $(x^1,...,x^K)$ be a cycle for $P(R^1) \cup P(R^2)$. By Proposition 3, $\exists k \in \{1,...,K-1\}$ such that $(x^{k+1},x^k) \in P(R^1)$ or $(x^1,x^K) \in P(R^1)$. Let us consider $(x^1,x^K) \in P(R^1)$ (the proof is similar in the other cases). Then, $(x^K,x^1) \in P(R^2)$ and $(x^1,x^K) \in P(R^1)$. Then, $C_{P(R^1,R^2)}(\{x^1,x^K\}) = \{x^1\} \neq \{x^K\} = C_{P(R^2,R^1)}(\{x^1,x^K\})$. This is a contradiction.■

Obviously, for two criteria, the acyclicity of $P(R^1) \cup P(R^2)$ is very demanding. Indeed, it implies that $P(R^1) \cup P(R^2)$ is asymmetric, i.e. that the two criteria are never in contradiction. This requirement is rarely met when we are concerned with the efficiency and equity criteria.

As we show next, the acyclicity of $P(R^1) \cup P(R^2)$ is also necessary for procedure β to be order independent. In fact, requiring order independence of procedure β is even more demanding than procedure α.

Proposition 13 *If $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1)}C_{P(R^2)}$, then $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1)}C_{P(R^2)} = C_{P(R^1)} \cap C_{P(R^1)}$.*

Proof. Let $S \in \mathcal{X}$. Let $x \in C_{P(R^2)}C_{P(R^1)}(S) = C_{P(R^1)}C_{P(R^2)}(S)$. By definition, $x \in C_{P(R^1)}(S)$ and $x \in C_{P(R^2)}(S)$.

Let $x \in C_{P(R^1)} \cap C_{P(R^1)}$. Then, by definition, $\forall y \in S, (y, x) \notin P(R^1) \cup P(R^2)$. Then, by definition, $x \in C_{P(R^2)}C_{P(R^1)}(S) = C_{P(R^1)}C_{P(R^2)}(S)$.■

A necessary and sufficient condition for $C_{P(R^2)}C_{P(R^1)}$ to coincide with $C_{P(R^1)}C_{P(R^2)}$ and satisfy non-empty was given in Houy (2007).

Proposition 14 *$C_{P(R^2)}C_{P(R^1)} = C_{P(R^1)}C_{P(R^2)}$ and $C_{P(R^2)}C_{P(R^1)}$ satisfies non-emptiness if and only if (i) $P(R^1) \cup P(R^2)$ is acyclic, and (ii) for all $x, y, z \in X$, if $(x, y), (y, z) \in P(R^1) \cup P(R^2)$ and for some $i \in \{1, 2\}$, $(x, y) \in P(R^i)$ and $(y, z) \notin P(R^i)$, then $(x, z) \in P(R^1) \cup P(R^2)$.*

From Proposition 13 and Lemma 1, we have the following corollary.

Corollary 7 *If $C_{P(R^2)}C_{P(R^1)} = C_{P(R^1)}C_{P(R^2)}$ and $C_{P(R^2)}C_{P(R^1)}$ satisfies non-emptiness, then $C_{P(R^1,R^2)} = C_{P(R^2,R^1)} = C_{P(R^2)}C_{P(R^1)} = C_{P(R^1)}C_{P(R^2)} = C_{P(R^1)\cup P(R^2)} = C_{P(R^2)} \cap C_{P(R^1)}$.*

Let us give an example to show that order independence and non-emptiness of procedure β is strictly more demanding that order independence and non-emptiness of procedure α. Let us have $\mathcal{X} = \{x, y, z\}$, $R^1 = \{(x, y)\}$ and $R^2 = \{(y, z)\}$. Then, $P(R^1, R^2) = P(R^2, R^1) = \{(x, y), (y, z)\}$. However, $C_{P(R^2)}C_{P(R^1)}(\{x, y, z\}) = \{x, z\}$ whereas $C_{P(R^1)}C_{P(R^2)}(\{x, y, z\}) = \{x\}$.

7 Maximal Elements in the Set of All Alternatives

In the previous sections, we have studied non-emptiness and order independence of lexicographic compositions of multiple criteria *for every finite subset* of X. Such analyzes are important for decision making when not all alternatives are available in each social or individual problem due to, for instance, technological, political, social, or time constraints. However, if all alternatives are attainable, we may need only to know the "optimal" alternatives in the whole set of alternatives. In this section, we focus on the maximal elements of the choice functions in the set X of all alternatives.

The next result provides a necessary and sufficient condition for the outcomes of the choice functions representing procedure α to be independent of the order of lexicographic applications of two criteria.

Proposition 15 *Let $R^1, R^2 \in \mathcal{R}$. $C_{P(R^1,R^2)}(X) = C_{P(R^2,R^1)}(X)$ if and only if for every $x \in X$, (i) $x \in C_{P(R^1)}(X)$ and $x \notin C_{P(R^2)}(X)$ imply that there exists $y \in X$ with $(y, x) \in P(R^2)$ and $(x, y) \notin P(R^1)$; and (ii) $x \in C_{P(R^2)}(X)$ and $x \notin C_{P(R^1)}(X)$ imply that there exists $y \in X$ with $(y, x) \in P(R^1)$ and $(x, y) \notin P(R^2)$. Furthermore, if $C_{P(R^1,R^2)}(X) = C_{P(R^2,R^1)}(X)$, then $C_{P(R^1,R^2)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$.*

Proof. The "if" part: Assume that the condition in the proposition holds.

(1) Let $x \in C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$. Then, for every $y \in X$, $(y, x) \notin P(R^1)$ and $(y, x) \notin P(R^2)$. Notice that for every $y \in X$, $(y, x) \in P(R^1, R^2) \cup P(R^2, R^1)$ holds only if $(y, x) \in P(R^1) \cup P(R^2)$. Hence, for every $y \in X$, $(y, x) \notin P(R^1, R^2) \cup P(R^2, R^1)$. Thus, $x \in C_{P(R^1,R^2)}(X) \cap C_{P(R^2,R^1)}(X)$. We have shown that $C_{P(R^1)}(X) \cap C_{P(R^2)}(X) \subseteq C_{P(R^1,R^2)}(X) \cap C_{P(R^2,R^1)}(X)$.

(2) Let $x \in X \setminus [C_{P(R^1)}(X) \cup C_{P(R^2)}(X)]$. Then, there exist $y \in X$ with $(y, x) \in P(R^1)$ and $z \in X$ with $(z, x) \in P(R^2)$. Therefore, $x \notin C_{P(R^1,R^2)}(X) \cup C_{P(R^2,R^1)}(X)$. Thus, $C_{P(R^1,R^2)}(X) \cup C_{P(R^2,R^1)}(X) \subseteq C_{P(R^1)}(X) \cup C_{P(R^2)}(X)$.

(3) Let $x \in C_{P(R^1)}(X)$ and $x \notin C_{P(R^2)}(X)$. Then, $x \notin C_{P(R^2,R^1)}(X)$. By the condition in the proposition, there exists $y \in X$ with $(y, x) \in P(R^2)$ and $(x, y) \notin P(R^1)$. Then, $(y, x) \in P(R^1, R^2)$. Hence, $x \notin C_{P(R^1,R^2)}(X)$. Thus, $x \notin C_{P(R^1,R^2)}(X) \cup C_{P(R^2,R^1)}(X)$.

(4) Similarly to (3), if $x \in C_{P(R^2)}(X)$ and $x \notin C_{P(R^1)}(X)$, then $x \notin C_{P(R^1,R^2)}(X) \cup C_{P(R^2,R^1)}(X)$.

It follows from (1), (2), (3) and (4) that $C_{P(R^1,R^2)}(X) = C_{P(R^2,R^1)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$.

The "only if" part: Assume that condition (i) in the proposition does not hold. That is, there exists $x \in X$ such that $x \in C_{P(R^1)}(X)$, $x \notin C_{P(R^2)}(X)$ and for every $y \in X$, $(y, x) \in P(R^2)$ implies $(x, y) \in P(R^1)$. Since $x \notin C_{P(R^2)}(X)$, we have $x \notin C_{P(R^2,R^1)}(X)$. Because $x \in C_{P(R^1)}(X)$ and for every $y \in X$, $(x, y) \notin P(R^1)$ implies $(y, x) \notin P(R^2)$, it follows that $x \in C_{P(R^1,R^2)}(X)$. Thus, $C_{P(R^1,R^2)}(X) \neq C_{P(R^2,R^1)}(X)$.

Similarly, if the condition (ii) does not hold, then $C_{P(R^1,R^2)}(X) \neq C_{P(R^2,R^1)}(X)$.■

Notice that $C_{P(R^1,R^2)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$ does not imply that $C_{P(R^1,R^2)}(X) = C_{P(R^2,R^1)}(X)$. As shown by Tadenuma (2002), in the classical division problem, $C_{P(R^P,R^F)}(X) \neq C_{P(R^F,R^P)}(X)$, where R^P is the weak Pareto domination, and R^F is the no-envy relation defined in Example 1.

For procedure β, an analogous result to Proposition 15 is given in the next proposition.

Proposition 16 *Let $R^1, R^2 \in \mathcal{R}$. $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X)$ if and only if for every $x \in X$, (i) $x \in C_{P(R^1)}(X)$ and $x \notin C_{P(R^2)}(X)$ imply that there exists $y \in C_{P(R^1)}(X)$ with $(y,x) \in P(R^2)$; and (ii) $x \in C_{P(R^2)}(X)$ and $x \notin C_{P(R^1)}(X)$ imply that there exists $y \in C_{P(R^2)}(X)$ with $(y,x) \in P(R^1)$. Furthermore, if $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X)$, then $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$.*

Proof. The "if" part: Assume that the condition in the proposition holds.
(1) Let $x \in C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$. Then, $x \in C_{P(R^1)}C_{P(R^2)}(X) \cap C_{P(R^2)}C_{P(R^1)}(X)$. Thus, $C_{P(R^1)}(X) \cap C_{P(R^2)}(X) \subseteq C_{P(R^1)}C_{P(R^2)}(X) \cap C_{P(R^2)}C_{P(R^1)}(X)$.
(2) Let $x \in X \setminus [C_{P(R^1)}(X) \cup C_{P(R^2)}(X)]$. Then, $x \notin C_{P(R^1)}C_{P(R^2)}(X) \cup C_{P(R^2)}C_{P(R^1)}(X)$. Therefore, $C_{P(R^1)}C_{P(R^2)}(X) \cup C_{P(R^2)}C_{P(R^1)}(X) \subseteq C_{P(R^1)}(X) \cup C_{P(R^2)}(X)$.
(3) Let $x \in C_{P(R^1)}(X)$ and $x \notin C_{P(R^2)}(X)$. Since $x \notin C_{P(R^2)}(X)$, we have $x \notin C_{P(R^1)}C_{P(R^2)}(X)$. By the condition given in the proposition, there exists $y \in C_{P(R^1)}(X)$ with $(y,x) \in P(R^2)$. Hence, $x \notin C_{P(R^2)}C_{P(R^1)}(X)$. Thus, $x \notin C_{P(R^1)}C_{P(R^2)}(X) \cup C_{P(R^2)}C_{P(R^1)}(X)$.
(4) Similarly to (3), if $x \in C_{P(R^2)}(X)$ and $x \notin C_{P(R^1)}(X)$, then $x \notin C_{P(R^1)}C_{P(R^2)}(X) \cup C_{P(R^2)}C_{P(R^1)}(X)$.

From (1), (2), (3) and (4), we obtain $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$.

The "only if" part: Assume that condition (i) in the proposition does not hold. Then, there exists $x \in X$ such that $x \in C_{P(R^1)}(X)$, $x \notin C_{P(R^2)}(X)$ and for every $y \in C_{P(R^1)}(X)$, $(y,x) \notin P(R^2)$. By $x \notin C_{P(R^2)}(X)$, we have $x \notin C_{P(R^1)}C_{P(R^2)}(X)$.

Since $x \in C_{P(R^1)}(X)$ and there exists no $y \in C_{P(R^1)}(X)$ with $(y, x) \in P(R^2)$, $x \in C_{P(R^2)}C_{P(R^1)}(X)$. Hence, $C_{P(R^1)}C_{P(R^2)}(X) \neq C_{P(R^2)}C_{P(R^1)}(X)$.

Similarly, if condition (ii) in the proposition does not hold, then $C_{P(R^1)}C_{P(R^2)}(X) \neq C_{P(R^2)}C_{P(R^1)}(X)$. ■

Note that $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$ does not imply $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X)$. In an example of the classical division problem in Tadenuma (2002, p.465), let R^P be the weak Pareto domination and R^F the no-envy relation as defined in Example 1. Then, it can be shown that $C_{P(R^F)}C_{P(R^P)}(X) = C_{P(R^F)}(X) \cap C_{P(R^P)}(X)$, but $C_{P(R^F)}C_{P(R^P)}(X) \neq C_{P(R^P)}C_{P(R^F)}(X)$.

It is clear that the condition in Proposition 16 implies the condition in Proposition 15. Hence, if $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X)$, then $C_{P(R^1,R^2)}(X) = C_{P(R^2,R^1)}(X)$, and hence, by Propositions 15 and 16, $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X) = C_{P(R^1,R^2)}(X) = C_{P(R^2,R^1)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$.

Corollary 8 *If $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X)$, then $C_{P(R^1)}C_{P(R^2)}(X) = C_{P(R^2)}C_{P(R^1)}(X) = C_{P(R^1,R^2)}(X) = C_{P(R^2,R^1)}(X) = C_{P(R^1)}(X) \cap C_{P(R^2)}(X)$.*

8 Conclusion

In this paper we have formalized two distinct procedures to lexicographically apply multiple criteria. We have shown conditions for these procedures to be non-empty, path independent and order independent.

The results suggest that to guarantee non-empty outcomes from procedure α, in which we first construct the lexicographic composition of two binary relations, and then select its maximal elements, is more difficult than from procedure β, in which we first select the set of maximal elements for the first binary relation, and then chooses from that set its maximal elements for the second binary relation. However, to guarantee path independent outcomes from procedure α is equivalent to guaranteeing path independent outcomes from procedure β.

For order independence of procedures α and β, the conditions are rather strong. However, contrary to non-emptiness, guaranteeing order independent outcomes from procedure β is more difficult than from procedure α. In particular, acyclicity of the union of two binary relations implies that there is no conflict between the two criteria such that x is better than y according to the first criterion whereas y is better than x for the second criterion. In many cases such as the efficiency-equity trade-off, this condition cannot be met. This suggests that in most cases we have to be concerned about the order of application of multiple criteria. However, if we are only interested in maximal elements in the set of all alternatives, then it is less difficult to obtain order independence.

Social or individual decision making often involves multiple criteria. The lexicographic applications of the criteria considered in this paper seem natural and reasonable ways to make decisions in such contexts. Other useful conditions for these choice procedures to be non-empty, path independent, or order independent should be worth investigating. It would also be interesting to examine how the choice procedures with multiple criteria can explain seemingly irrational social or individual choices in concrete problems.

References

Aizerman, M. A. (1985), "New Problems in the General Choice Theory," *Social Choice and Welfare* 2, 235-282.

Aizerman, M. A. and F. T. Aleskrov (1995), *Theory of Choice.* Amsterdam: North Holland.

Arrow, K. J. (1951), *Social Choice and Individual Values*, New York: Wiley. Second ed., 1963.

Arrow, K. J. (1977), "Extended Sympathy and the Possibility of Social Choice," *Americal Economic Review, Papers and Proceedings* 67, 219-225.

Blair, D. H., G. Bordes, J. S. Kelly and K. Suzumura, "Impossibility Theorems without Collective Rationality," *Journal of Economic Theory* 13, 361-379.

Chernoff, H. (1954), "Rational Selection of Decision Functions," *Econometrica* 22, 422-443.

Feldman, A. M. and A. Kirman (1974), "Fairness and Envy," *American Economic Review* 64, 996-1005.

Foley, D. (1967), "Resource Allocation and the Public Sector," *Yale Economic Essays* 7, 45-98.

Hammond, P. J. (1976), "Equity, Arrow's Conditions and Rawls' Difference Principle," *Econometrica* 44, 793-804.

Harsanyi, J. C. (1955), "Cardinal Welfare, Individual Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy* 63, 309-321.

Houy, N. (2007), "Rationality and Order-Dependent Sequential Rationality," *Theory and Decision* 62, 119-134.

Kolm, S.-C. (1972), *Justice et Equité*, Paris: Centre National de la Recherche Scientifique.

Manzini, P. and M. Mariotti (2005) "Rationalizing Boundedly Rational Choice: Sequential Rationalizability and Rational Shortlist Method," mimeo., Queen Mary, University of London.

Pattanaik, P. K. (1968), "Risk, Impersonality, and the Social Welfare Function," *Journal of Political Economy* 76, 1152-1169.

Pazner, E. A. and D. Schmeidler (1978), "Egalitarian Equivalent Allocations: A New Concept of Economic Equity," *Quarterly Journal of Economics* 92, 671-687.

Plott, C. R. (1973), "Path Independence, Rationality, and Social Choice," *Econometrica* 41, 1075-1091.

Sen, A. K. (1970), *Collective Choice and Social Welfare*, San Francisco: Holden-Day.

Sen A.K. (1985), *Commodities and Capabilities*, Amsterdam: North-Holland.

Suppes, P. (1966), "Some Formal Models of Grading Principles," *Synthese* 6, 284-306.

Suzumura, K. (1981a), "On Pareto-Efficiency and the No-Envy Concept of Equity," *Journal of Economic Theory* 25, 367-379.

Suzumura, K. (1981b), "On the Possibility of "Fair" Collective Choice Rule," *International Economic Review* 22, 351-364.

Suzumura, K. (1983), *Rational Choice, Collective Decisions, and Social Welfare*, Cambridge, U. K.: Cambridge University Press.

Tadenuma, K. (2002), "Efficiency First or Equity First? Two Principles and Rationality of Social Choice," *Journal of Economic Theory* 104, 462-472.

Tadenuma, K. (2005), "Egalitarian-Equivalence and the Pareto Principle for Social Preferences," *Social Choice and Welfare* 24, 455-473.