Discussion Paper # 2007-2

Uniform Continuity of the Value of Zero-Sum Games with Differential Information*

Ezra Einy[†], Ori Haimanko[‡], Diego Moreno[§], and Benyamin Shitovitz[¶]

Abstract

We establish uniform continuity of the value for zero-sum games with differential information, when the distance between changing information fields of each player is measured by the Boylan (1971) pseudo-metric. We also show that the optimal strategy correspondence is upper semi-continuous when the information fields of players change (even with the weak topology on players' strategy sets), and is approximately lower semi-continuous.

JEL Classification Number: C72.

Keywords: Zero-Sum Games, Differential Information, Value, Optimal Strategies, Uniform Continuity.

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*Revised version of Discussion Paper # 04-04 of Monaster Center for Economic Research, Ben-Gurion University of the Negev.

[†]Graduate School of Economics, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo 186-8601, Japan; and Department of Economics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel. E-mail: einy@bgu.ac.il

[‡]*Corresponding author.* Department of Economics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel. E-mail: orih@bgu.ac.il

[§]Departamento de Economia, Universidad Carlos III de Madrid, 28903 Getafe, Spain. E-mail: dmoreno@eco.uc3m.es

[¶]Department of Economics, University of Haifa, Haifa 31905, Israel. E-mail: binya@econ.haifa.ac.il

1 Introduction

Bayesian games, or games with differential information, describe situations in which there is uncertainty about players' payoffs, and different players have (typically) different private information about the realized state of nature ω that affects the payoffs. Private information of player i is often represented by a partition of the space Ω of all states of nature (in which case i knows to which element of the partition the realized ω belongs), or more generally, by a σ-field $\mathcal{F}^i$ of measurable sets (events) in Ω (in which case i knows, given any event in $\mathcal{F}^i$, whether it has occurred). It was shown by Simon (2003) that Bayesian Nash equilibrium (BNE) may fail to exist in games with differential information, as a result of discontinuity of the expected payoff function in Bayesian strategies of all players *simultaneously*. The situation changes, however, when attention is confined to two-person *zero-sum* games with differential information. Indeed, under quite general conditions, the expected payoff function is (weakly) continuous in Bayesian strategies of each player *separately*, and Sion (1958) minimax theorem needs only this form of continuity to guarantee existence of the value and of optimal strategies for each player.

This work concerns the behavior of the value of a zero-sum game with differential information when players' information endowments (fields) undergo small changes, and the distance between informations fields is measured by means of Boylan (1971) pseudo-metric. It turns out that the value has strong continuity properties. We show that, when the payoff function is Lipschitz-continuous in strategies at each state of nature,[1] a mild integrability assumption[2] on the state-dependent Lipschitz constant guarantees that the value is a *uniformly* continuous function of players' information fields (see Theorem 1). If, in addition, the state-dependent Lipschitz constant of the payoff function is bounded, then the value is in fact Lipschitz-continuous in information fields (see Corollary 1). Moreover, the correspondence describing players' optimal strategies as a function of information is upper semi-continuous, even with respect to the weak convergence topology on each player's set of strategies, and is approximately lower semi-continuous (see Theorem 3).

[1] This requirement is satisfied, for instance, by games which have a matrix-game form in all states of nature (see the Example).

[2] The assumption is that the state-dependent Lipschitz constant is *q-integrable* for $q > 1$. When this constant is merely integrable, the value is still continuous (see Theorem 2), but possibly not uniformly.

These continuity properties of the value (and optimal strategies) in zero-sum games stand in contrast to discontinuity of the BNE correspondence in general (non zero-sum) games with differential information. The NE correspondence is not lower semi-continuous – that is, BNE strategies/payoffs may not be approachable by BNE (or even ε-BNE) strategies/payoffs in games with slightly modified information endowments. This was shown by Monderer and Samet (1996)[3] in a setting similar to ours. The BNE are also not upper semi-continuous as was shown by Milgrom and Weber (1985) and Cotter (1991).

The continuity of the BNE correspondence has been investigated in two different set-ups. In this paper, we use the basic set-up of Monderer and Samet (1996), who work with information fields to describe players' varying private information, with fixed common prior belief about the distribution of the states of nature. (This follows a certain tradition of modelling information in economic theory; see, e.g., Allen (1983), Cotter (1986), Stinchcombe (1990), and Van Zandt (1993, 2002)). In other words, the underlying uncertainty in the game (represented by the common prior) is fixed, but information endowments of players (represented by information fields) are variable. However, there is a different approach to continuity of NE correspondences, which is with respect to the common prior belief (see, e.g., Milgrom and Weber (1985), Kajii and Morris (1998)). In this approach, contrary to ours, the underlying uncertainty (the common prior) is variable, but information endowments are fixed (the space of states of nature is assumed to be the cross product of *fixed* sets of players' types, and each player's private information is given by the knowledge of his type). Perturbing the underlying uncertainty influences the expected payoffs of *all* players, but does not affect their strategy sets. However, our setting emphasizes *differences* in information, allowing information endowments in the game to be perturbed in a way that directly affects only one individual player, or in a way that affects all players differently. Indeed, a change in the private information of both players induces (typically different) changes in players' strategy sets, due to the constraint of measurability of each player's strategies with respect to his information field. While the impact of these information changes on the structure of the game might appear to be significant, our theorems show that

[3] In fact, Monderer and Samet (1996) (as well as Kajii and Morris (1998) in a fixed-types model of differential information), are concerned precisely with the question of what topology on information endowments would lead to approximate lower semi-continuity of BNE. It must be significantly weaker than the Boylan topology.

the value and the optimal strategies in zero-sum games are nevertheless well behaved with respect to these changes.

In non-zero-sum games, upper semi-continuity of BNE is obtained at the cost of imposing certain restrictions on information structure in the game. Indeed, in the set-up of types, a sufficient *spread* of the common prior distribution on the product of players' types is needed for upper semi-continuity of BNE (see Milgrom and Weber (1985); the common prior is required to be absolutely continuous with respect to the product of its marginal distributions). And in the set-up of information fields, an analogous condition in Cotter (1991) also yields upper semi-continuity with respect to Boylan topology on information endowments, but only under assumption that all fields are generated by at most countable partitions of the space of states of nature. Our results show, however, that for the continuity of the value or upper semi-continuity of optimal strategies in zero-sum games no restrictions on information fields are necessary.

Our paper is organized as follows. The set-up is described in Section 2. Our results (Theorems 1, 2, 3 and Corollaries 1, 2) are stated and proved in Section 3; Remarks 1 and 2 appear at the end of this section. The Appendix contains the proof of a technical Lemma 2.

2 Preliminaries

We consider zero-sum games with two players, $i = 1, 2$. Games are played in an uncertain environment, which affects payoff functions of the players. The underlying uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is a set of states of nature, $\mathcal{F}$ is a countably generated σ-field of subsets of Ω, and μ is a countably additive probability measure on $(\Omega, \mathcal{F})$, which represents the *common prior belief* of the players about the distribution of the realized state of nature. The initial *information endowment* of player i is given by a σ-subfield $\mathcal{F}^i$ of $\mathcal{F}$.

Each player $i = 1, 2$ has a set S^i of *strategies*, which is a convex and compact subset of a Euclidean space R^{n_i}. We will assume, without loss of generality, that $\max_{s \in S^1 \cup S^2} \|s\| \leq 1$, where $\|\cdot\|$ stands for the Euclidean norm in R^{n_1} or R^{n_2}. One simple example of such strategy set S^i, to which we return later, is the $(n_i - 1)$-dimensional simplex of i's mixed strategies, provided player i has n_i pure strategies.

There is, in addition, a measurable[4] real valued *payoff function* $u : \Omega \times S^1 \times S^2 \to R$, such that $u\left(\cdot, s^1, s^2\right)$ is integrable for every $\left(s^1, s^2\right) \in S^1 \times S^2$. At every state of nature $\omega \in \Omega$, $u_\omega\left(s^1, s^2\right) \equiv u\left(\omega, s^1, s^2\right)$ represents the payoff received by player 1 (and the loss incurred by player 2) when each player i chooses to play s^i. We assume that each u_ω is a Lipschitz function with constant $K(\omega)$, that is,

$$\left|u_\omega\left(s^1, s^2\right) - u_\omega\left(t^1, t^2\right)\right| \leq K(\omega)(\left\|s^1 - t^1\right\| + \left\|s^2 - t^2\right\|) \tag{1}$$

for every $\left(s^1, s^2\right), \left(t^1, t^2\right) \in S^1 \times S^2$. We also assume that the state-dependent Lipschitz constant $K(\cdot)$ is $\mathcal{F}$-measurable, and that there exists $q \geq 1$ such that it is q-integrable (and, in particular, integrable):

$$\int_\Omega (K(\omega))^q \, d\mu(\omega) < \infty. \tag{2}$$

The probability space $(\Omega, \mathcal{F}, \mu)$, information endowments $\mathcal{F}^1$ and $\mathcal{F}^2$, strategy sets S^1 and S^2, and the payoff function u fully describe a *zero-sum Bayesian game.* To concentrate on the effects of changes in information endowments, we keep all the attributes of the game fixed, with the exception of $\mathcal{F}^1$ and $\mathcal{F}^2$ that are variable. Thus, we denote the game by $G(\mathcal{F}^1, \mathcal{F}^2)$, to emphasize its changeable characteristics.

A *Bayesian strategy* of player i is an $\mathcal{F}^i$-measurable function $x^i : \Omega \to S^i$. The set of all Bayesian strategies of player i will be denoted by $X^i\left(\mathcal{F}^i\right)$.

For $1 \leq p \leq \infty$, $L_p^n(\Omega, \mathcal{F}, \mu)$ denotes the Banach space of all $\mathcal{F}$-measurable functions[5] $x : \Omega \to R^n$ such that

$$\|x\|_p \equiv \left(\int_\Omega \|x(\omega)\|^p \, d\mu(\omega)\right)^{\frac{1}{p}} < \infty \tag{3}$$

(recall that $\|\cdot\|$ stands for the Euclidean norm on R^n) if $p < \infty$, and $\|x\|_\infty \equiv$ essential supremum of $\|x(\cdot)\| < \infty$ if $p = \infty$. For $1 < p < \infty$, the weak topology on $L_p^n(\Omega, \mathcal{F}, \mu)$ is the (weakest) one in which the linear functional[6] $\varphi_y(x) \equiv \int_\Omega x(\omega) \cdot y(\omega) \, d\mu(\omega)$ is continuous for any given $y \in L_q^n(\Omega, \mathcal{F}, \mu)$,

[4] The measurability is in all coordinates jointly (with respect to the Borel σ-fields in the second and third coordinates).

[5] Or, to be precise, their equivalence classes, where any two functions which are equal μ-almost everywhere are identified. This identification applies to Bayesian strategies as well.

[6] The dot stands for the inner product in R^n.

$q = \frac{p}{p-1}$. Note that $X^i(\mathcal{F})$ is a weakly closed subset of the unit ball in $L_p^{n_i}(\Omega, \mathcal{F}, \mu)$ (which is metrizable in the weak topology, since its dual $L_q^{n_i}(\Omega, \mathcal{F}, \mu)$ is separable due to our assumption on $\mathcal{F}$; the unit ball is also compact in the weak topology). The topology that this induces on $X^i(\mathcal{F})$ does not depend on $p > 1$. This follows easily from the fact that $\{x_k\}_{k=1}^{\infty} \subset X^i(\mathcal{F})$ converges to x in the $L_p^{n_i}(\Omega, \mathcal{F}, \mu)$-weak topology if and only if $\lim_{k\to\infty} \varphi_y(x_k) = \varphi_y(x)$ holds for all *bounded* $\mathcal{F}$-measurable functions y (which is in turn implied by the uniform boundedness of $\{x_k\}_{k=1}^{\infty}$ and x as functions in $X^i(\mathcal{F})$)[7].

In the sequel, this induced topology will be called the *weak topology* on $X^i(\mathcal{F})$. The weakly closed subset $X^i(\mathcal{F}^i)$ of $X^i(\mathcal{F})$ thus also becomes metrizable and compact.

The expected payoff of player 1 (and the expected loss of player 2) when $x^i \in X^i(\mathcal{F}^i)$ is chosen by i is

$$U(x^1, x^2) \equiv E\left(u.\left(x^1(\cdot), x^2(\cdot)\right)\right) = \int_\Omega u_\omega\left(x^1(w), x^2(w)\right) d\mu(\omega)$$

(the integral is well defined, due to integrability of each $u(\cdot, s^1, s^2)$, assumption (1), and integrability of $K(\cdot)$). This also defines U for all $(x^1, x^2) \in X^1(\mathcal{F}) \times X^2(\mathcal{F})$.

If $\min_{x^2\in X^2(\mathcal{F}^2)} \max_{x^1\in X^1(\mathcal{F}^1)} U(x^1, x^2)$ and $\max_{x^1\in X^1(\mathcal{F}^1)} \min_{x^2\in X^2(\mathcal{F}^2)} U(x^1, x^2)$ are well defined, and

$$\min_{x^2\in X^2(\mathcal{F}^2)} \max_{x^1\in X^1(\mathcal{F}^1)} U(x^1, x^2) = \max_{x^1\in X^1(\mathcal{F}^1)} \min_{x^2\in X^2(\mathcal{F}^2)} U(x^1, x^2), \tag{4}$$

then the common value $v = v(\mathcal{F}^1, \mathcal{F}^2)$ of the two expressions in (4) is called the *value* of the zero-sum Bayesian game $G(\mathcal{F}^1, \mathcal{F}^2)$. Note that v is the value of $G(\mathcal{F}^1, \mathcal{F}^2)$ if and only if there exists a pair of Bayesian strategies $(x^1, x^2) \in X^1(\mathcal{F}^1) \times X^2(\mathcal{F}^2)$ such that for every $(y^1, y^2) \in X^1(\mathcal{F}^1) \times X^2(\mathcal{F}^2)$

$$U(x^1, y^2) \geq U(x^1, x^2) = v(\mathcal{F}^1, \mathcal{F}^2) \geq U(y^1, x^2). \tag{5}$$

Strategy x^i is called *optimal* for player i.[8] Given $\varepsilon > 0$, strategies $(x^1, x^2) \in$

[7] In fact, this shows that the topology induced on $X^i(\mathcal{F})$ is the same as the weak topology on it, viewed as a subset of $L_1^{n_i}(\Omega, \mathcal{F}, \mu)$. We chose to describe the topology by using $p > 1$, however, in order to demonstrate metrizability and compactness of $X^i(\mathcal{F})$: only when $p > 1$ is the unit ball in $L_p^{n_i}(\Omega, \mathcal{F}, \mu)$ metrizable and compact in the weak topology, implying the same properties of its weakly closed subset $X^i(\mathcal{F})$.

[8] The definition of the value used here presupposes existence of optimal strategies, instead of merely requiring inf sup = sup inf in (4). However, conditions stated in the next paragraph will guarantee existence of the value in this more strict sense.

$X^1(\digamma^1) \times X^2(\digamma^2)$ are called *ε-optimal* in $G(\digamma^1, \digamma^2)$ if $U(x^1, y^2) + \varepsilon \geq v(\digamma^1, \digamma^2) \geq U(y^1, x^2) - \varepsilon$ for every $(y^1, y^2) \in X^1(\digamma^1) \times X^2(\digamma^2)$.

We shall assume that the state-dependent payoff function u_ω is concave in $s^1 \in S^1$ for a fixed $s^2 \in S^2$, and convex in $s^2 \in S^2$ for a fixed $s^1 \in S^1$. This will guarantee the existence of the value in $G(\digamma^1, \digamma^2)$:

Proposition. Under the above assumption,

(a) the expected payoff function U is weakly upper semi-continuous and concave in $x^1 \in X^1(\digamma)$ for a fixed $x^2 \in X^2(\digamma)$, and weakly lower semi-continuous and convex in $x^2 \in X^2(\digamma)$ for a fixed $x^1 \in X^1(\digamma)$;

and

(b) the game $G(\digamma^1, \digamma^2)$ possesses a value.

Proof. **(a)** Since $u_\omega(\cdot, s^2)$ is a continuous and concave function of s^1, and its maximum $\psi(\omega, s^2) \equiv \max_{s^1 \in S^1} u(\omega, s^1, s^2)$ is integrable in ω due to the integrability of the Lipschitz constant $K(\omega)$, Theorem 2.8 of Balder and Yannelis (1993) can be applied[9] to deduce weak upper semi-continuity of U in $x^1 \in X^1(\digamma)$. The concavity of U in $x^1 \in X^1(\digamma)$ is obvious. A mirror argument shows lower semi-continuity and convexity of U in $x^2 \in X^2(\digamma)$.

(b) Properties of U shown in (a) guarantee existence of the value in $G(\digamma^1, \digamma^2)$ by Sion (1958) minimax theorem, since $X^1(\digamma^1) \times X^2(\digamma^2)$ is weakly compact. ■

Example. The most prevalent payoff function that gives rise to such U comes from the usual matrix game. In a matrix game, each player i has n_i pure strategies, and S^i is the $(n_i - 1)$-dimensional simplex of i's mixed strategies. In each $\omega \in \Omega$, the payoff function is

$$u_\omega(s^1, s^2) = s^1 A(\omega) s^2, \tag{6}$$

where strategy $s^1 \in S^1$ is regarded as a row vector, $s^2 \in S^2$ – as a column vector, and $A(\omega)$ is an $n_1 \times n_2$-matrix, with $A(\omega)_{j,k}$ being the payoff of player

[9] Theorem 2.8 of Balder and Yannelis (1993) is a little too heavy for our purpose (it aims to show weaker upper semi-continuity of U by assuming u_ω to be only upper semi-continuous), but it is a convenient reference.

1 when he chooses pure strategy j and 2 – pure strategy k. Conditions (1) and (2) are guaranteed if $a(\omega) = \max_{j,k} |A_{j,k}(\omega)|$ is integrable. ■

Finally, we define convergence of players' information endowments by means of the following pseudo-metric (introduced in Boylan (1971)) on the family $\mathcal{F}^*$ of σ-subfields of $\mathcal{F}$:[10]

$$d(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A \in \mathcal{F}_1} \inf_{B \in \mathcal{F}_2} \mu(A \triangle B) + \sup_{B \in \mathcal{F}_2} \inf_{A \in \mathcal{F}_1} \mu(A \triangle B),$$

where $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the "symmetric difference" of A and B.

If $x^i \in X^i(\mathcal{F})$ and $\mathcal{F}' \in \mathcal{F}^*$, denote by $E(x^i \mid \mathcal{F}') \in X^i(\mathcal{F}')$ the conditional expectation of x^i with respect to the field $\mathcal{F}'$. The conditional expectation $E(x^i \mid \mathcal{F}')$ is well-behaved with respect to small changes in $\mathcal{F}'$:

Lemma 1. If $x^i \in X^i(\mathcal{F})$ and $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}^*$, then

$$\left\| E(x^i \mid \mathcal{F}_1) - E(x^i \mid \mathcal{F}_2) \right\|_1 \leq 16 n_i d(\mathcal{F}_1, \mathcal{F}_2). \tag{7}$$

Proof. If $n_i = 1$ (that is, if $S^i \subset [-1, 1]$), (7) is well known. (See, e.g., Rogge (1974) and Landers and Rogge (1986), who show that $\|E(f \mid \mathcal{F}_1) - E(f \mid \mathcal{F}_2)\|_1 \leq 8d(\mathcal{F}_1, \mathcal{F}_2)$ for all $\mathcal{F}$-measurable functions f with values in $[0, 1]$.) When $n_i > 1$,

$$\begin{aligned}
\left\| E(x^i \mid \mathcal{F}_1) - E(x^i \mid \mathcal{F}_2) \right\|_1 &= \int_\Omega \left\| E(x^i \mid \mathcal{F}_1) - E(x^i \mid \mathcal{F}_2) \right\| d\mu(\omega) \\
&\leq \int_\Omega \sum_{j=1}^{n_i} \left| E(x^i_j \mid \mathcal{F}_1) - E(x^i_j \mid \mathcal{F}_2) \right| d\mu(\omega) \\
&= \sum_{j=1}^{n_i} \left\| E(x^i_j \mid \mathcal{F}_1) - E(x^i_j \mid \mathcal{F}_2) \right\|_1 \\
&\leq 16 n_i d(\mathcal{F}_1, \mathcal{F}_2).
\end{aligned}$$

■

[10] If $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}^*$ contain the same sets of positive measure, then $d(\mathcal{F}_1, \mathcal{F}_2) = 0$. For this reason d is indeed a pseudo-metric rather than a metric. It would have become a metric if we passed to work with equivalence sets of σ-subfields, dropping the distinction between any such $\mathcal{F}_1, \mathcal{F}_2$.

3 Results

Given two pairs of fields in $\mathcal{F}^*$, $(\mathcal{F}_1^1, \mathcal{F}_1^2)$ and $(\mathcal{F}_2^1, \mathcal{F}_2^2)$ (where $\mathcal{F}_j^i$ is the information endowment of player $i = 1, 2$ in pair $j = 1, 2$), the distance between them will be measured by the following pseudo-metric:

$$\overline{d}\left((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2)\right) \equiv \max[d(\mathcal{F}_1^1, \mathcal{F}_2^1), d(\mathcal{F}_1^2, \mathcal{F}_2^2)].$$

Theorem 1. If the state-dependent Lipschitz constant $K(\cdot)$ of the payoff function is q-integrable for some $q > 1$, the value $v(\mathcal{F}^1, \mathcal{F}^2)$ is a uniformly continuous (in fact, Hölder-continuous) function of $(\mathcal{F}^1, \mathcal{F}^2) \in \mathcal{F}^* \times \mathcal{F}^*$ with respect to the pseudo-metric $\overline{d}$. Specifically, for any two $(\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2) \in \mathcal{F}^* \times \mathcal{F}^*$,

$$\left|v(\mathcal{F}_1^1, \mathcal{F}_1^2) - v(\mathcal{F}_2^1, \mathcal{F}_2^2)\right| \leq C\left[\,\overline{d}\left((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2)\right)\right]^{\frac{q-1}{q}}, \tag{8}$$

where $C > 0$ is a constant given by

$$C \equiv 4\left(8 \max\left(n_1, n_2\right)\right)^{\frac{q-1}{q}} \|K\|_q . \tag{9}$$

Proof. For any two given $(\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2) \in \mathcal{F}^* \times \mathcal{F}^*$, let $x^1 \in X^1(\mathcal{F}_1^1)$ be an optimal strategy of player 1 in the game $G(\mathcal{F}_1^1, \mathcal{F}_1^2)$, and pick $y^2 \in X^2(\mathcal{F}_2^2)$. Now denote $x_2^1 \equiv E(x^1 \mid \mathcal{F}_2^1) \in X^1(\mathcal{F}_2^1)$ and $y_1^2 \equiv E(y^2 \mid \mathcal{F}_1^2) \in X^2(\mathcal{F}_1^2)$. The optimality of x^1 in $G(\mathcal{F}_1^1, \mathcal{F}_1^2)$ implies

$$U(x^1, y_1^2) \geq v(\mathcal{F}_1^1, \mathcal{F}_1^2). \tag{10}$$

Note that

$$\left|U(x^1, y_1^2) - U\left(x_2^1, y^2\right)\right|$$

(by (1))

$$\leq \int_\Omega K(\omega)\left\|x^1(\omega) - x_2^1(\omega)\right\| d\mu(\omega) + \int_\Omega K(\omega)\left\|y_1^2(\omega) - y^2(\omega)\right\| d\mu(\omega)$$

(by the Hölder inequality, for $p = \frac{q}{q-1}$)

$$\leq \|K\|_q \left(\left\| x^1 - x_2^1 \right\|_p + \left\| y_1^2 - y^2 \right\|_p \right)$$

(since $\left\| x^1(\omega) - x_2^1(\omega) \right\|, \left\| y_1^2(\omega) - y^2(\omega) \right\| \leq 2$ for μ-almost every $\omega \in \Omega$)

$$\leq 2^{\frac{p-1}{p}} \|K\|_q \left(\left(\int_\Omega \left\| x^1(\omega) - x_2^1(\omega) \right\| d\mu(\omega) \right)^{\frac{1}{p}} + \left(\int_\Omega \left\| y_1^2(\omega) - y^2(\omega) \right\| d\mu(\omega) \right)^{\frac{1}{p}} \right)$$

$$= 2^{\frac{p-1}{p}} \|K\|_q \left(\left\| x^1 - x_2^1 \right\|_1^{\frac{1}{p}} + \left\| y_1^2 - y^2 \right\|_1^{\frac{1}{p}} \right)$$

$$= 2^{\frac{p-1}{p}} \|K\|_q \left(\left\| E(x^1 \mid \mathcal{F}_1^1) - E(x^1 \mid \mathcal{F}_2^1) \right\|_1^{\frac{1}{p}} + \left\| E(y^2 \mid \mathcal{F}_1^2) - E(y^2 \mid \mathcal{F}_2^2) \right\|_1^{\frac{1}{p}} \right)$$

(by (7) in Lemma 1)

$$\leq 2^{\frac{p-1}{p}} \left(16 \max(n_1, n_2)\right)^{\frac{1}{p}} \|K\|_q \left(\left[d(\mathcal{F}_1^1, \mathcal{F}_2^1) \right]^{\frac{1}{p}} + \left[d(\mathcal{F}_1^2, \mathcal{F}_2^2) \right]^{\frac{1}{p}} \right).$$

$$\leq 4 \left(8 \max(n_1, n_2)\right)^{\frac{1}{p}} \|K\|_q \left[\, \overline{d} \left((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2) \right) \right]^{\frac{1}{p}}.$$

To summarize, we have shown that

$$\left| U(x^1, y_1^2) - U\left(x_2^1, y^2\right) \right| \leq C \left[\, \overline{d} \left((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2) \right) \right]^{\frac{q-1}{q}}. \tag{11}$$

Together with (10), (11) implies that

$$U\left(x_2^1, y^2\right) \geq v(\mathcal{F}_1^1, \mathcal{F}_1^2) - C \left[\, \overline{d} \left((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2) \right) \right]^{\frac{q-1}{q}}.$$

This holds for every $y^2 \in X^2(\mathcal{F}_2^2)$, and hence it follows that

$$v(\mathcal{F}_2^1, \mathcal{F}_2^2) = \max_{y^1 \in X^1\left(\mathcal{F}_2^1\right)} \min_{y^2 \in X^2\left(\mathcal{F}_2^2\right)} U(y^1, y^2) \tag{12}$$

$$\geq \min_{y^2 \in X^2\left(\mathcal{F}_2^2\right)} U(x_2^1, y^2) \geq v(\mathcal{F}_1^1, \mathcal{F}_1^2) - C \left[\, \overline{d} \left((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2) \right) \right]^{\frac{q-1}{q}}. \tag{13}$$

Using similar arguments (when we start from an optimal strategy $x^2 \in X^2(\mathcal{F}_1^2)$ of player 2 in the game $G(\mathcal{F}_1^1, \mathcal{F}_1^2)$) we can show that, for $x_2^2 = E(x^2 \mid \mathcal{F}_2^2) \in X^2(\mathcal{F}_2^2)$, the following inequality

$$U\left(y^1, x_2^2\right) \leq v(\mathcal{F}_1^1, \mathcal{F}_1^2) + C \left[\, \overline{d} \left((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2) \right) \right]^{\frac{q-1}{q}}$$

holds for every $y^1 \in X^1\left(F_2^1\right)$. This leads to

$$v(F_2^1, F_2^2) = \min_{y^2 \in X^2\left(F_2^2\right)} \max_{y^1 \in X^1\left(F_2^1\right)} U(y^1, y^2) \tag{14}$$

$$\leq \max_{y^1 \in X^1\left(F_2^1\right)} U(y^1, x_2^2) \leq v(F_1^1, F_1^2) + C\left[\, \overline{d}\left((F_1^1, F_1^2), (F_2^1, F_2^2)\right)\right]^{\frac{q-1}{q}}. \tag{15}$$

The combination of (12)-(13) and (14)-(15) now implies (8). ■

The continuity of the value as a function of (F^1, F^2) is, of course, an immediate implication of Theorem 1:

Corollary 1. Suppose that $\{F_k^i\}_{k=1}^{\infty} \subset F^*$ is a sequence such that $\lim_{k\to\infty} F_k^i = F^i$ in the Boylan pseudo-metric, for $i = 1, 2$, and that the condition of Theorem 1 holds. Then $\lim_{k\to\infty} v(F_k^1, F_k^2) = v(F^1, F^2)$.

If $K(\cdot)$ is a bounded function, it is obvious that (2) holds for *every* $q > 1$, and thus q can be chosen to be arbitrarily high. The constant $C = C(q)$, defined in (9), converges to the limit

$$32 \max(n_1, n_2) \|K\|_{\infty}$$

when q approaches infinity. Inequality (8) of Theorem 1 thus provides us with the following corollary:

Corollary 2. If $K(\cdot)$ is a bounded function, the value $v(F^1, F^2)$ is a *Lipschitz* function of $(F^1, F^2) \in F^* \times F^*$, with respect to the pseudo-metric $\overline{d}$.

It is natural to ask whether the value is continuous when $K(\cdot)$ is merely *integrable*. Our next theorem shows that the continuity still obtains under this more general assumption. However, it does not follow from Theorem 1 (since we do not have uniform continuity in this case) and has to be established directly (using similar techniques).

Theorem 2. If the state-dependent Lipschitz constant $K(\cdot)$ is integrable, and if $\{\mathcal{F}_k^i\}_{k=1}^{\infty} \subset F^*$ is a sequence such that $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ in the Boylan pseudo-metric, for $i = 1, 2$, then $\lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v(\mathcal{F}^1, \mathcal{F}^2)$.

Proof. We will show that the limit v' of any convergent subsequence of $\{v(\mathcal{F}_k^1, \mathcal{F}_k^2)\}_{k=1}^{\infty}$ (which we assume, w.l.o.g., to be the sequence itself) is equal to $v(\mathcal{F}^1, \mathcal{F}^2)$. Let x_k^1 be an optimal strategy of player 1 in the game $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$, for every $k \geq 1$. As was mentioned, $X^1(\mathcal{F})$ is metrizable and compact, and therefore there is a subsequence of $\{x_k^1\}_{k=1}^{\infty}$ (which we again let, w.l.o.g., to be the sequence itself) that converges weakly to some $x^1 \in X^1(\mathcal{F})$. By Lemma 2 in the Appendix x^1 is $\mathcal{F}^1$-measurable, which implies that $x^1 \in X^1(\mathcal{F}^1)$.

Now fix $y^2 \in X^2(\mathcal{F}^2)$, and, for every $k \geq 1$, let $y_k^2 \equiv E(y^2 \mid \mathcal{F}_k^2) \in X^2(\mathcal{F}_k^2)$. Since x_k^1 is an optimal strategy of 1 in $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$,

$$U(x_k^1, y_k^2) \geq v(\mathcal{F}_k^1, \mathcal{F}_k^2). \tag{16}$$

Since $\lim_{k\to\infty} y_k^2 = y^2$ in $L_1^{n_2}(\Omega, \mathcal{F}, \mu)$ by (7), there is a subsequence of $\{y_k^2\}_{k=1}^{\infty}$ that converges pointwise to y^2 μ-almost everywhere; w.l.o.g., the sequence itself converges pointwise. Note that

$$\left|U(x_k^1, y_k^2) - U\left(x^1, y^2\right)\right| \leq \left|U(x_k^1, y_k^2) - U\left(x_k^1, y^2\right)\right| + \left|U(x_k^1, y^2) - U\left(x^1, y^2\right)\right|$$

(by (1))

$$\leq \int_{\Omega} K(\omega) \left\|y_k^2(\omega) - y^2(\omega)\right\| d\mu(\omega) + \left|U(x_k^1, y^2) - U\left(x^1, y^2\right)\right|.$$

The first term in the above expression converges to zero as $k \to \infty$ by the bounded convergence theorem, and the second term also converges to zero since U is weakly continuous in each variable separately. Thus,

$$\lim_{k\to\infty} U(x_k^1, y_k^2) = U\left(x^1, y^2\right), \tag{17}$$

and together with (16) this implies

$$U(x^1, y^2) \geq \lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v'; \tag{18}$$

this inequality holds for every $y^2 \in X^2(\mathcal{F}^2)$. Thus,

$$v(\mathcal{F}^1, \mathcal{F}^2) = \max_{y^1 \in X^1(\mathcal{F}^1)} \min_{y^2 \in X^2(\mathcal{F}^2)} U(y^1, y^2) \tag{19}$$

$$\geq \min_{y^2 \in X^2(\mathcal{F}^2)} U(x^1, y^2) \geq v'. \tag{20}$$

Using similar arguments (when we start from finding a limit point x^2 of a sequence $\{x_k^2\}_{k=1}^{\infty}$ of optimal strategies of player 2 in games $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$) we can show that

$$U(y^1, x^2) \leq \lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v' \tag{21}$$

for every $y^1 \in X^1(\mathcal{F}^1)$. This leads to

$$v(\mathcal{F}^1, \mathcal{F}^2) = \min_{y^2 \in X^2(\mathcal{F}^2)} \max_{y^1 \in X^1(\mathcal{F}^1)} U(y^1, y^2) \tag{22}$$

$$\leq \max_{y^1 \in X^1(\mathcal{F}^1)} U(y^1, x^2) \leq v'. \tag{23}$$

The combination of (19)-(20) and (22)-(23) now implies $v' = v(\mathcal{F}^1, \mathcal{F}^2)$. This establishes $\lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v(\mathcal{F}^1, \mathcal{F}^2)$. ■

Remark 1 (Monotonic Convergence of Information Fields). Theorem 2 also applies in the important case when information fields of players converge *monotonically* (i.e., for each player i, $\{\mathcal{F}_k^i\}_{k=1}^{\infty} \subset F^*$ is a sequence of fields such that either $\mathcal{F}_1^i \subset \mathcal{F}_2^i \subset ... \subset \mathcal{F}^i$ and $\mathcal{F}^i$ is generated by $\bigcup_{k=1}^{\infty} \mathcal{F}_k^i$, or $\mathcal{F}_1^i \supset \mathcal{F}_2^i \supset ... \supset \mathcal{F}^i$ and $\mathcal{F}^i = \bigcap_{k=1}^{\infty} \mathcal{F}_k^i$). Although monotonic convergence of information fields does not necessarily imply convergence in the Boylan pseudo-metric (as remarked in Boylan (1971)), the proof of Theorem 2 remains valid for the monotonic convergence. The only change is in the argument showing $L_1^{n_2}(\Omega, \mathcal{F}, \mu)$-convergence of $y_k^2 \equiv E(y^2 \mid \mathcal{F}_k^2) \in X^2(\mathcal{F}_k^2)$ to y^2: instead of appealing to (7), one has to use the martingale convergence theorem (for increasing or decreasing martingales; see, e.g., Theorems 2 and 3 in Section 2 of Parry (2004)). Similarly, our next Theorem 3 also applies to monotonically converging information fields. ■

The following theorem follows quite easily from the proof of Theorem 2.

Theorem 3. Suppose that $\{\mathcal{F}_k^i\}_{k=1}^{\infty} \subset F^*$ is a sequence such that $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ in the Boylan pseudo-metric, for $i = 1, 2$.

1. The optimal strategy correspondence is *upper semi-continuous* for each player. That is, if $i \in \{1,2\}$, and $\{x_k^i\}_{k=1}^{\infty} \in \prod_{k=1}^{\infty} X^i\left(\mathcal{F}_k^i\right)$ is a sequence that converges to x^i in the weak topology and in which, for every k, x_k^i is an optimal strategy of i in the game $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$, then x^i is an optimal strategy of i in $G(\mathcal{F}^1, \mathcal{F}^2)$.

2. The optimal strategy correspondence is *approximately lower semi-continuous* for each player. That is, if $i \in \{1,2\}$ and x^i is an optimal strategy of i in $G(\mathcal{F}^1, \mathcal{F}^2)$, then there exist sequences $\{\varepsilon_k\}_{k=1}^{\infty} \subset [0,\infty)$ and $\{x_k^i\}_{k=1}^{\infty} \in \prod_{k=1}^{\infty} X^i\left(\mathcal{F}_k^i\right)$ such that, for every k, x_k^i is an ε_k-optimal strategy of i in $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$, $\lim_{k\to\infty} \varepsilon_k = 0$, and $\lim_{k\to\infty} x_k^i = x^i$ in $L_1^{n_i}(\Omega, \mathcal{F}, \mu)$.

Proof. We will establish both assertions of the theorem for $i = 1$ only, since the case of $i = 2$ requires entirely analogous arguments. We therefore fix $i = 1$ for the rest of the proof.

1. Since $\lim_{k\to\infty} x_k^1 = x^1$ weakly, the entire first part of the proof of Theorem 2 (leading to (18)) can be utilized to show that $U(x^1, y^2) \geq \lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2)$ for every $y^2 \in X^2\left(\mathcal{F}^2\right)$. However, by Theorem 2, $\lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v(\mathcal{F}^1, \mathcal{F}^2)$, and so x^1 is indeed an optimal strategy of 1 in $G(\mathcal{F}^1, \mathcal{F}^2)$.

2. Denote

$$x_k^1 \equiv E(x^1 \mid \mathcal{F}_k^1) \in X^1\left(\mathcal{F}_k^1\right) \text{ and } \varepsilon_k \equiv \sup_{y^2 \in X^2\left(\mathcal{F}_k^2\right)} \left(v(\mathcal{F}_k^1, \mathcal{F}_k^2) - U(x_k^1, y^2)\right) \geq 0$$

for every k. Thus, x_k^1 is an ε_k-optimal strategy of 1 in $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$. By (7), $\lim_{k\to\infty} x_k^1 = x^1$ in the $L_1^{n_1}(\Omega, \mathcal{F}, \mu)$-topology. We will now show that $\lim_{k\to\infty} \varepsilon_k = 0$. Indeed, suppose by the way of contradiction that this is not so. Then there exists an increasing subsequence $\{k_l\}_{l=1}^{\infty}$ of indices such that

$$\liminf_{l\to\infty} U(x_{k_l}^1, y_{k_l}^2) < \lim_{l\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v(\mathcal{F}^1, \mathcal{F}^2) \tag{24}$$

for some $\left\{y_{k_l}^2\right\}_{l=1}^{\infty} \subset \prod_{l=1}^{\infty} X^i\left(\mathcal{F}_{k_l}^2\right)$. By metrizability and compactness of $X^2(\mathcal{F})$ there is a subsequence of $\left\{y_{k_l}^2\right\}_{l=1}^{\infty}$ which converges weakly to some $y^2 \in X^2(\mathcal{F})$ (w.l.o.g., the sequence itself converges to y^2). By Lemma 2 in the Appendix, $y^2 \in X^2\left(\mathcal{F}^2\right)$. Since $\lim_{l\to\infty} x_{k_l}^1 = x^1$ in $L_1^{n_1}(\Omega, \mathcal{F}, \mu)$ and $\lim_{l\to\infty} y_{k_l}^2 = y^2$ weakly, it can be shown as in the proof of (17) that $\lim_{l\to\infty} U(x_{k_l}^1, y_{k_l}^2) = U(x^1, y^2)$. But $U(x^1, y^2) \geq v(\mathcal{F}^1, \mathcal{F}^2)$, and therefore (24) is contradicted. We conclude that $\lim_{k\to\infty} \varepsilon_k = 0$. ■

Remark 2 (Optimal Strategies are not Lower Semi-Continuous). Part 2 of Theorem 3 cannot be strengthened because the optimal strategy correspondence is not *lower* semi-continuous in general. That is, it may be the case that $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ in the Boylan pseudo-metric for $i = 1, 2$, and x^i is an optimal strategy of i in $G(\mathcal{F}^1, \mathcal{F}^2)$, but there is no sequence $\{x_k^i\}_{k=1}^{\infty}$ of optimal strategies of i in $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$ that converges to x^i in $L_1^{n_i}(\Omega, \mathcal{F}, \mu)$ or even in the weak topology. Indeed, consider the example where $\Omega = [-1, 1]$, $\mathcal{F}$ is the σ-field of Borel sets in Ω, μ is the normalized Lebesgue measure on Ω, $S^1 = [0, 1]$, $S^2 = \{0\}$, and $u(\omega, s^1, s^2) = \omega s^1$. Now, for each $k = 1, 2, 3, \ldots$, let $\mathcal{F}_k^1 = \mathcal{F}_k^2$ be the finite σ-field generated by the intervals $\left[-1, -1 + \frac{1}{k}\right]$ and $(-1 + \frac{1}{k}, 1]$, and let $\mathcal{F}^1 = \mathcal{F}^2 = \{\emptyset, \Omega\}$. Then clearly $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ for $i = 1, 2$. However, consider a pair $(x^1, x^2) \equiv (0, 0)$ of optimal strategies in the game $G(\mathcal{F}^1, \mathcal{F}^2)$. Since the optimal strategy x_k^1 of 1 in the game $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$ satisfies $x_k^1(\omega) = 1$ for every $\omega \in (-1 + \frac{1}{k}, 1]$, there exists no sequence of optimal strategies of 1 in $\{G(\mathcal{F}_k^1, \mathcal{F}_k^2)\}_{k=1}^{\infty}$ that converges to x^1 in either $L_1^{n_1}(\Omega, \mathcal{F}, \mu)$ or in the weak topology. ■

4 Appendix

Lemma 2. Let $\{\mathcal{F}_k\}_{k=0}^{\infty} \subset \mathcal{F}^*$ be a sequence such that $\lim_{k\to\infty} \mathcal{F}_k = \mathcal{F}_0$ in the Boylan pseudo-metric. If $\{x_k\}_{k=1}^{\infty} \subset \prod_{k=1}^{\infty} X^i(\mathcal{F}_k)$ is a sequence of functions that converges weakly to $x \in X^i(\mathcal{F})$, then x is $\mathcal{F}_0$-measurable (that is, $x \in X^i(\mathcal{F}_0)$).

Proof. Without loss of generality, assume that

$$\sum_{k=1}^{\infty} d(\mathcal{F}_k, \mathcal{F}_0) < \infty \tag{25}$$

(otherwise consider instead some subsequence $\{\mathcal{F}_{k_l}\}_{l=1}^{\infty}$ with $\sum_{l=1}^{\infty} d(\mathcal{F}_{k_l}, \mathcal{F}_0) < \infty$). For every k denote by $\mathcal{G}_k$ the σ-field $\bigvee_{n=k}^{\infty} \mathcal{F}_k$, that is, the minimal σ-subfield of $\mathcal{F}$ which contains each one of $\{\mathcal{F}_n\}_{n=k}^{\infty}$. It follows from (25) by Corollary 2 of Van Zandt (1993) that $\lim_{k\to\infty} \mathcal{G}_k = \mathcal{F}_0$.

By applying the Banach-Saks theorem for the sequence $\{x_n\}_{n=k}^{\infty}$ that converges weakly to x, for every $k \geq 1$, one can find a sequence $\{\overline{x}_k\}_{k=1}^{\infty}$ such that: (a) $\overline{x}_k$ is a convex combination of $\{x_n\}_{n=k}^{\infty}$ and therefore $\overline{x}_k \in X^i(\mathcal{G}_k)$; and (b) $\{\overline{x}_k\}_{k=1}^{\infty}$ converges to x *strongly* (that is, in the $\|.\|_p$ norm for some $p \geq 1$). By Lemma 1 in Einy et al (2005), the strong limit of $\{\overline{x}_k\}_{k=1}^{\infty}$ is measurable with respect to $\lim_{k\to\infty} \mathcal{G}_k = \mathcal{F}_0$. We conclude that $x \in X^i(\mathcal{F}_0)$. ■

Acknowledgement: The authors wish to thank Robert Simon and Nicholas Yannelis for their helpful comments.

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