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Strategic Complexity in Repeated Extensive Games

by
Nozomu Muto

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Abstract

This paper studies a two-player machine (finite automaton) game in which an extensive game with perfect information is infinitely repeated. We introduce a new measure of strategic complexity named "multiple complexity", which considers the responsiveness of a strategy to information as well as the number of states of machines. In contrast to Piccione and Rubinstein (1993), we prove that a machine game may include non-trivial Nash equilibria. In the sequential-move prisoners' dilemma, cooperation can be sustained in an equilibrium of the machine game.

1 Introduction

Strategic complexity is an important concept in the theory of bounded rationality. This is based on the idea that people prefer less "complex" strategies. It is, however, not necessarily obvious which strategies are considered to be less complex. In the repeated game context, Aumann (1981) first proposed a measure of strategic complexity using finite automata (Moore machines), followed by the seminal works of Neyman (1985) and Rubinstein (1986). A

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†Graduate School of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601, Japan. E-mail: em041208@srv.cc.hit-u.ac.jp

	C	D
C	$2, 2$	$-1, 3$
D	$3, -1$	$0, 0$

Table 1: Two-player prisoners' dilemma game

finite automaton outputs a player's action at every "state" of it. In the literature, the complexity of a finite automaton is considered to be the number of states that it contains. We call it the counting-states complexity.

The notion of a strategic complexity reduces the outcomes of Nash equilibria in a repeated game. Let us consider an example of the infintely repeated game of a two-player prisoners' dilemma. The payoff matrix is given by Table 1. Abreu and Rubinstein (1988) showed that the set of outcomes on the equilibrium path is $\{(C, C), (D, D)\}$ or $\{(C, D), (D, C)\}$.[1]

Piccione and Rubinstein (1993) extended the analysis of the strategic complexity to the case of a two-player repeated game of an extensive-form stage game Γ. When Γ is a game with perfect information, they proved that any Nash equilibrium of the machine game consists of an infinite repetition of a Nash equilibrium of Γ.

However, this result critically depends on the counting-states complexity. Piccione and Rubinstein (1993) formulated an output function of an automaton as a mapping from the set of states in the automaton to the set of stage-game strategies. In this framework, the counting-states complexity of an automaton can not take into account a "complexity" of a stage-game strategy itself with respect to its responsiveness to information in the stage-game. For example, let us consider the sequential-move prisoners' dilemma described in Figure 1. Consider the following two machines for player 2. One is a one-state machine which outputs a stage-game strategy $s_2 : A_1 \to A_2$ such that $s_2(C) = s_2(D) = C$. The other is also a one-state machine which outputs s_2' such that $s_2'(C) = C$ and $s_2'(D) = D$. Because both machines have only one state, they are considered to have the same complexity, one. In our opinion, it is natural to consider that the second machine is more complex than the first one.

This paper provides a new complexity measure, which we call the multiple complexity (M-comlexity). The multiple complexity takes into account a

[1]See Piccione (1992) for the improved version of the proof.

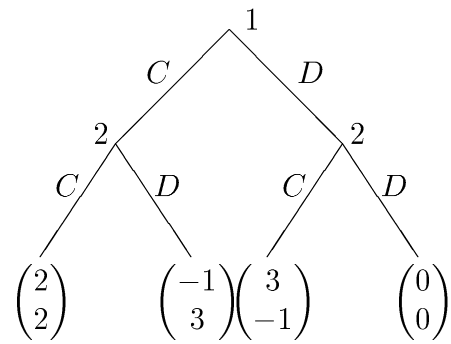

Figure 1: Sequential-move prisoners' dilemma game

complexity of a behavior rule in a period as well as the number of states of the automaton. We derive a necessary and sufficient condition for a Nash equilibrium of the machine game under the multiple complexity. The condition implies that in any equilibrium of the machine game the number of states in the automata is smaller than or equal to the number of every player's actions. In an example of the sequential-move prisoner's dilemma, cooperation can be sustained in an equilibrium of the machine game.

This paper proceeds as follows. In section 2, we present necessary definitions. In section 3, we prove the main result. In section 4, we discuss our result.

2 Definitions

Let $G = (A_1, A_2; u_1, u_2)$ be a two-player strategic game, A_i be a finite set of pure actions for player i $(= 1, 2)$, and u_i be player i's payoff function defined on $A_1 \times A_2$. Let $A = A_1 \times A_2$. Let G^∞ be the infinitely repeated game of G. If player i's stage-game payoff at period t is u_i^t, player i's payoff π_i in G^∞ is $\sum_{t=1}^{\infty} \delta^{t-1} u_i^t$ for a discount factor $0 < \delta < 1$.

In G^∞, player i's finite automaton is represented by a four-tuple $M_i = (Q_i, q_i^1, \lambda_i, \mu_i)$ in which Q_i is a finite set of states, $q_i^1 \in Q_i$ is the initial state, $\lambda_i : Q_i \to A_i$ is the output function, and $\mu_i : Q_i \times A \to Q_i$ is the transition function. Let $\mathcal{M}_i$ be the set of all automata of player i, and $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. The number of states in an automaton $M_i \in \mathcal{M}_i$, i.e. $\#Q_i$, is called the *counting-states (CS-) complexity* and denoted by $\mathrm{comp}_{cs}(M_i)$.[2]

A machine game is a game in which each player chooses her automaton and obtains the sum of discounted payoffs in G^∞. In the machine game for G^∞, players consider complexity of their automata as well as their payoffs. We assume the following class of preferences in a machine game, introduced by Abreu and Rubinstein (1988).

Definition 1. *A preference relation $\succsim_i$ of player i in the machine game satisfies all of the following criteria.*

For $M_1, M_1' \in \mathcal{M}_1$ and $M_2, M_2' \in \mathcal{M}_2$,

1. *If $\pi_i(M_1, M_2) = \pi_i(M_1', M_2')$ and $\mathrm{comp}_{cs}(M_i) = \mathrm{comp}_{cs}(M_i')$, then $(M_1, M_2) \sim_i (M_1', M_2')$.*
2. *If $\pi_i(M_1, M_2) > \pi_i(M_1', M_2')$ and $\mathrm{comp}_{cs}(M_i) = \mathrm{comp}_{cs}(M_i')$, then $(M_1, M_2) \succ_i (M_1', M_2')$.*
3. *If $\pi_i(M_1, M_2) = \pi_i(M_1', M_2')$ and $\mathrm{comp}_{cs}(M_i) < \mathrm{comp}_{cs}(M_i')$, then $(M_1, M_2) \succ_i (M_1', M_2')$.*

Various preference relations satisfy the conditions in Definition 1. For instance, the lexicographic preference in which the second criteria is substituted by "if $\pi_i(M_1, M_2) > \pi_i(M_1', M_2')$, then $(M_1, M_2) \succ_i (M_1', M_2')$" is a special case of Definition 1.

In the machine game, the following result is fundamental.

Proposition 1 (Abreu and Rubinstein, 1988). *Suppose that a pair of automata $(M_1, M_2) \in \mathcal{M}$ is a Nash equilibrium of the machine game. Let q_i^t be player i's state at period t, and a_i^t be player i's output at period t. Then the following three statements are true.*

1. $\mathrm{comp}_{cs}(M_1) = \mathrm{comp}_{cs}(M_2)$.
2. *$q_1^t = q_1^{t'}$ if and only if $q_2^t = q_2^{t'}$ for any two periods t, t'.*
3. *$a_1^t = a_1^{t'}$ if and only if $a_2^t = a_2^{t'}$ for any two periods t, t'.*

Let Γ be a two-player extensive game with perfect information in which player 1 first chooses an action $a_1 \in A_1$ and player 2 chooses an action $a_2 \in A_2$

[2]For a finite set S, $\#S$ denotes the cardinality of S.

after observing a_1. Let S_i $(i = 1, 2)$ be the set of player i's strategies in Γ. Since player 1 is the first-mover and player 2 is the second-mover, we have $S_1 = A_1$ and $S_2 = \{s_2 \,|\, s_2 : A_1 \to A_2\}$. Let Γ^∞ be the infinitely repeated game of Γ. Automata in Γ^∞ will be defined as follows. Since player 1, who moves first, chooses an action in Γ in the same way as in G, automata for player 1 in Γ^∞ and G^∞ are identical. An automaton for player 2 in Γ^∞ is defined by $M_2 = (Q_2, q_2^1, \lambda_2, \mu_2) \in \mathcal{M}_2^\Gamma$. The output function λ_2 is a function from Q_2 to S_2. The range of λ_2 is the set of stage-game strategies of player 2 in Γ. The set of automata for player i in Γ^∞ is denoted by $\mathcal{M}_i^\Gamma$, and let $\mathcal{M}^\Gamma = \mathcal{M}_1^\Gamma \times \mathcal{M}_2^\Gamma$. Under the CS-complexity, Piccione and Rubinstein (1993) showed that there are only trivial equilibria of the machine game for Γ^∞ which consist of an infinite repetition of a Nash equilibrium of Γ.

We are now ready to introduce a new measure of complexity for automata in Γ^∞.

Definition 2. *For player i's automaton $M_i = (Q_i, q_i^1, \lambda_i, \mu_i) \in \mathcal{M}_i^\Gamma$, the multiple (M-) complexity of M_i is defined by*

$$\text{comp}_m(M_i) = \sum_{q_i \in Q_i} \#\{a_i(\lambda_i(q_i), s_j) | s_j \in S_j\} \qquad (j \neq i)$$

where $a_i(s_i, s_j) \in A_i$ is player i's action induced by the pair of strategies (s_i, s_j) in Γ.

Note that $a_1(\lambda_1(q_1), s_2) = \lambda_1(q_1)$ for any $s_2 \in S_2$, and that $a_2(a_1, \lambda_2(q_2)) = (\lambda_2(q_2))(a_1)$ for every $a_1 \in A_1$. Thus $\text{comp}_m(M_1) = \text{comp}_{cs}(M_1)$ for any player 1's automaton $M_1 \in \mathcal{M}_1^\Gamma$. For a stage-game strategy $s_2 : A_1 \to A_2$, let $c(s_2) = \#\{s_2(a_1) \in A_2 \,|\, a_1 \in A_1\}$, namely the cardinality of the range of s_2. Then note that $\text{comp}_m(M_2) = \sum_{q_2 \in Q_2} c(\lambda_2(q_2))$ for player 2's automaton $M_2 \in \mathcal{M}_2^\Gamma$.

The multiple complexity considers a "complexity" of outputs as well as the counting-states complexity.[3] In other words, $c(s_2)$ is considered to be a measure of a complexity of s_2. The intuition is that when player 2 has more opportunities to change her actions according to player 1's actions, her automaton is considered to be more complex. The simplest output of player

[3]The multiple complexity does not consider complexity of transition functions. When the stage game is in a strategic form, Banks and Sundaram (1990) showed that a stage-game Nash equilibrium is played at every period on any equilibrium of the repeated game with a measure of complexity which considers transition functions.

2's automata is a stage-game strategy which always takes the same action. In Γ^∞, we will assume the preference relation in Definition 1 with respect to the M-complexity instead of the CS-complexity.

Suppose that $M_2 = (Q_2, q_2^1, \lambda_2, \mu_2) \in \mathcal{M}_2^\Gamma$ satisfies $c(\lambda_2(q_2)) = 1$ for all $q_2 \in Q_2$. Then, at every state, player 2 chooses an action independent of an action of player 1. This may be interpreted that player 2 moves without observing player 1's output. Thus player 2 moves as if she played the corresponding simultaneous-move game G. In other words, an automaton in the simultaneous-move game is regarded as an automaton in the sequential-move game. More formally, for a given automaton $M_2 = (Q_2, q_2^1, \lambda_2, \mu_2) \in \mathcal{M}_2$, define an output function $\tilde{\lambda}_2 : Q_2 \to \{s_2 : A_1 \to A_2\}$ to be $\big(\tilde{\lambda}_2(q_2)\big)(a_1) = \lambda_2(q_2)$ for every $a_1 \in A_1$. Then the mapping $(Q_2, q_2^1, \lambda_2, \mu_2) \mapsto (Q_2, q_2^1, \tilde{\lambda}_2, \mu_2)$ is an injection from $\mathcal{M}_2$ to $\mathcal{M}_2^\Gamma$. In this way, $\mathcal{M}_2$ is regarded as a subset of $\mathcal{M}_2^\Gamma$.

3 Results

Lemma 1. *Every automaton $M_2 = (Q_2, q_2^1, \lambda_2, \mu_2) \in \mathcal{M}_2^\Gamma$ for player 2 satisfies the following properties:*

1. $\mathrm{comp}_m(M_2) \geq \mathrm{comp}_{cs}(M_2)$,
2. $\mathrm{comp}_m(M_2) = \mathrm{comp}_{cs}(M_2)$ *if and only if* $c(\lambda_2(q_2)) = 1$ *for all* $q_2 \in Q_2$.

Proof. Both properties are obvious since $c(\lambda_2(q_2)) \geq 1$. □

For a pair of automata $(M_1, M_2) \in \mathcal{M}^\Gamma$, let $a^t = (a_1^t, a_2^t) \in A_1 \times A_2$ and $q^t = (q_1^t, q_2^t) \in Q_1 \times Q_2$ be the pair of outputs and states, respectively, at period t induced by (M_1, M_2).

Lemma 2. *For $(M_1, M_2) \in \mathcal{M}^\Gamma$, suppose that $a_1^t = a_1^{t'}$ implies $q_1^t = q_1^{t'}$ for all periods t and t'. Then, there exists an automaton $\bar{M}_2 \in \mathcal{M}_2$ for player 2 such that $\mathrm{comp}_m(M_2) = \mathrm{comp}_{cs}(\bar{M}_2)$ and (M_1, M_2) and $(M_1, \bar{M}_2)$ generate the same action profiles.*

Proof. Let $M_1 = (Q_1, q_1^1, \lambda_1, \mu_1) \in \mathcal{M}_1$ and $M_2 = (Q_2, q_2^1, \lambda_2, \mu_2) \in \mathcal{M}_2^\Gamma$. Define $\bar{M}_2 = (\bar{Q}_2, \bar{q}_2^1, \bar{\lambda}_2, \bar{\mu}_2) \in \mathcal{M}_2$ as follows:

$$\bar{Q}_2 = \bigcup_{q_2 \in Q_2} \{(q_2, a_2) \in Q_2 \times A_2 \mid a_2 = \lambda_2(q_2)(a_1) \text{ for some } a_1 \in A_1\},$$
$$\bar{q}_2^1 = (q_2^1, a_2^1),$$
$$\bar{\lambda}_2(q_2, a_2) = a_2,$$

$$\bar{\mu}_2((q_2, a_2), a) = \begin{cases} (q_2^{t'+1}, a_2^{t'+1}) & \text{if } ((q_2, a_2), a) = ((q_2^t, a_2^t), a^t) \text{ for some } t \\ & \text{ and there is } t' \leq t-1 \\ & \quad \text{with } ((q_2^t, a_2^t), a^t) = ((q_2^{t'}, a_2^{t'}), a^{t'}), \\ (q_2^{t+1}, a_2^{t+1}) & \text{if } ((q_2, a_2), a) = ((q_2^t, a_2^t), a^t) \text{ for some } t \\ & \text{ and the above does not hold,} \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

$\mathrm{comp}_m(M_2) = \mathrm{comp}_{cs}(\bar{M}_2)$ is easily verified from the definition of $\bar{Q}_2$. Let $(\bar{a}_1^t, \bar{a}_2^t)$ be the pair of actions at period t induced by $(M_1, \bar{M}_2)$. We will prove $(\bar{a}_1^t, \bar{a}_2^t) = (a_1^t, a_2^t)$ for all t by induction. For $t = 1$, $\bar{a}_1^1 = a_1^1$ and $\bar{a}_2^1 = \bar{\lambda}_2(q_2^1, a_2^1) = a_2^1$. Next fix t and assume that $(\bar{a}_1^k, \bar{a}_2^k) = (a_1^k, a_2^k)$ for all $k = 1, \ldots, t$. $\bar{a}_1^{t+1} = a_1^{t+1}$ directly follows from the assumption. If there does not exist $1 \leq t' \leq t-1$ such that $((q_2^t, a_2^t), a^t) = ((q_2^{t'}, a_2^{t'}), a^{t'})$, then

$$\begin{aligned} \bar{a}_2^{t+1} &= \bar{\lambda}_2(\bar{\mu}_2((q_2^t, a_2^t), a^t)) \\ &= \bar{\lambda}_2(q_2^{t+1}, a_2^{t+1}) \\ &= a_2^{t+1}. \end{aligned}$$

If there exists $1 \leq t' \leq t-1$ such that $((q_2^t, a_2^t), a^t) = ((q_2^{t'}, a_2^{t'}), a^{t'})$, then the assumption of lemma implies $q_1^t = q_1^{t'}$. Therefore $q_1^{t+1} = q_1^{t'+1}$ and $a_1^{t+1} = a_1^{t'+1}$. This yields

$$\begin{aligned} a_2^{t+1} &= \lambda_2(\mu_2(q_2^t, a^t))(a_1^{t+1}) \\ &= \lambda_2(\mu_2(q_2^{t'}, a^{t'}))(a_1^{t'+1}) \\ &= a_2^{t'+1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \bar{a}_2^{t+1} &= \bar{\lambda}_2(\bar{\mu}_2((q_2^t, a_2^t), a^t)) \\ &= \bar{\lambda}_2(q_2^{t'+1}, a_2^{t'+1}) \\ &= a_2^{t'+1}. \end{aligned}$$

Thus $\bar{a}_2^{t+1} = a_2^{t+1}$ is proved. (M_1, M_2) and $(M_1, \bar{M}_2)$ generate the same action profiles. □

In the following Lemma we use an analogous argument to the lemma in Piccione and Rubinstein (1993).

Lemma 3. *Suppose that $(M_1, M_2) \in \mathcal{M}^\Gamma$ is an equilibrium in Γ^∞ under the M-complexity. Then*

$$\text{comp}_{cs}(M_1) = \text{comp}_{cs}(M_2) = \text{comp}_m(M_2).$$

Particularly, the second equality implies that player 2 moves without observing player 1's action at any state of M_2.

Proof. Let $M_1 = (Q_1, q_1^1, \lambda_1, \mu_1)$ and $M_2 = (Q_2, q_2^1, \lambda_2, \mu_2)$. For given M_1, consider player 2's payoff maximization problem of the repeated game. This Markovian decision problem has a stationary solution $\sigma_2 : Q_1 \to A_2$. Then, define player 2's automaton $M_2' = (Q_1, q_1^1, \lambda_2', \mu_2')$ with $\lambda_2'(q_1)(\cdot) = \sigma_2(q_1)$ and $\mu_2'(q_1, \cdot) = \mu_1(q_1, (\lambda_1(q_1), \sigma_2(q_1)))$. Since $c(\lambda_2'(q_1)) = 1$ for any $q_1 \in Q_1$, $\text{comp}_m(M_2')$ is equal to $\text{comp}_{cs}(M_2')$, and to $\text{comp}_{cs}(M_1)$ by the definition of M_2'. Since M_2 is a best reply to M_1, $(M_1, M_2) \succsim_2 (M_1, M_2')$. On the other hand, by the definition of M_2', $\pi_2(M_1, M_2) \leq \pi_2(M_1, M_2')$. Therefore by Definition 1, it must hold that $\text{comp}_m(M_2) \leq \text{comp}_m(M_2')$. Hence, $\text{comp}_m(M_2) \leq \text{comp}_{cs}(M_1)$.

Considering player 1's Markovian decision problem for given M_2 shows $\text{comp}_{cs}(M_1) \leq \text{comp}_{cs}(M_2)$. By combining all the inequalities above and in Lemma 1, the lemma is proved. □

Recall that we regard $\mathcal{M}_2$ as a subset of $\mathcal{M}_2^\Gamma$. When $(M_1, M_2) \in \mathcal{M}^\Gamma$ is an equilibrium in Γ^∞ under the multiple complexity, this lemma implies that $M_2 \in \mathcal{M}_2$. Therefore (M_1, M_2) is an equilibrium also in G^∞. By this fact, Proposition 1 can be applied to Γ^∞, and the following result is obtained.

Lemma 4. *Suppose that $(M_1, M_2) \in \mathcal{M}^\Gamma$ is an equilibrium in Γ^∞ under the multiple complexity. Let $(a_1^t, a_2^t) \in A_1 \times A_2$ be the pair of outputs at period t, and $(q_1^t, q_2^t) \in Q_1 \times Q_2$ be the pair of states at period t. At any period t, t',*

1. *$q_1^t = q_1^{t'}$ if and only if $q_2^t = q_2^{t'}$,*
2. *$a_1^t = a_1^{t'}$ if and only if $a_2^t = a_2^{t'}$.*

From Lemma 4, we now obtain a necessary and sufficient condition of Nash equilibria in the machine game of Γ^∞.

Theorem 1. *For a pair of automata* $(M_1, M_2) \in \mathcal{M}_1^\Gamma \times \mathcal{M}_2^\Gamma$*, let* $(a_1^t, a_2^t) \in A_1 \times A_2$ *and* $(q_1^t, q_2^t) \in Q_1 \times Q_2$ *be the pair of outputs and states, respectively, at period* t *induced by* (M_1, M_2)*.* (M_1, M_2) *is an equilibrium in* Γ^∞ *under the multiple complexity if and only if* $M_2 \in \mathcal{M}_2$*,* (M_1, M_2) *is an equilibrium in* G^∞*, and furthermore* $a_i^t = a_i^{t'}$ *implies* $q_i^t = q_i^{t'}$ *for all* t *and* t' $(i = 1, 2)$*.*

Proof. First suppose that (M_1, M_2) is an equilibrium in Γ^∞ under the multiple complexity. By Lemma 3, (M_1, M_2) is an equilibrium in G^∞. Let $M_2 = (Q_2, q_2^1, \lambda_2, \mu_2) \in \mathcal{M}_2$. Note that $\text{comp}_m(M_2) = \#\{q_2^t \mid t = 1, 2, \dots\}$. Define an automaton $M_2' = (Q_2', q_2', \lambda_2', \mu_2') \in \mathcal{M}_2^\Gamma$ such that $\text{comp}_{cs}(M_2') = 1$ as follows:

- $Q_2' = \{q_2'\}$,
- $\lambda_2'(q_2')(a_1^t) = a_2^t$,
- $\mu_2'(q_2', \cdot) = q_2'$.

Note that this definition is well-defined by the second condition of Lemma 4, and that (M_1, M_2') and (M_1, M_2) generate the same action profiles. Suppose that $a_2^t = a_2^{t'}$ but $q_2^t \neq q_2^{t'}$ at some t, t'. Then $\#\{a_2^t\} < \#\{q_2^t\}$. By the definition of the multiple complexity, $\text{comp}_m(M_2') = \#\{a_2^t\}$. Therefore $\text{comp}_m(M_2') < \text{comp}_m(M_2)$, which implies that M_2 is not a best reply to M_1. By contradiction, if $a_2^t = a_2^{t'}$ then $q_2^t = q_2^{t'}$. With this fact, Lemma 4 implies that if $a_1^t = a_1^{t'}$ then $q_1^t = q_1^{t'}$.

Second suppose that (M_1, M_2) is an equilibrium in G^∞, and that $a_i^t = a_i^{t'}$ implies $q_i^t = q_i^{t'}$ at any t, t'. Since all the states of M_1 is used on the equilibrium path of (M_1, M_2), the second supposition means that the automaton M_1 has the property that different states output different actions. Therefore for any $M_2' \in \mathcal{M}_2^\Gamma$, the condition in Lemma 2 holds true with respect to (M_1, M_2'). Assume that there is $M_2' \in \mathcal{M}_2^\Gamma$ such that $(M_1, M_2') \succ_2 (M_1, M_2)$. Then there is $\bar{M}_2' \in \mathcal{M}_2$ as in Lemma 2 such that $(M_1, \bar{M}_2') \succ_2 (M_1, M_2)$, contradicting the assumption that (M_1, M_2) is an equilibrium in G^∞. Hence (M_1, M_2) is an equilibrium in Γ^∞ under M-complexity. □

Corollary 1. *If* (M_1, M_2) *is an equilibrium in* Γ^∞ *under the M-complexity. Then*

$$\text{comp}_{cs}(M_1) = \text{comp}_{cs}(M_2) \leq \min(\#A_1, \#A_2).$$

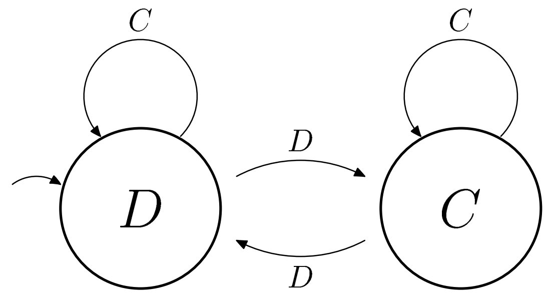

Figure 2: The Tat-for-Tit automaton

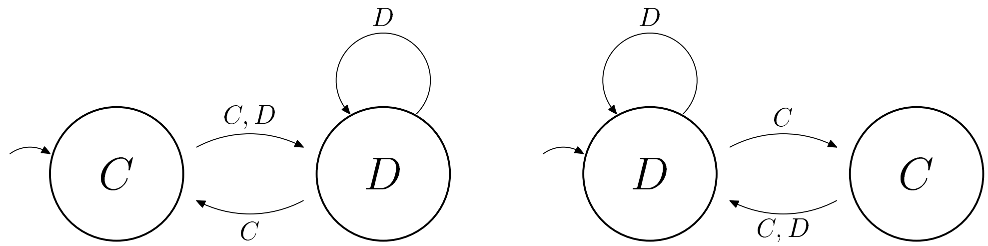

Figure 3: An automaton pair which induces the repetition of (C, D) and (D, C)

Proof. Because (M_1, M_2) is an equilibrium in Γ^∞, $\text{comp}_{cs}(M_1) = \text{comp}_{cs}(M_2)$ by Lemma 3. By Theorem 1, $\text{comp}_{cs}(M_i) \leq \#A_i$ for $i = 1, 2$. $\square$

Example 1. Let Γ be a sequential-move prisoners' dilemma described by Figure 1. Assume that the preferences are lexicographic, and that the discount factor δ is sufficiently close to one. By Corollary 1, automata with at most two states must appear in an equilibrium in Γ^∞ under the M-complexity, because there are only two possible actions C, D in the game. When the number of state is one, it is obvious that an equilibrium must be such that both players infinitely repeat D.

Suppose that the number of states is two. By Lemma 4, the actions and states have to be cyclic with period 2 or cyclic with period 1 after the second period. First consider the case of period 1. Since we have already considered the repetition of D, consider the repetition of C. One example is when both players implement a well-known automaton called Tat-for-Tit

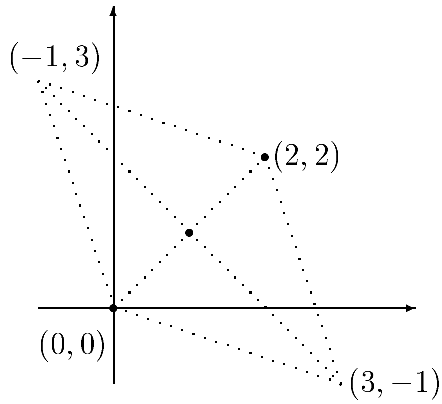

Figure 4: The equilibrium payoffs in the sequential-move prisoners' dilemma

shown in Figure 2.[4] It can be an equilibrium in G^∞.[5] By Theorem 1, it is also an equilibrium of Γ^∞. In this equilibrium (C, C) is played repeatedly after period 2. When the CS-complexity is adopted, player 2 can deviate to an automaton with only one state which outputs the stage-game strategy describing C, D when player 1 chooses C, D, respectively. Under the M-complexity, the complexity of this automaton is 2. Hence player 2 has no incentive of deviation, and the equilibrium survives.

Second consider the case of period 2. By Theorem 1, the equilibrium in Γ^∞ is regarded as that in G^∞. Therefore without loss of generality, we assume that player 1 plays C at the first period. Then the action profiles are $\{(C, C), (D, D), (C, C), (D, D), \ldots\}$ or $\{(C, D), (D, C), (C, D), (D, C), \ldots\}$. In the former case, either player deviates to an automaton which always outputs D, gaining more payoffs. The latter case is realized by the pair of automata shown in Figure 3.

Therefore the set of equilibrium payoffs are described in Figure 4 when the discount factor δ is almost equal to one.

Example 2. Next consider the battle of the sexes game G with a payoff matrix shown in Table 2. The pure-strategy Nash equilibria of this game are

[4]The terminology "Tat-for-Tit" is adopted from Binmore and Samuelson (1992).

[5]See Abreu and Rubinstein (1988) for the proof of sufficiency of equilibria in the simultaneous-move games.

	B	O
B	$3,1$	$0,0$
O	$0,0$	$1,3$

Table 2: The battle of the sexes

(B,B) and (O,O). In the equilibria in G^∞, if (B,O) or (O,B) is played at some period, then only $\{(B,O),(O,B)\}$ appears on the path. Therefore the player can deviate to an automaton always playing the same action, in which she gains strictly positive payoff. Thus either (B,B) or (O,O) is played in equilibria.

Let Γ be the sequential-move game corresponding to G. By Theorem 1, automata with one or two states must be used in the equilibria in Γ^∞ under the M-complexity. When the number of state is one, the equilibrium outcome is repetition of the same action profile at every period. When the number of states is two, there is an equilibrium in which (B,B) and (O,O) are played alternately. These are all the outcomes on the equilibria.

4 Discussion

If we introduce the multiple complexity in the machine game of a repeated sequential-move game, there exists a Nash equilibrium of the machine game which has more than one state. This result is different from that of Piccione and Rubinstein (1993) in which the counting-states complexity is considered. An intuition for this result is that player 2 can reduce the number of states of her automaton by employing a stage-game strategy depending on player 1's actions, keeping the multiple complexity unchanged.

In this paper, we have formulated the output function of an automaton as a mapping from the set of states to the set of stage-game strategies, based on the reduction of an extensive game to its normal from. Let us discuss how the multiple complexity is related to an alternative formulation in an automaton of an extensive game introduced in Piccione and Rubinstein (1993).

Let Γ be a two-player extensive game with perfect recall. Let U_i be the set of player i's information sets in Γ, $A(u_i)$ be the set of actions available at $u_i \in U_i$, and $A(U_i) = \bigcup_{u_i \in U_i} A(u_i)$. Let E be the set of end-nodes in Γ. A player i's automata in Γ^∞ is defined by $(Q_i, q_i^1, \tilde{\lambda}_i, \tilde{\mu}_i)$ with output function

$\tilde{\lambda}_i : Q_i \times U_i \to A(U_i)$ and transition function $\tilde{\mu}_i : Q_i \times (U_i \cup E) \to Q_i$. Transition occurs at the time an information set is reached and before the action is taken.

When Γ is a sequential-move game with perfect information, U_1 consists of a single information set and U_2 can be identified with A_1. Moreover, $A(U_2)$ is identified with A_2. Thus we can have $\tilde{\lambda}_1 : Q_1 \to A_1$, $\tilde{\lambda}_2 : Q_2 \times A_1 \to A_2$ in the above definition. For an output function $\tilde{\lambda}_2$ and $S_2 = \{A_1 \to A_2\}$, let us define $\lambda_2 : Q_2 \to S_2$ to be $\lambda_2(q_2)(a_1) = \tilde{\lambda}_2(q_2, a_1)$. Then $\tilde{\lambda}_2$ and λ_2 assign the actions for a pair (q_2, a_1). In this formulation, the multiple complexity is given by

$$\text{comp}_m(M_2) = \sum_{q_2 \in Q_2} \#\{\tilde{\lambda}_2(q_2, a_1) \in A_2 \,|\, a_1 \in A_1\}.$$

In this representation, the multiple complexity is complementary to the notion of the response complexity introduced by Chatterjee and Sabourian (2000). While their response complexity counts the cardinality of the range of the mapping $\tilde{\lambda}_2(\cdot, a_1)$, our multiple complexity counts the cardinality of the range of the mapping $\tilde{\lambda}_2(q_2, \cdot)$.

Finally, it would be a challenging problem to characterize a Nash equilibrium of the machine game for a general extensive game. The identification above can be applied to a repeated game of an extensive stage game Γ with perfect recall, that is, $\lambda_i : Q_i \to S_i$ with $S_i = \{s_i : U_i \to A(U_i)\}$. Thus the definition of the M-complexity is possible in Γ^∞ if $\tilde{\lambda}_i$ is defined everywhere on $Q_i \times U_i$. However, when players move more than once in a stage game, one has to be careful in formulating transition functions. A further analysis is left to future research.

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