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THE RANK OF A SUB-MATRIX OF COINTEGRATION

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Abstract

This paper proposes a test of the rank of the sub-matrix of \( \beta \), where \( \beta \) is a cointegrating matrix. In addition, the sub-matrix of \( \beta_\perp \), an orthogonal complement to \( \beta \), is investigated. We show that information on the rank of the sub-matrix of \( \beta \) and/or \( \beta_\perp \) is useful in several situations. We construct the test statistic by using the eigenvalues of the quadratic form of the sub-matrix. We show that the test statistic has a limiting chi-squared distribution when the data is non-trending, and we propose a conservative test when the data is trending. Finite sample simulations show that, although the simulation settings are limited, the proposed test works well.

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1. Introduction

A vector autoregressive (VAR) process has often been used to model a multivariate economic time series and, following the seminal work of Engle and Granger (1987), a cointegrating relation has been incorporated into the VAR model. A typical \( n \)-dimensional VAR model of order \( m \) is

\[
x_t = d + A_1 x_{t-1} + \cdots + A_m x_{t-m} + \varepsilon_t,
\]

for \( t = 1, \cdots, T \), where \( \{ \varepsilon_t \} \) is independently and identically distributed (i.i.d.) with mean zero and a positive definite matrix \( \Sigma \) and \( I_n - A_1 z - \cdots - A_m z^m = 0 \) has roots outside the unit circle or equal to 1. The model (1) can be written in the error correction (EC) format,

\[
\Delta x_t = d + \alpha \beta' x_{t-1} + \sum_{j=1}^{m-1} \Gamma_j \Delta x_{t-j} + \varepsilon_t,
\]

where \( \alpha \) and \( \beta \) are \( n \times r \) matrices with rank \( r \), \( \Delta = 1 - L \), and \( L \) denotes the lag operator. We assume \( 0 < r < n \) and then there are \( r \) cointegrating relations. The exact condition of the existence of cointegration is given by Johansen (1992). We also assume that the cointegrating rank \( r \) is known or estimated by some testing procedure, such as the maximum likelihood (ML) test proposed by Johansen (1988, 1991) or the Lagrange multiplier (LM) test by Lütkepohl and Saikkonen (2000) and Saikkonen and Lütkepohl (2000). Other testing procedures of the cointegrating rank are reviewed by Hubrich, Lütkepohl, and Saikkonen (2001) and papers therein.

In this paper, we investigate the tests of the rank of \( \beta_1 \), the sub-matrix of \( \beta \), and the rank of \( \beta_{\perp,1} \), the sub-matrix of \( \beta_{\perp} \), where \( \beta = [\beta_1, \beta_2]' \) and \( \beta_{\perp} = [\beta_{\perp,1}, \beta_{\perp,2}]' \), with \( \beta_{\perp} \) being an orthogonal complement to \( \beta \). In practical analysis, we sometimes encounter cases where we need to know the rank of \( \beta_1 \) and/or \( \beta_{\perp,1} \). For example, the cointegrating matrix is sometimes normalized as \( \beta^* = \beta (a' \beta)^{-1} \), as proposed by Johansen (1988, 1991) and Paruolo (1997), where \( a \) is an \( n \times r \) matrix with full column rank, and the prototype normalization is represented by \( a = [I_r, 0]' \). However, there is no guarantee that \( a' \beta \) has full rank. In such a situation, we would like to know whether the first \( r \) rows of \( \beta \) have full rank. Another example is the Granger non-causality test. As shown in Toda and Phillips (1993), when there is a cointegrating relationship, in general the Wald statistic of the Granger non-
causality test from the last $n_3$ variables of $x_t$ to the first $n_1$ variables has a non-standard limiting distribution, depending on nuisance parameters. However, if the last $n_3$ rows of $\beta$ have full row rank, the Wald statistic is asymptotically $\chi^2$ distributed. In Section 2, we will illustrate these situations where information of the rank of $\beta_1$ and/or $\beta_{1,1}$ is useful in practical application.

Tests of the rank of a matrix have been investigated in the literature and recent econometric developments can be seen in works by Cragg and Donald (1996, 1997) and Robin and Smith (2000), among others. Although these papers proposed tests of the rank of a matrix, they assume that the estimator of the matrix is $T^{1/2}$ consistent and has a limiting normal distribution with a non-stochastic variance matrix. However, the estimator of the cointegrating matrix is $T$ (or $T^{3/2}$) consistent and has an asymptotic non-standard distribution. As a result, we cannot apply existing testing procedures to the cointegrating matrix.

The paper is organized as follows. Section 2 illustrates situations in which we need to know the rank of $\beta_1$ and/or $\beta_{1,1}$. It is these situations that motivated us to investigate the test of the rank. In Section 3, we propose tests of the rank of $\beta_1$ and $\beta_{1,1}$ for non-trending data. We will show that the proposed test statistics have limiting $\chi^2$ distributions. Section 4 considers the case of trending data. In this case, the test statistics do not necessarily converge to $\chi^2$ distributions. To overcome this situation, we propose tests that are conservative. Section 5 investigates the finite sample properties of the tests. Section 6 concludes the paper.

In regard to notation, we use $\text{vec}(A)$ to stack the rows of a matrix $A$ into a column vector, $\lceil x \rceil$ to denote the largest integer $\leq x$, $\bar{a} = a(a'a)^{-1}$ for a full column rank matrix $a$. $\xrightarrow{p}$, $\xrightarrow{d}$, and $\Rightarrow$ signify convergence in probability, convergence in distribution, and weak convergence of the associated probability measures. We denote the rank of $A$ by $\text{rk}(A)$ and the column space of $A$ by $\text{sp}(A)$. We write integrals like $\int_0^1 X(s)dY(s)'$ simply as $\int XdY'$ to achieve notational economy, and all integrals are from 0 to 1 except where otherwise noted.

2. Examples of Situations where Information on the Rank is Useful

2.1. Identifying normalizations
From the cointegrating matrix, $\beta$, we know directions that make the I(1) vector process stationary, but, in general, $\beta$ cannot be identified because any pair of $\alpha c$ and $\beta c^{-1}$ for a non-singular matrix $c$ is equivalent to the pair of $\alpha$ and $\beta$ in the model (2). Johansen (1988, 1991) and Paruolo (1997) proposed the identifying normalization such that $\beta^* = \beta(a'\beta)^{-1}$ and $\hat{\beta}^* = \hat{\beta}(a'\hat{\beta})^{-1}$ for the ML estimator $\hat{\beta}$, where $a$ is an $n \times r$ matrix with full column rank. This normalization is useful in practical analysis because the limiting distribution of the normalized estimator has been derived by these authors and we can use it for statistical inference. However, there is no guarantee that $a'\beta$ has full rank. For example, the typical normalization is represented by $a = [I_r, 0]'$ and then $\beta$ is normalized as $\beta(a'\beta)^{-1} = \beta\beta_1^{-1}$ where $\beta_1$ is the first $r$ rows of $\beta$. Although $\beta$ has full column rank $r$, there is no guarantee that $\beta_1$ has full rank, and then, as discussed in Paruolo (1997), we have to carefully chose the normalizing matrix $a$. In this case, the test of the rank of $\beta_1$ is useful to confirm that $\beta_1$ has full rank. If the rank of $\beta_1$ is decided to be $r$ by the statistical test, we will use the normalizing matrix $a$. Otherwise, we have to choose another normalizing matrix. Since we may encounter the same identifying problem for $\beta_\perp$, we will also consider the test of the rank of the sub-matrix of $\beta_\perp$.

2.2. The Granger non-causality test

To test for Granger non-causality, we may use either the levels VAR model (1) or the EC format (2). Suppose that we are interested in whether the last $n_3$ variables in $x_t$ are Granger-caused by the first $n_1$ variables. We write $x_t = [x_{1t}', x_{2t}', x_{3t}]'$ and partition $\alpha$, $\beta$ and $A_i$ for $i = 1, \cdots, m$ conformably with $x_t$. First, we consider the test with the levels VAR model (1). Then, the null hypothesis of non-causality is formulated as

$$A_{i1}^{31} = \cdots = A_{im}^{31} = 0,$$

where $A_{i1}^{31}$ is the $n_3 \times n_1$ lower-left sub-matrix of $A_i$ for $i = 1, \cdots, m$. If we estimate the model (1) by the least squares method and construct the Wald statistic in a usual form, the test statistic is shown by Toda and Phillips (1993) to have a non-standard limiting distribution and to depend on nuisance parameters in general. However, Toda and Phillips also showed that, if $rk(\beta_1) = n_1$ where $\beta_1$ is the first $n_1$ rows of $\beta$, the Wald statistic
converges in distribution to $\chi^2_{n_1n_3m}$. Then, if we pretest the rank of $\beta_1$ and find that it has full row rank $n_1$, we can use the Wald statistic in a standard form.

On the other hand, if we estimate the model (2) by the ML method, the null hypothesis of non-causality is formulated as

$$\Gamma_{1}^{31} = \cdots = \Gamma_{m-1}^{31} = 0 \quad \text{and} \quad \bar{A}^{31} = 0,$$

where $\Gamma_i^{31}$ and $\bar{A}^{31}$ are the $n_3 \times n_1$ lower-left matrices of $\Gamma_i$ and $\bar{A}$ for $i = 1, \cdots, m - 1$ with $\bar{A} = \alpha \beta'$. As in the case of the levels VAR, the Wald statistic for non-causality has a non-standard limiting distribution in general but Toda and Phillips (1993, Theorem 3) showed that if $rk(\beta_1) = n_1$ or $rk(\alpha_3) = n_3$, then the Wald statistic has a limiting $\chi^2_{n_1n_3m}$ distribution. Consequently, information on $rk(\beta_1)$ is useful in this case. Note that the existing testing procedure may be available for the test of $rk(\alpha_3)$ because the ML estimator of $\alpha$ has a limiting normal distribution with a variance matrix being a Kronecker product structure. See, for example, Corollary 3.1. of Robin and Smith (2000).

Although other testing procedures for Granger non-causality are proposed in the literature, such as the fully modified (FM) method by Phillips (1995) and Phillips and Hansen (1990), and the lag-augmented (LA) method by Dolado and Lütkepohl (1996), Toda and Yamamato (1995) and Kurozumi and Yamamoto (2000), they have some deficiencies. For example, Yamada and Toda (1997, 1998) showed that the Granger non-causality test based on the FM method suffers from a large size distortion, while the LA method estimates the model with an artificially augmented lag that causes loss of power because of inefficiency. On the other hand, Yamamoto (2002) showed that the finite sample properties of the standard Wald statistic are fairly good when $\beta_1$ has full row rank. As a result, before testing for Granger non-causality, we recommend testing the rank of $\beta_1$ and, if it has full row rank, the standard Wald statistic should be used.

2.3. The test for long-run non-causality

A test for long-run non-causality was proposed in Bruneau and Jondeau (1999) and was developed into the test for block long-run non-causality by Yamamoto and Kurozumi (2001, 2002). The long-run causality is defined by considering the $h$-step ahead forecast with
Let us consider the companion form of the model (1).

\[ X_t = FX_{t-1} + E_t, \]

where \( X_t = [x'_t, x'_{t-1}, \ldots, x'_{t-m+1}]' \), \( E_t = [\varepsilon'_t, 0', \ldots, 0']' \) and \( F \) is defined consistently with the expression (1). The \( h \)-step ahead prediction of \( x_{t+h} \), given \( X_t \), is expressed as \( x_{t+h|t} = MF^hX_t \), where \( M = [I_n, 0, \cdots, 0] \), and the coefficient matrix of the long-run prediction is defined as

\[ \tilde{B} = \lim_{h \to \infty} (MF^h). \]

Let us partition \( x_t \) in the same way as in the previous section. The hypothesis of the long-run non-causality of \( x_{3t} \) to \( x_{1t} \) is given by

\[ R_L BR_R = 0, \]

where \( R_L = [I_{n_1}, 0] \) and \( R_R = I_n \otimes R^*_R \) with \( R^*_R = [0, I_{n_3}]' \). Yamamoto and Kurozumi (2001, 2002) derived the limiting distribution of the ML estimator of \( \tilde{B}, \hat{B} \), which is asymptotically normal. The natural way to test the above hypothesis seems to be to construct the Wald statistic. However, the asymptotic variance matrix of \( \hat{B} \) is singular and consequently the Wald statistic might also have a singular variance matrix. If a variance matrix has full rank, the usual Wald statistic is used to test the hypothesis, while we may construct the test statistic using a generalized inverse of the matrix if it is singular and we know the rank of the variance matrix. The important point is that singularity of the variance matrix depends only on \( rk(\beta_3) \) and \( rk(\beta_\perp, 1) \). In other words, we can identify the rank of the variance matrix if we know the rank of \( \beta_3 \) and \( \beta_\perp, 1 \). Then, the tests of \( rk(\beta_3) \) and \( rk(\beta_\perp, 1) \) play an important role in the long-run non-causality test.

3. Test of the Rank of the Sub-Matrix for Non-Trending Data

3.1. The model with \( d = 0 \)

In this section we consider a test of rank for non-trending data with \( d = 0 \). The model considered in this section is

\[ \triangle x_t = \alpha\beta'x_{t-1} + \sum_{j=1}^{m-1} \Gamma_j \triangle x_{t-j} + \varepsilon_t. \quad (3) \]
We estimate the model (3) by the ML method assuming that \( \{ \varepsilon_t \} \) is Gaussian, although asymptotic properties are preserved under more general assumptions. We denote the ML estimator with \( \hat{\cdot} \). For example, the ML estimator of \( \beta \) is denoted by \( \hat{\beta} \). Using the result that \( T^{-1/2} \sum_{t=1}^{[Tr]} \varepsilon_t \Rightarrow W(r) \) for \( 0 \leq r \leq 1 \) by the functional central limit theorem, where \( W(\cdot) \) is an \( n \)-dimensional Brownian motion with a variance matrix \( \Sigma \), Johansen (1988, 1996) showed that \( \tilde{\beta} = \hat{\beta} (\hat{\beta}' \hat{\beta})^{-1} d \int G_0 G_0' ds \int G_0 dV' \), (4)

where \( G_0(\cdot) = \hat{\beta}_1' CW(\cdot) \) with \( C = \beta_1'(\alpha_1' \Gamma \beta_1')^{-1} \alpha_1' \), \( \Gamma = I_n - \sum_{i=1}^{m-1} \Gamma_i \), \( V(\cdot) = (\alpha' \Sigma^{-1} \alpha)^{-1} \alpha' \Sigma^{-1} W(\cdot) \) and \( G_0(\cdot) \) and \( V(\cdot) \) are independent. He also showed that \( \tilde{\alpha} = \hat{\alpha} \hat{\beta}' \bar{\beta} \), \( \hat{\Sigma} \) and \( \hat{\Gamma}_i \) \((i = 1, \cdots, m - 1)\) are consistent estimators of \( \alpha \), \( \Sigma \) and \( \Gamma_i \), respectively.

Let us partition \( \beta \) as \( \beta' = [\beta_1', \beta_2'] \) where \( \beta_1 \) and \( \beta_2 \) are \( n_1 \times r \) and \( (n - n_1) \times r \) matrices, respectively. Similarly, we partition \( \beta_1' = [\beta_{1,1}', \beta_{1,2}'] \) conformably. Our interest lies in finding the rank of \( \beta_1 \) and thus, we consider the following testing problem.

\[ H_0 : rk(\beta_1) = f \text{ v.s. } H_1 : rk(\beta_1) > f. \] (5)

Note that the rank of \( \beta_1 \) is at most \( p \equiv \min(n_1, r) \).

To test the rank of \( \beta_1 \), we follow the same strategy as Robin and Smith (2000), who test the rank of a matrix and investigate its quadratic form. In our situation, we construct a quadratic form of \( \beta_1 \). The advantage of considering a quadratic form is that the eigenvalues are non-negative real values, even if those of \( \beta_1 \) are complex values. Then, the null hypothesis \( H_0 \) becomes equivalent to the existence of \( f \) positive real and \( n_1 - f \) zero eigenvalues.

Let \( \Psi \) and \( \Phi \) be \( r \times r \) and \( n_1 \times n_1 \) possibly stochastic matrices that are symmetric and positive definite almost surely (a.s.). Since they are full rank matrices (a.s.), the rank of \( \beta_1 \) is equal to the rank of \( \Phi^{-1} \beta_1 \Psi \beta_1' \) (a.s.). Therefore, the test of the rank of \( \beta_1 \) is equivalent to that of \( \Phi^{-1} \beta_1 \Psi \beta_1' \), and then we consider the rank of the latter matrix. Note that, although this strategy is basically the same as that followed by Robin and Smith (2000), we cannot directly use their result because they assume that the estimated matrix is asymptotically normally distributed with a convergence rate \( T^{1/2} \), while \( \hat{\beta}_1 \) is shown to be \( T \) consistent and the limiting distribution is mixed Gaussian.

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For the test of the rank of $\beta_1$, we define $\Psi = \alpha' \Sigma^{-1} \alpha$ and

$$\Phi = [\beta_1, \beta_{\perp, 1} (\beta_{\perp}' \beta_{\perp})^{-1}] \begin{bmatrix} (\beta' \beta)^{-1} & 0 \\ 0 & \left( \int G_0 G_0' ds \right)^{-1} \end{bmatrix} \begin{bmatrix} \beta_1' \\ \beta_{\perp}' \beta_{\perp} \end{bmatrix}^{-1} \beta_{\perp, 1}.$$  \hspace{1cm} (6)

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}$ be the ordered eigenvalues of $\Phi^{-1} \beta_1 \Psi \beta_1'$, which are the solution of the determinant equation

$$|\beta_1 \Psi \beta_1' - \lambda \Phi| = 0. \hspace{1cm} (7)$$

Then, under $H_0$, $\lambda_1 \geq \cdots \geq \lambda_f > 0$ and $\lambda_{f+1} = \cdots = \lambda_{n_1} = 0$ (a.s.).

We construct a sample analogue of (7) using the LM estimator and investigate the limiting distributions of the eigenvalues. The sample analogue of (7) is given by

$$|\hat{\beta}_1 \hat{\Psi} \hat{\beta}_1' - \hat{\lambda} \hat{\Phi}| = 0,$$  \hspace{1cm} (8)

where $\hat{\beta}_1$ is the first $n_1$ rows of $\hat{\beta}$, $\hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha}$ and

$$\hat{\Phi} = [\hat{\beta}_1, \hat{\beta}_{\perp, 1} (\hat{\beta}_{\perp}' \hat{\beta}_{\perp})^{-1}] \begin{bmatrix} (\hat{\beta}' \hat{\beta})^{-1} & 0 \\ 0 & \left( \frac{1}{T} \hat{\beta}_{\perp}' S_{11} \hat{\beta}_{\perp} \right)^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1' \\ \hat{\beta}_{\perp}' \hat{\beta}_{\perp} \end{bmatrix}^{-1} \hat{\beta}_{\perp, 1}, \hspace{1cm} (9)

$$= \hat{\beta}_1 (\hat{\beta}' \hat{\beta})^{-1} \hat{\beta}_1' + \hat{\beta}_{\perp, 1} \left( \frac{1}{T} \hat{\beta}_{\perp}' S_{11} \hat{\beta}_{\perp} \right)^{-1} \hat{\beta}_{\perp, 1} \hat{\beta}_{\perp, 1}^{-1}, \hspace{1cm} (10)$$

where $S_{11} = T^{-1} \sum_{t=1}^{T} R_{1t} R_{1t}'$, with $R_{1t}$ being the regression residual of $X_{t-1}$ on $\triangle x_{t-1}, \cdots, \triangle x_{t-m+1}$, and we denote the ordered eigenvalues of (9) as $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{n_1}$. Note that when $n_1 > r$, the smallest $n_1 - r$ eigenvalues are obviously equal to 0, that is, $\hat{\lambda}_{f+1} = \cdots = \hat{\lambda}_{n_1} = 0$. We can easily see from the expressions (6) and (9) that $\Phi$ and $\hat{\Phi}$ are positive definite (a.s.), while the expression (10) is simpler and may be used to construct $\hat{\Phi}$ in practice.

To test the rank of $\beta_1$, we consider the following test statistic.

$$L_T = T^2 \sum_{i=f+1}^{p} \hat{\lambda}_i = T^2 \sum_{i=f+1}^{n_1} \hat{\lambda}_i,$$  \hspace{1cm} (11)

which rejects the null hypothesis when $L_T$ takes large values. The second equality is established because $p = \min(n_1, r)$ and $\hat{\lambda}_{p+1} = \cdots = \hat{\lambda}_{n_1} = 0$ when $n_1 > r$.

**Theorem 1** Let $\hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha}$ and $\hat{\Phi}$ be given by (10). If $f < p$, under $H_0$, $L_T \overset{d}{\longrightarrow} \chi^2_{(n_1-f)(r-f)}$. 

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Remark 1: Since the determinant equation (8) converges to (7) in distribution, the estimated ordered eigenvalues of (8) also converge in distribution to those of (7). Then, under the alternative, \( \hat{\lambda}_{f+1} \xrightarrow{d} \lambda_{f+1} > 0 \) (a.s.), so that \( T^2 \hat{\lambda}_{f+1} \) goes to infinity. Therefore, the test statistic \( \mathcal{L}_T \) is consistent.

Next, we consider a test of the rank of the sub-matrix of \( \beta_\perp \). The testing problem is

\[
H_{0\perp} : \text{rk}(\beta_\perp, 1) = g \quad \text{v.s.} \quad H_{1\perp} : \text{rk}(\beta_\perp, 1) > g.
\]

For the same reason as in the test of \( \beta_1 \), we investigate the rank of \( \hat{\Phi}^{-1} \beta_\perp \hat{\Psi} \beta_\perp' \), where \( \hat{\Phi} \) and \( \hat{\Psi} \) are \( (n - r) \times (n - r) \) and \( n_1 \times n_1 \) full rank matrices (a.s.). Similar to (7), we consider the following determinant equation.

\[
|\beta_\perp \hat{\Psi} \beta_\perp' - \mu \hat{\Phi}| = 0, \quad (11)
\]

where \( \hat{\Psi} = \int G_0 G_0' ds \) and

\[
\hat{\Phi} = [\beta_\perp, \beta_1 (\beta')^{-1}]
\begin{bmatrix}
(\beta_\perp \beta_\perp')^{-1} & 0 \\
0 & (\alpha' \Sigma^{-1} \alpha)^{-1}
\end{bmatrix}
\begin{bmatrix}
\beta_\perp' \beta_\perp' \\
(\beta_\perp' \beta_\perp)
\end{bmatrix},
\]

and the sample analogue of (11) is given by

\[
|\hat{\beta}_\perp \hat{\Psi} \hat{\beta}_\perp' - \hat{\mu} \hat{\Phi}| = 0, \quad (12)
\]

where \( \hat{\Psi} = T^{-1} \hat{\beta}_\perp S_{11} \hat{\beta}_\perp \) and

\[
\hat{\Phi} = [\hat{\beta}_\perp, \hat{\beta}_1 (\hat{\beta}')^{-1}]
\begin{bmatrix}
(\hat{\beta}_\perp \hat{\beta}_\perp')^{-1} & 0 \\
0 & (\hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha})^{-1}
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_\perp' \hat{\beta}_\perp' \\
(\hat{\beta}_\perp' \hat{\beta}_\perp)
\end{bmatrix}.
\]

Let \( \mu_1 \geq \cdots \geq \mu_{n_1} \) and \( \hat{\mu}_1 \geq \cdots \geq \hat{\mu}_{n_1} \) be ordered eigenvalues of (11) and (12), respectively. Under the null hypothesis, the smallest \( (n_1 - g) \) eigenvalues, \( \mu_{g+1}, \cdots, \mu_{n_1} \), are all zeros (a.s.) and we then construct the following test statistic.

\[
\mathcal{L}_T = T^2 \sum_{i=g+1}^{q} \hat{\mu}_i = T^2 \sum_{i=g+1}^{n_1} \hat{\mu}_i,
\]

which rejects the null hypothesis when it takes large values, where \( q = \min(n_1, n - r) \). The second equality is established because \( \hat{\mu}_{n-r+1} = \cdots \hat{\mu}_{n_1} = 0 \) are obvious solutions when \( n_1 > (n - r) \). The following theorem gives the limiting distribution of \( \mathcal{L}_T \).

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Theorem 2 Let \( \hat{\Psi} = T^{-1}\hat{\beta}'_1S_1\hat{\beta}_\perp \) and \( \hat{\Phi} \) be given by (13). If \( g < q \), under \( H_{0\perp} \), \( \mathcal{L}_T \xrightarrow{d} \chi^2_{(n_1-g)(n-r-g)} \).

Note that the consistency of the test is shown in a similar way as in Remark 1.

Given the above two theorems, we can test the rank of \( \beta_1 \) and \( \beta_{\perp,1} \). In addition, we may consider the procedure to decide the rank of the sub-matrix, as the cointegrating rank is selected sequentially using the test of the cointegrating rank. For example, to decide the rank of \( \beta_1 \), we firstly test the null of \( f = 0 \). If the null hypothesis is accepted, the rank of \( \beta_1 \) is decided to be zero. Otherwise, we then test the hypothesis of \( f = 1 \). We sequentially continue to test the rank of \( \beta_1 \) until the null hypothesis is accepted. When the null of \( f = p - 1 \) is rejected, we consider that \( \beta_1 \) has full rank. Similarly, the rank of \( \beta_{\perp,1} \) can be decided by the same procedure.

3.2. The model with \( d \neq 0 \)

In the previous section, we considered the model with \( d = 0 \) for non-trending data. However, in practice, we sometimes consider the model (2) with \( d \neq 0 \) but the level of data has no linear trend. In this case, the constant term can be expressed as \( d = \alpha \rho_0 \) where \( \rho_0 \) is a \( r \times 1 \) coefficient vector, so that the model (2) becomes

\[
\Delta x_t = \alpha \beta^+ x_{t-1}^+ + \sum_{j=1}^{m-1} \Gamma_j \Delta x_{t-j} + \varepsilon_t, \tag{14}
\]

where \( \beta^+ = [\beta', \rho_0]' \) and \( x_{t-1}^+ = [x_{t-1}', 1]' \). See Johansen (1991, 1996). The maximum likelihood estimator of \( \beta^+ \) can be obtained by the reduced rank regression of \( \Delta x_t \) on \( x_{t-1}^+ \) corrected for \( \Delta x_{t-1}, \ldots, \Delta x_{t-m+1} \), and the estimator of the cointegrating matrix is the first \( n \) rows of \( \hat{\beta}^+ \).

To test the rank of the sub-matrix of \( \beta \) for the model (14), we construct the test statistic \( \mathcal{L}_T \) with \( \hat{\Phi} \) defined by

\[
\hat{\Phi} = \hat{\beta}_1(\hat{\beta}' \hat{\beta})^{-1}\hat{\beta}'_1 + \hat{\beta}_{\perp,1}(\hat{\beta}'_{\perp,1} \hat{\beta}_{\perp,1})^{-1}L'(\Upsilon_T'S_T\Upsilon_T)^{-1}L(\hat{\beta}'_{\perp,1} \hat{\beta}_{\perp,1})^{-1}\hat{\beta}'_{\perp,1}, \tag{15}
\]

where \( L \) and \( \Upsilon_T \) are \( (n-r+1) \times (n-r) \) and \( (n+1) \times (n-r+1) \) matrices defined by

\[
L = \begin{bmatrix} I_{n-r} \\ 0 \end{bmatrix}, \quad \Upsilon_T = \begin{bmatrix} T^{-1/2} \hat{\beta}_1 & 0 \\ 0 & 1 \end{bmatrix},
\]

9
and $S_{11}^+ = T^{-1} \sum_{t=1}^T R_{11}^+ R_{11}^{+\prime}$, with $R_{11}^+$ being the regression residual of $x_{t-1}^+$ on $\triangle x_{t-1}, \cdots, \triangle x_{t-m+1}$.

**Theorem 3** Consider the model (14) and let $\hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha}$ and $\hat{\Phi}$ be given by (15). If $f < p$, under $H_0$, $L_T \xrightarrow{d} \chi^2_{(n_1-f)(r-f)}$.

For the test of the sub-matrix of $\beta_\perp$, we construct the test statistic $L_{\perp T}$ using $\hat{\Psi} = \{L'(\Upsilon_T' S_{11}^+ \Upsilon_T)^{-1} L\}^{-1}$.

**Theorem 4** Consider the model (14) and let $\hat{\Psi} = \{L'(\Upsilon_T' S_{11}^+ \Upsilon_T)^{-1} L\}^{-1}$ and $\hat{\Phi}$ be given by (13). If $g < q$, under $H_{0\perp}$, $L_{\perp T} \xrightarrow{d} \chi^2_{(n_1-g)(n-r-g)}$.

In practical analysis, we will obtain $\hat{\beta}$ by the reduced rank regression and we have to calculate $\hat{\beta}_\perp$ from $\hat{\beta}$. If $d = 0$, $\hat{\beta}_\perp$ can be easily obtained as explained in Johansen (1996, p.95). When $d = \alpha \rho_0$, one of the methods to calculate $\hat{\beta}_\perp$ is as follows: first we calculate the orthogonal projection matrix of $\hat{\beta}$, $M = I_n - \hat{\beta}(\hat{\beta}' \hat{\beta})^{-1} \hat{\beta}'$. Then, by the singular value decomposition, $M$ is expressed as $M_l M_\lambda M_r'$ where $M_l$ and $M_r$ are $n \times (n-r)$ orthogonal matrices and $M_\lambda$ is an $(n-r) \times (n-r)$ diagonal matrix with positive diagonal elements. Since $sp(M) = sp(M_l)$ and they are orthogonal to $\hat{\beta}$, we can use $M_l$ as $\hat{\beta}_\perp$.

4. The Test of the Rank of the Sub-Matrix for Trending Data

When the data is trending, $x_t$ can be expressed as the sum of the stochastic trend, the deterministic trend, and the I(0) component such that

$$x_t = C \sum_{i=1}^t \varepsilon_i + \tau t + C_1(L)(\varepsilon_t + d) + x_0^*, \quad (16)$$

where $\tau = Cd$, $C_1(L) = (C(L) - C(1))/(1 - L)$ with $C(L)$ being a lag polynomial when $\triangle x_t$ is represented as the vector moving-average process like $\triangle x_t = C(L)(d + \varepsilon_t)$, and $x_0^*$ is a stochastic component such that $\beta' x_0^* = 0$. See Johansen (1991, 1996) for more details.

In this case, $\beta_\perp$ is decomposed to $\tau$, the coefficient of a linear trend in (16), and $\gamma$, an $n \times (n-r-1)$ matrix that is orthogonal to $\tau$. As shown in Chapter 13.2 of Johansen (1996), $\tilde{\beta}$ can be expressed as

$$\tilde{\beta} = \beta + \gamma (\gamma' \gamma)^{-1} U_{1T} + \frac{1}{T^{1/2}} \tau' \tau^{-1} U_{2T}, \quad (17)$$

10
where
\[
T \begin{bmatrix} U_1 T \\ U_2 T \end{bmatrix} \xrightarrow{d} \left( \int G G' ds \right)^{-1} \int GdV' = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},
\]
say.

where \( G(r) = [G_1'(r), G_2'(r)]' \) with \( G_1(r) = G_0(r) - \int G_0 ds \) and \( G_2(r) = r - 1/2. \) We denote \( \Omega = \int GG' ds \) and partition it into 2 \times 2 blocks conformably with \([U_1', U_2']'. \) We express the \((i,j)\) block element of \((\int GG' ds)^{-1}\) as \( \Omega_{ij} \) for \( i, j = 1 \) and 2. In this section, we need the estimator of \( \Omega_{11} \), which is given by
\[
\hat{\Omega}_{11} = T \hat{\gamma}' S_{11}^{-1} \hat{\gamma},
\]
and \( S_{11} \) is defined in the same way as in the previous section, with \( R_{1t} \) being the regression residual of \( X_{t-1} \) on a constant and \( \Delta x_{t-1}, \cdots, \Delta x_{t-m+1} \). Convergence of \( \hat{\Omega}_{11} \) to \( \Omega_{11} \) is proved in Lemma 2 (iii) in the appendix, while the consistency of other ML estimators, such as \( \tilde{\alpha}, \hat{\Sigma}, \) and \( \hat{\Gamma}_i \), is shown by Johansen (1991, 1996).

Let us consider the testing problem (5). Under the null hypothesis, we can find the \( f \) linearly independent column vectors in \( \beta_1 \) and we define \( \beta_1^* \) as an \( n_1 \times f \) matrix whose columns consist of those \( f \) vectors. We also define an \( n_1 \times (n_1 - f) \) matrix \( \delta^* \) whose columns span the space orthogonal to the columns of \( \beta_1^* \) so that \( \delta^* \beta_1^* = 0. \) In the following, we show that the direction of \( \delta^* \) is important in deciding the convergence rate of \( \tilde{\beta}_1 \) and it also affects the limiting property of the test statistic.

Since \( \tilde{\beta}_1 \) is the first \( n_1 \) rows of \( \tilde{\beta} \), it is expressed from (17) as
\[
\tilde{\beta}_1 = \beta_1 + \gamma_1 (\gamma' \gamma)^{-1} U_1 T + \frac{1}{T^{1/2}} \tau_1 (\tau' \tau)^{-1} U_2 T.
\]

Suppose that an \( n_1 \times 1 \) vector \( \tau_1^* \) exists that is orthogonal to \( \gamma_1 \) (\( \tau_1'^* \gamma_1 = 0 \)) and belongs to the column space of \( \delta^* \). Here, note that, since the \( n \times n \) matrix \([\beta, \gamma, \tau]\) has full rank, the first \( n_1 \) rows of this matrix, \([\beta_1, \gamma_1, \tau_1]\), must have full row rank, which implies that \( a'[\beta_1, \gamma_1, \tau_1] \neq 0 \) for any non-zero vector \( a \). Then, because \( \tau_1^* \) is orthogonal to both \( \beta_1 \) and \( \gamma_1 \) by the assumption, we have \( \tau_1'^*[\beta_1, \gamma_1, \tau_1] = [0, 0, \tau_1'^* \tau_1] \neq 0, \) so that \( \tau_1'^* \tau_1 \neq 0. \) This implies that
\[
T^{3/2} \tau_1'^* \tilde{\beta}_1 = \tau_1'^* \tau_1 (\tau' \tau)^{-1} (TU_{2T}) \xrightarrow{d} \tau_1'^* \tau_1 (\tau' \tau)^{-1} U_2 = X_2', \text{ say,} \quad (18)
\]
while for an \( n_1 \times (n - r - 1) \) matrix \( \delta_0^* \) whose columns span the orthogonal complement to \( \tau_1^* \) in \( sp(\delta^*) \),

\[
T \delta_0^* \hat{\beta}_1 = \delta_0^* \gamma_1(\gamma' \gamma)^{-1} (TU_{1T}) \xrightarrow{d} \delta_0' \gamma_1(\gamma' \gamma)^{-1} U_1 = X_1', \text{ say.} \tag{19}
\]

On the other hand, if there exists no vector in \( sp(\delta^*) \) that is orthogonal to \( \gamma_1 \), we have

\[
T \delta_0^* \hat{\beta}_1 = \delta_0^* \gamma_1(\gamma' \gamma)^{-1} (TU_{1T}) + \delta_0^* \gamma_1(\tau' \tau)^{-1} (T^{1/2}U_{2T}) \xrightarrow{d} \delta_0^* \gamma_1(\gamma' \gamma)^{-1} U_1 = X', \text{ say.} \tag{20}
\]

Therefore, the convergence rate of \( \hat{\beta}_1 \) depends on whether a vector \( \tau_1^* \) orthogonal to \( \gamma_1 \) exists in \( sp(\delta^*) \).

The existence of \( \tau_1^* \) indicates that the column space of \( [\beta_1, \gamma_1] \) does not include \( \tau_1^* \) because \( \tau_1' \beta_1 = 0 \) and \( \tau_1' \gamma_1 = 0 \). We also note that the rank of \( [\beta_1, \gamma_1] \) must be \( n_1 - 1 \) or \( n_1 \) because \( [\beta_1, \gamma_1, \tau_1] \) has full rank \( n_1 \). Then, from another point of view, we can say that the rank of \( [\beta_1, \gamma_1] \) is \( n_1 - 1 \) if a vector \( \tau_1^* \) exists, while the non-existence of \( \tau_1^* \) is equivalent to \( rk([\beta_1, \gamma_1]) = n_1 \). Thus, we have to consider the asymptotic property separately according to the two cases where the rank of \( [\beta_1, \gamma_1] \) is \( n_1 \) and \( n_1 - 1 \).

In the following theorem, the test statistic is constructed from the eigenvalues of (8) using the same \( \hat{\Psi} \) as in the previous section and either

\[
\hat{\Phi} = \left[ \hat{\beta}_1, \hat{\gamma}_1(\gamma' \gamma)^{-1} \right] \begin{bmatrix} (\beta' \beta)^{-1} & 0 \\ 0 & \hat{\Omega}^{11} \end{bmatrix} \begin{bmatrix} (\gamma' \gamma)^{-1} \gamma_1' \\ \gamma_1' \end{bmatrix} \tag{21}
\]

or

\[
\hat{\Phi} = \left[ \hat{\beta}_1, \hat{\gamma}_1(\gamma' \gamma)^{-1} \tau_1 \right] \begin{bmatrix} (\beta' \beta)^{-1} & 0 \\ 0 & \hat{\Omega}^{11} \end{bmatrix} \begin{bmatrix} (\gamma' \gamma)^{-1} \gamma_1' \\ \tau_1' \end{bmatrix} \tag{22}
\]

**Theorem 5** *(i-a) Let \( \hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} \) and \( \hat{\Phi} \) be given by (21). If \( rk([\beta_1, \gamma_1]) = n_1 \) and \( f < p \), under \( H_0, \mathcal{L}_T \xrightarrow{d} \chi^2_{(n_1-f)(r-f)} \).

*(i-b) Let \( \hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} \) and \( \hat{\Phi} \) be given by (22). If \( rk([\beta_1, \gamma_1]) = n_1 \) and \( f < p \), under \( H_0, \mathcal{L}_T \) converges in distribution to a random variable that is bounded above by \( \chi^2_{(n_1-f)(r-f)} \).

*(ii) Let \( \hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} \) and \( \hat{\Phi} \) be given by (22). If \( rk([\beta_1, \gamma_1]) = n_1 - 1 \) and \( f < p \), under \( H_0, \mathcal{L}_T \xrightarrow{d} \chi^2_{(n_1-f-1)(r-f)} \).
Remark 2: In the case of (i-b), the test statistic converges in distribution to \( \chi^2_{(n_1-f)(r-f)} \) if and only if \( \delta^{*'} \tau_1 = 0 \), which is equivalent to the case where \( \tau_1 \in sp(\beta_1^*) = sp(\beta_1) \).

See the proof in the appendix. In general, the test using (22) is a conservative test if \( rk([\beta_1, \gamma_1]) = n_1 \).

From Theorem 5, if we know the rank of \([\beta_1, \gamma_1]\), we can construct the test statistic that converges to a \( \chi^2 \) distribution by appropriately using (21) or (22). However, such information is not available in practice. Notice that if \( rk([\beta_1, \gamma_1]) = n_1 - 1 \), \( \hat{\Phi} \) given by (21) may violate the condition that it is a full rank matrix, and in that case, the test statistic converges, not to the same \( \chi^2 \) distribution as given by Theorem 5 (ii), but to a random variable that depends on a nuisance parameter. Then, the test using (21) is not available in practice. On the other hand, if we use \( \hat{\Phi} \) given by (22), we can test the hypothesis by referring to a \( \chi^2 \) distribution irrespective of the rank of \([\beta_1, \gamma_1]\), although the test may be conservative and the degree of freedom changes depending on the rank of \([\beta_1, \gamma]\). Then, noting that the critical value of \( \chi^2_{(n_1-f)(r-f)} \) in Theorem 5 (i-b) is larger than that of \( \chi^2_{(n_1-f-1)(r-f)} \) in Theorem 5 (ii), we propose to test the null of \( rk(\beta_1) = f \) as follows:

1. We construct the test statistic \( L_T \) using (22).
2. If \( L_T \) is larger than the critical value of \( \chi^2_{(n_1-f)(r-f)} \), we reject the null hypothesis.
3. If \( L_T \) is smaller than the critical value of \( \chi^2_{(n_1-f-1)(r-f)} \), we accept the null hypothesis.

In this procedure, we may encounter a case where the test statistic is larger than the critical value of \( \chi^2_{(n_1-f-1)(r-f)} \) but smaller than that of \( \chi^2_{(n_1-f)(r-f)} \), that is, the case where \( c_{(n_1-f-1)(r-f)} \leq L_T \leq c_{(n_1-f)(r-f)} \), where \( c_{(n_1-f-1)(r-f)} \) and \( c_{(n_1-f)(r-f)} \) are corresponding critical values. To cope with such a case, the following corollary is useful.

**Corollary 1** Let \( \hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} \) and \( \hat{\Phi} \) be given by (22). Suppose that the rank of \( \beta_1 \) is \( f \) \((< p)\).

(i) If \( rk([\beta_1, \gamma_1]) = n_1 \), \( T^2 \hat{\lambda}_p \) converges in distribution to a random variable that is bounded above by \( \lambda^*_\text{min} \), where \( \lambda^*_\text{min} \) is the smallest non-zero eigenvalue of

\[
|X^* X^{*'} - \lambda^* I_{n_1-f}| = 0,
\]
where \(X^*\) is a \((r - f) \times (n_1 - f)\) matrix with \(\text{vec}(X^*) \sim N(0, I_{(r - f)(n_1 - f)})\).

(ii) If \(rk([\beta_1, \gamma_1]) = n_1 - 1\), \(T^2 \hat{\lambda}_p\) converges in probability to zero.

Table 1 shows the percentage points of \(\lambda^*_\text{min}\) for the case when \((n_1 - f) \geq (r - f)\). Since the non-zero eigenvalues of \(X^*X^*\) are the same as those of \(X^*X^*\), we can refer to the percentage points of \((r - f, n_1 - f)\) when \((n_1 - f) \leq (r - f)\).

Using the above corollary, we can cope with the situation where \(c_{(n_1 - f - 1)(r - f)} \leq c_{(n_1 - f)(r - f)}\). If \(T^2 \hat{\lambda}_p\) is smaller than the \((10, 5\) or \(1\%)\) percentage point of \(\lambda^*_\text{min}\), we reject the hypothesis of \(rk([\beta_1, \gamma_1]) = n_1\). In that case, \(c_{(n_1 - f - 1)(r - f)}\) is an appropriate critical value for \(L_T\), so that the null of \(rk(\beta_1) = f\) is rejected. On the other hand, if \(T^2 \hat{\lambda}_p\) is larger than the critical point of \(\lambda^*_\text{min}\), we accept the hypothesis of \(rk([\beta_1, \gamma_1]) = n_1\), so that the rank of \(\beta_1\) is decided to be \(f\).

Next, we investigate a test of the rank of \(\beta_{1,1}\). When the data is trending, \(\beta_{1,1}\) can be decomposed into \([\gamma_1, \tau_1]\) where \(\gamma_1\) and \(\tau_1\) are the first \(n_1\) rows of \(\gamma\) and \(\tau\), respectively. Then, testing the rank of \(\beta_{1,1}\) is equivalent to testing the rank of \([\gamma_1, \tau_1]\) and, therefore, we construct a test statistic from \([\hat{\gamma}_1, \hat{\tau}_1]\). Note that \(\hat{\beta}_{1,1}\) is the first \(n_1\) rows of \(\hat{\beta}_1\) and is not necessarily numerically equal to \([\hat{\gamma}_1, \hat{\tau}_1]\), although they span the same column space.

Let us consider the same determinant equation as (12) with \(\hat{\beta}_{1,1}\) replaced by \([\hat{\gamma}_1, \hat{\tau}_1]\) and

\[
\hat{\Psi} = \begin{bmatrix} (\hat{\Omega}^{11})^{-1} & 0 \\ 0 & 1 \end{bmatrix},
\]

(23)

\[
\hat{\Phi} = [\hat{\gamma}_1, \hat{\tau}_1, \hat{\beta}_1 (\hat{\beta}' \hat{\beta})^{-1}] \begin{bmatrix} (\hat{\gamma}' \hat{\gamma})^{-1} & 0 & 0 \\ 0 & (\hat{\tau}' \hat{\tau})^{-2} & 0 \\ 0 & 0 & (\hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha})^{-1} \end{bmatrix} \begin{bmatrix} \hat{\gamma}'_1 \\ \hat{\tau}'_1 \\ (\hat{\beta}' \hat{\beta})^{-1} \hat{\beta}'_1 \end{bmatrix}
\]

= \(\hat{\gamma}_1 (\hat{\gamma}' \hat{\gamma})^{-1} \hat{\gamma}'_1 + \hat{\tau}_1 (\hat{\tau}' \hat{\tau})^{-2} \hat{\tau}'_1 + \hat{\beta}_1 (\hat{\beta}' \hat{\beta})^{-1} (\hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha})^{-1} (\hat{\beta}' \hat{\beta})^{-1} \hat{\beta}'_1\).

We construct the test statistic \(L_{1,1}\) in the same way as in the previous section.

**Theorem 6** Let \(\hat{\Psi}\) and \(\hat{\Phi}\) be given by (23) and (24). If \(g < q\), under \(H_{0,1}\), \(L_{1,1}\) converges in distribution to a random variable that is bounded above by \(\chi^2_{(n_1 - g)(n - g - r)}\).

In a similar way to the non-trending data case, we can use the results of Theorems 5 and 6 to decide the ranks of \(\beta_1\) and \(\beta_{1,1}\) sequentially.
5. Simulation Results

In this section, we investigate the finite sample properties of the tests proposed in the previous sections. We consider the following four-dimensional error-correction model as a data generating process (DGP).

\[ \Delta x_t = d_0 + \alpha \beta' x_{t-1} + \varepsilon_t, \]

where \( \{\varepsilon_t\} \sim i.i.d. N(0, I_4) \). The following three DGPs are considered.

**DGP1:**
\[
\begin{align*}
\alpha &= \begin{bmatrix} -0.5 & 0.3 \\ 0.0 & -0.3 \\ -0.3 & -0.8 \\ -0.5 & 0.8 \end{bmatrix}, \\
\beta &= \begin{bmatrix} 1.0 & 0.0 \\ -1.0 & 0.0 \\ 1.0 & 1.0 \\ 0.0 & -0.5 \end{bmatrix}, \\
\beta_\perp &= \begin{bmatrix} 1.0 & 0.0 \\ 1.0 & 0.0 \\ 0.0 & 0.5 \\ 0.0 & 1.0 \end{bmatrix}, \\
d_0 &= \begin{bmatrix} -0.5 \\ 1.0 \\ 0.5 \\ -1.0 \end{bmatrix},
\end{align*}
\]

where DGP3 is the same as the simulation example in Reinsel and Ahn (1992). We test the rank of the first \( 2 \times 2 \) matrices of \( \beta \) and \( \beta_\perp \). Then, for DGP1, DGP2, and DGP3, the true rank of \( \beta_1 \) is 1, 1, and 2 respectively, while that of \( \beta_{\perp,1} \) is 2, 1, and 2, respectively. We set \( x_0 = 0 \) and discard the first 100 observations in all experiments. The number of replication is 1,000, and the level of significance is set equal to 0.05.

Table 2 reports the simulation results of the tests for non-trending data with \( d = 0 \) and \( d = \alpha \rho_0 \), and the tests for trending data. For non-trending data, we set \( d_0 = 0 \). Notice that the rank of \( \beta_1 \) equals 1 for DGP1 and DGP2, and then the corresponding entries in the table are rejection frequencies under the null hypothesis. For the test of non-trending data with \( d = 0 \), although the null hypothesis is rejected slightly more often than the nominal level, 0.05, the test seems to work well. On the other hand, the other entries for the test of \( \text{rk}(\beta_1) \) correspond to the power, which is close to 1. Similarly, the test of \( \text{rk}(\beta_{\perp,1}) \) seems to have good finite sample properties both under the null and the alternative hypotheses.
It can be seen in the middle of Table 2 that the tests for non-trending data with \( d = \alpha \rho_0 \) performs similarly to the case with \( d = 0 \).

The results of the tests for trending data are reported in the bottom of Table 2. We can see that the test works well both under the null and the alternative hypotheses, although the empirical size of \( \mathcal{L}_T \) is slightly lower than the nominal level, 0.05. The reason may be that the test is conservative in these cases.

To further investigate power properties of the tests, we use DGP2 with the (2, 2) element of \( \beta \) replaced by \( c_1 > 0 \) for the test of \( rk(\beta_1) \) and DGP2 with the (2, 2) element of \( \beta_\perp \) replaced by \( c_2 > 0 \) for the tests of \( rk(\beta_\perp, 1) \). We test for the null hypothesis of \( rk(\beta_1) = 1 \) or \( rk(\beta_\perp, 1) = 1 \) for each specification of a constant term \( d \). The simulation results are reported in Table 3. From the table, it seems difficult to detect the correct rank for \( c_1 < 0.01 \), even when the sample size is 200. However, when \( c_1 \) becomes larger than 0.01, the power of the test increases and it attains almost 1 when \( c_1 \) is 0.1 and \( T = 200 \).

For the test of \( rk(\beta_\perp, 1) \), we have to use different \( \beta \) depending on \( \beta_\perp \) to generate a process. In our simulation, we normalized \( \beta \) so that the (1, 1) and (3, 2) elements of \( \beta \) become 1. From the table, we can see that the power tends to increase when \( c_2 \) becomes larger than 0.01. We also note that the power of the test for trending data is lower compared with other tests.

We also conducted the simulation using DGP2 with the (1, 2) element of \( \beta \) or \( \beta_\perp \) replaced by non-zero value, but the relative performances of the tests are very similar to the results in Table 3 and we do not report to save the space.

6. Conclusion

In this paper, we proposed a test of the rank of the sub-matrix of cointegration. The test statistic is constructed by using the eigenvalues of the quadratic form of the sub-matrix. For non-trending data, the test statistic converges in distribution to a \( \chi^2 \) distribution under the null hypothesis, while for trending data, the test is conservative in general. Finite sample simulations reveal that, although the simulation settings are limited, the proposed test works well both under the null and the alternative hypotheses with a moderate sample
size, \( T = 100 \) and 200.
Appendix

In this appendix, we use the notation $H$ and $J$ alternately for different definitions if there is no confusion.

**Proof of Theorem 1:** First, note that we can replace $\tilde{\beta}_1$ and $\beta$ by $\tilde{\beta}_1$ and $\beta$ in (8), where $\tilde{\beta}_1$ is the first $n_1$ rows of $\tilde{\beta}$, because $\tilde{\beta} \tilde{\alpha}' = \beta \alpha'$ and $\tilde{\beta}_1(\beta' \beta)^{-1}\beta_1' = \beta_1(\beta' \beta)^{-1}\beta_1'$. The latter relation is established because $\tilde{\beta}$ is obtained by the non-singular transformation of the columns of $\beta$ (see (4)) and $\beta_1(\beta' \beta)^{-1}\beta_1'$ does not depend on the normalization of $\tilde{\beta}$ and $\beta_1$.

We also define

$$\tilde{\beta}_\perp = \beta_\perp - \beta(\beta' \beta)^{-1}\beta' \beta_\perp.$$  (24)

Since $\beta' \beta_\perp = 0$ and $\beta_\perp$ has full column rank, the columns of $\tilde{\beta}_\perp$ span the orthogonal complement to $sp(\beta)$, so that $\tilde{\beta}_\perp$ and $\beta_\perp$ span the same column space. This implies that $\tilde{\beta}_\perp$ can be obtained by the non-singular transformation of the columns of $\beta_\perp$. Then, we can also replace $\tilde{\beta}_\perp$ by $\beta_\perp$ in (8).

Under the null hypothesis, $rk(\beta_1)$ is $f$ and then an $n_1 \times f$ matrix $\beta_1^*$ exists with rank $f$ such that $sp(\beta_1) = sp(\beta_1^*)$. We denote the orthogonal complement to $\beta_1^*$ by $\delta^*$. That is, $\delta^*$ is an $n_1 \times (n_1 - f)$ matrix with rank $(n_1 - f)$ such that $\delta^* \beta^*_1 = 0$. Note that the $n_1 \times n_1$ square matrix $[\beta^*_1, \delta^*]$ has full rank $n_1$.

**Lemma 1:** (i) $\tilde{\beta} \xrightarrow{p} \beta$, $\tilde{\alpha} \xrightarrow{p} \alpha$, $\tilde{\Sigma} \xrightarrow{p} \Sigma$.

(ii) $T \delta^* \tilde{\beta}_1 = T \delta^* (\tilde{\beta}_1 - \beta_1) \xrightarrow{d} \delta^* \beta_\perp (\beta_1^* \beta_\perp)^{-1}(\int G_0 G_0' ds)^{-1} \int G_0 dV' = X_0'$, say.

(iii) $T(\tilde{\beta}_\perp - \beta_\perp) \xrightarrow{d} \beta(\beta' \beta)^{-1} \int dV G_0(\int G_0 G_0' ds)^{-1}$.

(iv) $T^{-1} \beta_\perp S_{11} \beta_\perp \xrightarrow{d} \int G_0 G_0' ds$.

Proof: (i) is proved by Johansen (1988, 1996).

(ii) As shown in Chapter 13.2 of Johansen (1996), $\tilde{\beta}$ can be expressed as $\tilde{\beta} = \beta + \beta_\perp (\beta_1^* \beta_\perp)^{-1} U_T$ for non-trending data, where $TU_T$ converges in distribution to $(\int G_0 G_0' ds)^{-1} \int G_0 dV'$. Since $\tilde{\beta}_1$ is the first $n_1$ rows of $\tilde{\beta}$, we have $\tilde{\beta}_1 = \beta_1 + \beta_\perp (\beta_1^* \beta_\perp)^{-1} U_T$, so that

$$T \delta^* \tilde{\beta}_1 = T \delta^* (\tilde{\beta}_1 - \beta_1)$$

$$= \delta^* \beta_\perp (\beta_1^* \beta_\perp)^{-1} (TU_T)$$
\[ \frac{d}{dt} \delta^* \beta_{\perp,1}(\bar{\beta}_1 \beta_{\perp})^{-1} \left( \int G_0 G_0' ds \right)^{-1} \int G_0 dV', \]

where the first equation holds because \( \delta^* \beta_1 = 0 \).

(iii) This convergence holds because \( T(\bar{\beta}_1 - \beta_1) = -\beta(\bar{\beta}' \beta)^{-1}(\bar{\beta} - \beta)' \beta_{\perp} T = -\beta(\bar{\beta}' \beta)^{-1}(TU_T)' \) from (24).

(iv) Note that

\[
\frac{1}{T} \beta_1 S_{11} \bar{\beta}_1 = \frac{1}{T} \beta'_1 S_{11} \bar{\beta}_1 + \frac{1}{T} (\bar{\beta}_1 - \beta_1) S_{11} \bar{\beta}_1 + \frac{1}{T} \beta_1 S_{11} (\bar{\beta}_1 - \beta_1) + \frac{1}{T} (\beta_1 - \beta_1)' S_{11} (\bar{\beta}_1 - \beta_1). \]

The first term converges in distribution to \( \int G_0 G_0' ds \) from Johansen (1988, 1996), while the remaining terms converge in probability to zero because \( (\bar{\beta}_1 - \beta_1) \) and \( S_{11} \) are of order \( T^{-1} \) and \( T \), respectively. \( \square \)

Now, let us consider the determinant equation (8). Since (8) is equivalent to

\[ |H'| |\bar{\beta}_1 \Psi \bar{\beta}'_1 - \lambda \bar{\Phi}|H| = 0, \quad (25) \]

where \( H \) is any \( n \times n \) non-singular matrix, we consider (25) with \( H = [\beta^*_1, T\delta^*] \). Then, using Lemma 1, we have

\[
H' \bar{\beta}_1 \Psi \bar{\beta}'_1 H = \begin{bmatrix}
\beta^*_1 \bar{\beta}_1 \Psi \bar{\beta}'_1 \beta^*_1 & \beta^*_1 \bar{\beta}_1 \Psi (\bar{\beta}'_1 T) \\
(T \bar{\delta}' \bar{\beta}_1) \Psi \bar{\beta}'_1 \beta^*_1 & (T \bar{\delta}' \bar{\beta}_1) \Psi (\bar{\beta}'_1 T)
\end{bmatrix}
\[ \xrightarrow{d} \]

\[
\begin{bmatrix}
\beta^*_1 \beta_1 \Psi \beta_1 \Psi X_0 & \beta^*_1 \beta_1 \Psi X_0 \\
X_0 \Psi \beta_1 \Psi X_0 & X_0 \Psi X_0
\end{bmatrix} \quad (26) \]

To investigate the asymptotic behavior of \( H' \bar{\Phi} H \), we consider \( \bar{\Phi} \) with the same expression as (9). Note that

\[
H' \left[ \bar{\beta}_1, \bar{\beta}_{\perp,1}(\bar{\beta}_1 \bar{\beta}_{\perp})^{-1} \right] = \begin{bmatrix}
\beta^*_1 \bar{\beta}_1 & \beta^*_1 \bar{\beta}_{\perp,1}(\bar{\beta}'_1 \bar{\beta}_{\perp})^{-1} \\
T \delta^* \bar{\beta}_1 & T \delta^* \bar{\beta}_{\perp,1}(\bar{\beta}'_1 \bar{\beta}_{\perp})^{-1}
\end{bmatrix}
\[ = \]

\[
\begin{bmatrix}
O_p(1) & O_p(1) \\
O_p(1) & T \delta^* \bar{\beta}_{\perp,1}(\bar{\beta}_1 \bar{\beta}_{\perp})^{-1} + o_p(T)
\end{bmatrix} \]

because \( \beta^*_1 \bar{\beta}_1 \xrightarrow{p} \beta^*_1 \beta_1, \beta^*_1 \bar{\beta}_{\perp,1} \xrightarrow{p} \beta^*_1 \beta_{\perp,1}, T \delta^* \bar{\beta}_1 = O_p(1) \) and \( T \delta^* \bar{\beta}_{\perp,1} = T \delta^* \beta_{\perp,1} + o_p(T) \) by Lemma 1. Then, \( \lambda H' \bar{\Phi} H \) is asymptotically equivalent to

\[
T^2 \lambda \begin{bmatrix}
0 & 0 \\
0 & \delta^* \bar{\beta}_{\perp,1}(\bar{\beta}_1 \bar{\beta}_{\perp})^{-1}(\int G_0 G_0' ds)^{-1}(\beta_1 \beta_{\perp})^{-1}(\bar{\beta}_1 \bar{\beta}_{\perp})^{-1} \beta^* \end{bmatrix}. \]

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Then, the equation (25) is asymptotically equal to
\[
\begin{bmatrix}
\beta_1'\beta_1\Psi\beta_1'\beta_1 & \beta_1'\beta_1\Psi X_0 \\
X_0'\Psi\beta_1'\beta_1 & X_0'\Psi X_0
\end{bmatrix} - T^2\hat{\lambda} \begin{bmatrix} 0 \\ 0 \end{bmatrix} 
\delta_\perp'(\beta_\perp'\beta_\perp)^{-1}\left(\int G_0 G_0' ds\right)^{-1}(\beta_\perp'\beta_\perp)^{-1}\beta_\perp'\delta_*^r
\]
\[
= |\beta_1'\beta_1\Psi\beta_1'\beta_1| 
\times \left|X_0'\{\Psi - \Psi\beta_1'\beta_1(\beta_1'\beta_1\Psi\beta_1'\beta_1)^{-1}\beta_1'\beta_1\Psi\}\right| X_0
\]
\[
- T^2\hat{\lambda}\delta_\perp'(\beta_\perp'\beta_\perp)^{-1}\left(\int G_0 G_0' ds\right)^{-1}(\beta_\perp'\beta_\perp)^{-1}\beta_\perp'\delta_*^r = 0.
\]
Therefore, the eigenvalues \(\hat{\lambda}_{r+1} \cdots \hat{\lambda}_p\) converge in probability to zeros and are of order \(T^2\).

Here, notice that, in the same way as Johansen (1988, p.246), we can find a \(n \times (r-f)\) matrix with rank \((r-f)\) such that
\[
JJ' = \Psi - \Psi\beta_1'\beta_1(\beta_1'\beta_1\Psi\beta_1'\beta_1)^{-1}\beta_1'\beta_1\Psi,
\]
with \(J'(\beta_1'\beta_1) = 0\) and \(J'\Psi^{-1}J = I_{r-f}\), implying that \(J'(\alpha'\Sigma^{-1}\alpha)^{-1}J = I_{r-f}\) because \(\Psi = \alpha'\Sigma^{-1}\alpha\). Then, since \(|\beta_1'\beta_1\Psi\beta_1'\beta_1| \neq 0\), (27) becomes
\[
\left|X_0'JJ'X_0 - T^2\hat{\lambda}\delta_\perp'(\beta_\perp'\beta_\perp)^{-1}\left(\int G_0 G_0' ds\right)^{-1}(\beta_\perp'\beta_\perp)^{-1}\beta_\perp'\delta_*^r \otimes I_{r-f}\right| = 0.
\]
Since the variance matrix of \(X_0'J\) conditioned on \(G_0(\cdot)\) is
\[
\delta_\perp'(\beta_\perp'\beta_\perp)^{-1}\left(\int G_0 G_0' ds\right)^{-1}(\beta_\perp'\beta_\perp)^{-1}\beta_\perp'\delta_*^r \otimes I_{r-f},
\]
we can see that \(T^2\hat{\lambda}\) converges in distribution to \(\lambda^*\), which is a solution of
\[
|X_0'X_0^* - \lambda^*I_{n_1-f}| = 0,
\]
where \(X_0^*\) is an \((n_1 - f) \times (r-f)\) matrix with \(vec(X_0^*) \sim N(0, I_{n_1-f} \otimes I_{r-f})\). Then, \(\mathcal{L}_T\) converges in distribution to the trace of \(X_0'X_0^*\), which proves Theorem 1. □

**Proof of Theorem 2:** In the same way as the proof of Theorem 1, we replace \(\hat{\beta}, \hat{\beta}_\perp\) and \(\hat{\beta}_{\perp,1}\) by \(\hat{\beta}, \hat{\beta}_\perp\) and \(\hat{\beta}_{\perp,1}\).

Under the null hypothesis, an \(n_1 \times g\) matrix \(\beta_{\perp,1}^*\) exists such that \(sp(\beta_{\perp,1}^*) = sp(\beta_{\perp,1})\) and \(rk(\beta_{\perp,1}^*) = g\). We can also find an \(n_1 \times (n_1 - g)\) matrix \(\eta^*\) with rank \((n_1 - g)\) that is orthogonal to \(\beta_{\perp,1}^*\). Here, we consider the following determinant equation.
\[
|H'||\hat{\beta}_{\perp,1}\hat{\Psi}\hat{\beta}_{\perp,1} - \hat{\mu}\hat{\Phi}|H| = 0,
\]

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where \( H = [\beta_{1,1}^*, T\eta^*] \). Since \( \beta_{1,1}^* \) is the first \( n_1 \) rows of \( \beta_{1,1}^* \), we obtain, using Lemma 1 (iii),

\[
T\eta'^*\beta_{1,1}^* = T\eta'^*(\beta_{1,1}^* - \beta_{1,1}^*) \xrightarrow{d} -\eta'^*\beta_1(\beta'\beta)^{-1} \int dV G_0' \left( \int G_0' d\sigma \right)^{-1} = Y_0', \text{ say},
\]

and \( \beta_{1,1}^* \beta_{1,1}^* - \beta_{1,1}^* \beta_{1,1}^* \rightarrow P \beta_{1,1}^* \beta_{1,1}^* \). Then,

\[
H' \beta_{1,1}^* \Psi' \beta_{1,1}^* H \xrightarrow{d} \begin{bmatrix}
\beta_{1,1}^* \beta_{1,1}^* \Psi' \beta_{1,1}^* \\
Y_0' \beta_{1,1}^* \beta_{1,1}^* \Psi Y_0
\end{bmatrix}.
\]

In addition, we can see that \( \beta_{1,1}^* \beta_{1,1}^* \rightarrow P \beta_{1,1}^* \beta_{1,1}^* \), \( T\eta'^* \beta_{1,1}^* = O_p(1) \) and \( T\eta'^* \beta_{1,1}^* = T\eta'^* \beta_{1,1}^* + o_p(T) \). Then, similar to the previous proof, the determinant equation (31) is asymptotically equivalent to

\[
\begin{bmatrix}
\beta_{1,1}^* \beta_{1,1}^* \Psi' \beta_{1,1}^* \\
Y_0' \beta_{1,1}^* \beta_{1,1}^* \Psi Y_0
\end{bmatrix} - T^2 \bar{\mu} \begin{bmatrix} 0 & 0 \\ 0 & \eta'^* \beta_1(\beta'\beta)^{-1}(\alpha'\Sigma^{-1}\alpha)^{-1}(\beta'\beta)^{-1} \beta' \eta^* \end{bmatrix}
\]

\[
= \begin{bmatrix}
\beta_{1,1}^* \beta_{1,1}^* \Psi' \beta_{1,1}^* \\
Y_0' \beta_{1,1}^* \beta_{1,1}^* \Psi Y_0
\end{bmatrix} - T^2 \bar{\mu} \eta'^* \beta_1(\beta'\beta)^{-1}(\alpha'\Sigma^{-1}\alpha)^{-1}(\beta'\beta)^{-1} \beta' \eta^* = 0.
\]

Then, \( \bar{\mu}_{g+1}, \cdots, \bar{\mu}_q \) are of order \( T^{-2} \). For a given \( G_0(\cdot) \), we can find an \( (n-r) \times (n-r-g) \) matrix with rank \( (n-r-g) \) such that

\[
JJ' = \Psi' - \Psi' \beta_{1,1}^* \beta_{1,1}^* (\beta_{1,1}^* \beta_{1,1}^* \Psi' \beta_{1,1}^* \beta_{1,1}^*)^{-1} \beta_{1,1}^* \beta_{1,1}^* \Psi
\]

with \( J'(\beta_{1,1}^* \beta_{1,1}^*) = 0 \) and \( J'(f G_0 G_0' ds)^{-1} = I_{n-r-g} \) because \( \Psi = \int G_0 G_0' ds \). Then, (32) becomes

\[
\begin{bmatrix}
Y_0' JJ' Y_0 - T^2 \bar{\mu} \eta'^* \beta_1(\beta'\beta)^{-1}(\alpha'\Sigma^{-1}\alpha)^{-1}(\beta'\beta)^{-1} \beta' \eta^* 
\end{bmatrix} = 0.
\]

Since the variance matrix of \( Y_0' J \) conditioned on \( G_0(\cdot) \) is given by

\[
\eta'^* \beta_1(\beta'\beta)^{-1}(\alpha'\Sigma^{-1}\alpha)^{-1}(\beta'\beta)^{-1} \beta' \eta^* \otimes I_{n-r-g},
\]

we can see that \( T^2 \bar{\mu} \) converges in distribution to \( \mu^* \), which is a solution of

\[
|Y_0' Y^* - \mu^* I_{n_1-g}| = 0,
\]
where $Y_0^{*'}$ is an $(n_1 - g) \times (n - r - g)$ matrix with $\text{vec}(Y_0^{*'}) \sim N(0, I_{n_1 - g} \otimes I_{n - r - g})$. This proves Theorem 2. □

**Proof of Theorems 3 and 4:** Let $\hat{\beta}^+ = [\hat{\beta}', \hat{\rho}]'$ and $\tilde{\beta}^+ = [\tilde{\beta}', \tilde{\rho}]'$. Exactly in the same way as the proof of Lemma 13.2 in Johansen (1996), we can show that

$$
\begin{bmatrix}
T\beta_1' - 0 \\
0 \\
T^{1/2}
\end{bmatrix}
(\bar{\beta}^+ - \hat{\beta}^+)
\xrightarrow{d}
\left(\int G_0^+ G_0^{+'} ds\right)^{-1} \int G_0^+ dV',
$$

where $G_0^+ = [G_0', 1]'$. Then, since $\tilde{\beta}$ is the first $n$ rows of $\tilde{\beta}^+$, we have

$$
T\beta_1' (\tilde{\beta} - \hat{\beta}) \xrightarrow{d} L' \left(\int G_0^+ G_0^{+'} ds\right)^{-1} \int G_0^+ dV',
$$

whose conditional variance is given by $L' \left(\int G_0^+ G_0^{+'} ds\right)^{-1} L$. Since $\tilde{\beta} = \beta + \beta_1 (\beta_1' \beta_1)^{-1} \beta_1' \tilde{\beta}$ as expressed in Johansen (1996, p. 179), we have

$$
T\delta_1' \tilde{\gamma}_1 \xrightarrow{d} L' \left(\int G_0^+ G_0^{+'} ds\right)^{-1} \int G_0^+ dV'.
$$

We also have

$$
\tilde{\Upsilon}_T S_{11}^+ \tilde{\Upsilon}_T \xrightarrow{d} \int G_0^+ G_0^{+'} ds,
$$

which is proved as Lemma 1(iv), where $\tilde{\Upsilon}_T$ is $\Upsilon_T$ with $\tilde{\beta}_1$ replaced by $\tilde{\beta}_1$. Then, Theorems 3 and 4 is proved in the same way as Theorems 1 and 2. □

**Proof of Theorem 5:** First, we give the following lemma.

**Lemma 2** (i) $\tilde{\gamma} \xrightarrow{p} \gamma$ and $\tilde{\gamma}_1 \xrightarrow{p} \gamma_1$, where

$$
\tilde{\gamma} = \gamma - \gamma_1 (\tilde{\gamma}_1' \gamma_1)^{-1} \tilde{\gamma}_1' \gamma,
$$

and $\tilde{\gamma}_1$ is the first $n_1$ rows of $\tilde{\gamma}$ with $\gamma_1 = [\beta, \tau]$ and $\tilde{\gamma}_1 = [\tilde{\beta}, \tilde{\tau}]$.

(ii) $T^{1/2} (\hat{\tau} - \tau) \xrightarrow{d} CW(1)$.

(iii) $\tilde{\Omega}_{11} \xrightarrow{d} \Omega_{11}$, where $\tilde{\Omega}_{11}$ is defined as $\hat{\Omega}_{11}$ with $\gamma$ replaced by $\tilde{\gamma}$.

**Proof:** (i) Since $\tilde{\gamma}_1 \xrightarrow{p} \gamma_1$ and $\gamma_1' \gamma = 0$, $\tilde{\gamma}$ converges in probability to $\gamma$.

(ii) This is proved by Johansen (1991, 1996).
(iii) Letting

$$T\tilde{\gamma}'S_{11}^{-1}\tilde{\gamma} = (T^{1/2}\tilde{\gamma}'K)(K'S_{11}K)^{-1}(T^{1/2}K'\tilde{\gamma}).$$

From Lemma 10.3 in Johansen (1996), $T^{-1}[\tilde{\gamma}, T^{-1/2}\tilde{\gamma}]'S_{11}[\tilde{\gamma}, T^{-1/2}\tilde{\gamma}]$ converges in distribution to $\Omega$ while $\beta'S_{11}\beta$ converges in probability to a positive definite matrix, $\Sigma_{\beta}$, and $[\tilde{\gamma}, T^{-1/2}\tilde{\gamma}]'S_{11}\beta = O_p(1)$. Then,

$$K'S_{11}K \xrightarrow{d} \begin{bmatrix} (\beta'\beta)^{-1}\Sigma_{\beta}(\beta'\beta)^{-1} & 0 \\ 0 & \Omega \end{bmatrix}. \quad (35)$$

In addition, we can see that

$$T\beta'\tilde{\gamma} = -T\beta'(\tilde{\gamma}\perp) \tilde{\gamma}_{\perp}^{-1} \tilde{\gamma}_{\perp}' \gamma = -[\beta'\beta, 0](\gamma_{\perp}^{-1} [\beta' - \beta']'T)(\gamma_{\perp}^{-1} [\tilde{\gamma}' - \tilde{\gamma}]'T) \xrightarrow{d} -U'_{1}\beta$$

because $\beta'\tau = 0$ and $\tilde{\gamma}'\gamma = O_p(T^{1/2})$. Using this result, we have

$$T^{1/2}K'\tilde{\gamma} = \begin{bmatrix} T^{1/2}(\beta'\beta)^{-1}\beta'\tilde{\gamma} \\ (\gamma_{\gamma}^{-1} \gamma_{\gamma})^{-1} \gamma_{\gamma}^{-1} \gamma_{\gamma}' \tilde{\gamma} \\ T^{-1/2}(\tau'\tau)^{-1} \tau'\tilde{\gamma} \end{bmatrix} \xrightarrow{p} \begin{bmatrix} 0 \\ I_{m-r-1} \\ 0 \end{bmatrix}. \quad (36)$$

From (35) and (36), $\tilde{\Omega}^{11}$ converges in distribution to $\Omega^{11}$. □

For the same reason as the previous proofs, we replace $\hat{\beta}$, $\hat{\beta}_1$, $\hat{\gamma}$ and $\hat{\gamma}_1$ by $\tilde{\beta}$, $\tilde{\beta}_1$, $\tilde{\gamma}$ and $\tilde{\gamma}_1$.

(i-a) We consider the same determinant equation as (25) with $H = [\beta_*', T\delta^*]$. Using (20), we can see that $H'\tilde{\beta}_1\tilde{\Psi}\tilde{\beta}_1'\gamma$ converges to the same limit as (26), replacing $X_0$ by $X$. On the other hand, because $\beta_*'\tilde{\beta}_1 \xrightarrow{p} \beta_1'\beta_1$, $\beta_*'\tilde{\gamma}_1 \xrightarrow{p} \beta_1'\gamma_1$, $T\delta^*\tilde{\beta}_1 = O_p(1)$, $T\delta^*\tilde{\gamma}_1 = T\delta^*\gamma_1 + o_p(T)$ and $\tilde{\Omega}^{11} \xrightarrow{d} \Omega^{11}$ by Lemma 2, $\lambda H'\tilde{\beta}H$ is asymptotically equivalent to

$$T^2\lambda \begin{bmatrix} 0 & 0 \\ 0 & \delta^*\gamma_1 (\gamma_{\gamma}^{-1}\Omega^{11}(\gamma_{\gamma}^{-1}\gamma_{\gamma})^{-1}\gamma_{\gamma})^{-1}\delta^* \end{bmatrix}. $$

Then, similar to (27), as far as the limiting distribution is concerned, it is sufficient to consider

$$|\beta_*'\beta_1 \tilde{\Psi}\beta_*'\beta_1'| \left| X'J'JX - T^2\lambda\delta^*\gamma_1 (\gamma_{\gamma}^{-1}\Omega^{11}(\gamma_{\gamma}^{-1}\gamma_{\gamma})^{-1}\gamma_{\gamma})^{-1}\delta^* \right| = 0,$$
where $J$ is the same $r \times (r - f)$ matrix as in (28). Since $vec(X'J)$ conditioned on $G(\cdot)$ is normally distributed with a variance matrix $\delta' \gamma_1 (\gamma' \gamma)^{-1} \Omega_{11} (\gamma' \gamma)^{-1} \gamma_1 \delta^* \otimes I_{r-f}$, $\mathcal{L}_T$ converges in distribution to $\chi^2(a_{n-f})$. 

(i-b) We consider the same determinant equation as in the proof of (i-a). We can easily see that $H' \tilde{\beta}_1 \tilde{\Psi} \tilde{\beta}_1^* H$ converges to the same limit as (i-a), while $\hat{\lambda} H' \tilde{\Phi} H$ is asymptotically equivalent to

$$T^2 \tilde{\lambda} \begin{bmatrix} 0 & 0 \\ 0 & \delta' \gamma_1 (\gamma' \gamma)^{-1} \Omega_{11} (\gamma' \gamma)^{-1} \gamma_1 \delta^* + \delta' \tau_1 \tau_1^* \delta^* \end{bmatrix} ,$$

because $\beta_1^r \tau_1 \overset{p}{\rightarrow} \beta_1^s \tau_1$ and $T \delta^r \tau_1 = T \delta^s \tau_1 + o_p(T)$ by Lemma 2. Then, similar to (27), as far as the limiting distribution is concerned, it is sufficient to consider,

$$|\beta_1^r \beta_1^s \beta_1^s | X' J J' X - T^2 \tilde{\lambda} \left\{ \delta' \gamma_1 (\gamma' \gamma)^{-1} \Omega_{11} (\gamma' \gamma)^{-1} \gamma_1 \delta^* + \delta' \tau_1 \tau_1^* \delta^* \right\} = 0,$$

where $J$ is the same $r \times (r - f)$ matrix as (28). Here, note that, in general, for a given symmetric and positive definite matrix $A$ and a vector $b$,

$$(A + bb')^{-1} = A^{-1} - A^{-1} bb'A^{-1}/(1 + b'A^{-1}b),$$

and then,

$$c'(A + bb')^{-1}c \leq c'A^{-1}c$$

for any non-zero vector $c$. By substituting $\delta' \gamma_1 (\gamma' \gamma)^{-1} \Omega_{11} (\gamma' \gamma)^{-1} \gamma_1 \delta^*$ and $\delta' \tau_1$ for $A$ and $b$, we obtain, for a given $G(\cdot)$,

$$tr \left( J' X \{ \delta' \gamma_1 (\gamma' \gamma)^{-1} \Omega_{11} (\gamma' \gamma)^{-1} \gamma_1 \delta^* + \delta' \tau_1 \tau_1^* \delta^* \} -1 X' J \right) \leq tr \left( J' X \{ \delta' \gamma_1 (\gamma' \gamma)^{-1} \Omega_{11} (\gamma' \gamma)^{-1} \gamma_1 \delta^* \} -1 X' J \right) = tr(X^* X^*) = \chi^2_{(r-f) (n_1 - f)} ,$$

where $X^*$ is a $(r - f) \times (n_1 - f)$ matrix with $vec(X^*) \sim N(0, I_{(r-f)(n_1-f)})$. The equality is established if and only if $\delta' \tau_1 = 0$.

(ii) Let us consider the determinant equation (25) with $H = [\beta_1^r, T^* \delta_0^* , T \tau_1^*]$. Using (18) and (19), we have

$$H' \tilde{\beta}_1 \tilde{\Psi} \tilde{\beta}_1^* H \overset{d}{\rightarrow} \begin{bmatrix} \beta_1^r \beta_1^s \beta_1^s \beta_1^r & \beta_1^r \beta_1^s \beta_1^r X_1 \\ X_1^t \beta_1^s \beta_1^s & X_1^t \beta_1^s X_1 \end{bmatrix} .$$

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On the other hand, because $\beta_1^T\hat{\beta}_1 \xrightarrow{p} \beta_1^T\beta_1$, $\beta_1^T\hat{\gamma}_1 \xrightarrow{p} \beta_1^T\gamma_1$, $\beta_1^T\hat{\tau}_1 \xrightarrow{p} \beta_1^T\tau_1$, $T\delta_0^T\hat{\beta}_1 = O_p(1)$, $T\delta_0^T\hat{\gamma}_1 = T\delta_0^T\gamma_1 + o_p(T)$, $T\delta_0^T\hat{\tau}_1 = T\delta_0^T\tau_1 + o_p(T)$, $T\tau_1^T\hat{\beta}_1 = O_p(T^{-1/2})$, $T\tau_1^T\hat{\gamma}_1 = O_p(T^{-1/2})$, $T\tau_1^T\hat{\tau}_1 = T\tau_1^T\tau_1 + o_p(T)$ and $\tilde{\Omega}^{11} \xrightarrow{d} \Omega^{11}$ by Lemma 2, $\hat{\lambda}H^T\hat{\Phi}H$ is asymptotically equivalent to

$$T^2\hat{\lambda} \begin{pmatrix} 0 & \delta_0^T\gamma_1(\gamma'\gamma)^{-1}\Omega^{11}(\gamma'\gamma)^{-1}\gamma_1^T\delta_0 + \delta_0^T\tau_1\gamma_1^T\delta_0 & 0 \\ \tau_1^T\gamma_1^T\delta_0 & \tau_1^T\gamma_1^T\delta_0 & 0 \end{pmatrix}. $$

Then, after some algebra, we can see that (25) is asymptotically equal to

$$|\beta_1^T\hat{\beta}_1\Psi[\beta_1^T]|^{-2}\lambda(\tau_1^T\tau_1)^2\|X_1J'X_1 - T^2\hat{\lambda}H^T\hat{\Phi}H\| = 0,$$

where $J$ is the same $r \times (r - f)$ matrix as (28). This determinant equation implies that there are $f$ non-zero eigenvalues, $p - f - 1$ eigenvalues of order $T^{-2}$, and one eigenvalue of order smaller than $T^{-2}$. In the same way as in the previous proofs of the theorems, we can see that

$$T^2 \sum_{i=f+1}^{p-1} \frac{d}{\lambda_i} \xrightarrow{d} \chi^2_{(n_1-f-1)(r-f)}.$$

Since $T^2\hat{\lambda}_p \xrightarrow{p} 0$, we have

$$\mathcal{L}_T = T^2 \sum_{i=f+1}^{p} \frac{d}{\lambda_i} = T^2 \sum_{i=f+1}^{p-1} \frac{d}{\lambda_i} + o_p(1) \xrightarrow{d} \chi^2_{(n_1-f-1)(r-f)}.$$

We can also see that $\hat{\lambda}_p$ is of order $T^3$ if we choose $H = [\beta_1^T, T\delta_0^T, T^{3/2}\tau_1^*]^T$. $\square$

**Proof of Corollary 1:** (i) When $rk[\beta_1, \gamma_1] = n_1$, from (38) in the proof of Theorem 5, $T^2\hat{\lambda}_p$ converges in distribution to the $p - f$-th eigenvalue (the smallest non-zero eigenvalue) of

$$J'X \left\{ \delta^T\gamma_1(\gamma'\gamma)^{-1}\Omega^{11}(\gamma'\gamma)^{-1}\gamma_1^T\delta^* + \delta^T\tau_1\gamma_1^T\delta^* \right\}^{-1} X'J.$$

Here, note that, in general, for a given positive definite matrix $A$, a vector $b$ and a matrix $D$,

$$D' A^{-1} D = D' (A + bb')^{-1} D + D' A^{-1} bb' A^{-1} D / (1 + b' A^{-1} b),$$

where we used the relation (37). By Theorem 9 of Magnus and Neudecker (1988, p.208), we can see that the $p - f$-th eigenvalue of $D' A^{-1} D$ is larger than that of $D' (A + bb')^{-1} D$. Then,
by substituting $\delta^*\gamma_1(\gamma'\gamma)^{-1}\Omega^{11}(\gamma'\gamma)^{-1}\gamma'_1\delta^*$, $\delta^*\tau_1$ and $X'J$ for $A$, $b$ and $D$, respectively, we can see that the limiting distribution of $T^2\hat{\lambda}_p$ is bounded above by $\lambda^*_{\min}$, the smallest non-zero eigenvalue of

$$J'X \left\{\delta^*\gamma_1(\gamma'\gamma)^{-1}\Omega^{11}(\gamma'\gamma)^{-1}\gamma'_1\delta^*\right\}^{-1} X'J = X^*X^*,$$

where $X^*$ is an $(r-f) \times (n_1-f)$ matrix with $\text{vec}(X^*) \sim N(0,I_{(n-r)(n_1-f)})$. Note that $T^2\hat{\lambda}_p \xrightarrow{d} \lambda^*_{\min}$ if and only if $\delta^*\tau_1 = 0$.

(ii) This is proved in Theorem 5 (ii). \(\square\)

**Proof of Theorem 6**: Similar to the proof of Theorem 2, under the null hypothesis there exists an $n_1 \times g$ matrix $\beta^*_\perp,1$ with rank $g$ whose columns span the same space as $\text{sp}(\beta^*_\perp,1)$, and an $n_1 \times (n_1-g)$ matrix $\eta^*$ with rank $(n_1-g)$ that is orthogonal to $\beta^*_\perp,1$. For the same reason as before, we replace $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\gamma}_1$ by $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\gamma}_1$.

First, we give the convergence result of $\tilde{\gamma}_1$ and $\tilde{\tau}_1$, where

$$\tilde{\gamma}_1 = \gamma_1 - \gamma_{\perp,1}(\gamma'_\perp\gamma_{\perp})^{-1}\gamma'_\perp\gamma,$$  \tag{40}

with $\gamma_{\perp,1} = [\beta_1,\tau_1]$.

**Lemma 3** (i) $T\eta^*\tilde{\gamma}_1 \xrightarrow{d} -\eta^*\beta_1(\beta'\beta)^{-1}U'_1 = Y'$, say.

(ii) $T\eta^*\tilde{\tau}_1 \xrightarrow{p} 0$.

**Proof**: (i) Since $\eta^*\gamma_1 = 0$ and $\eta^*\tau_1 = 0$, we have, using (40),

$$T\eta^*\tilde{\gamma}_1 = T\eta^*(\tilde{\gamma}_1 - \gamma_1) = -[\eta^*\beta_1,0](\gamma_{\perp,1})^{-1}\begin{bmatrix} (\tilde{\beta} - \beta)'\gamma T \\ (\tilde{\tau} - \tau)'\gamma T \end{bmatrix} \xrightarrow{d} -\eta^*\beta_1(\beta'\beta)^{-1}U'_1,$$

where the last convergence is established because $(\tilde{\tau} - \tau)$ is $O_p(T^{-1/2})$ and $\tilde{\beta}'\tau$ is $O_p(T^{-3/2})$.

(ii) First, note that, because $\tilde{\tau}_1 = \hat{\beta}'_{\perp,1}(\hat{\alpha}'_{\perp,1}\hat{\Gamma}\hat{\beta}_{\perp,1})^{-1}\hat{\alpha}'_{\perp,1}\tilde{\mu}$, $\tilde{\tau}_1$ is invariant to each normalization of $\hat{\alpha}_{\perp,1}$ and $\hat{\beta}_{\perp,1}$. Then, we can express $\tilde{\tau}_1$ as

$$\tilde{\tau}_1 = \hat{\beta}'_{\perp,1}(\hat{\alpha}'_{\perp,1}\hat{\Gamma}\hat{\beta}_{\perp,1})^{-1}\hat{\alpha}'_{\perp,1}\tilde{\mu}.$$
From the expression (24), we can see that

\[
T\eta'\tilde{\beta}_{\perp,1} = T\eta'\tilde{\beta}_{\perp,1} - \beta_{\perp,1}
\]

\[
= -\eta'\beta_1(\tilde{\beta}'\beta)^{-1} \left[ (\tilde{\beta} - \beta)'\gamma T, (\tilde{\beta} - \beta)'\tau T \right]
\]

\[
\overset{d}{\longrightarrow} -\eta'\beta_1(\beta)'^{-1}U_1'[I_{n-r-1}, 0]. \tag{41}
\]

Next, from the definition of \(\tau\), we can see that

\[
\bar{\gamma}'\tau = [I_{n-r-1}, 0](\alpha'_{\perp,1} \Gamma\beta_{\perp,1})^{-1}\alpha'_{\perp,1}\mu.
\]

Since the left-hand side is zero from the orthogonality between \(\gamma\) and \(\tau\), the first \(n - r - 1\) rows of \((\alpha'_{\perp,1} \Gamma\beta_{\perp,1})^{-1}\alpha'_{\perp,1}\mu\) are zero. Then, because each estimator is consistent, we have

\[
[I_{n-r-1}, 0](\alpha'_{\perp,1} \hat{\Gamma}\tilde{\beta}_{\perp,1})^{-1}\alpha'_{\perp,1}\mu \overset{p}{\longrightarrow} 0. \tag{42}
\]

Combining (41) and (42), we obtain

\[
T\eta'\hat{\tau}_1 = (T\eta'\tilde{\beta}_{\perp,1})(\hat{\alpha}'_{\perp,1} \hat{\Gamma}\hat{\beta}_{\perp,1})^{-1}\hat{\alpha}'_{\perp,1}\mu \overset{p}{\longrightarrow} 0. \quad \square
\]

Similar to the proof of Theorem 2, we consider the same determinant equation as (31) with \(H = [\beta'_{\perp,1}, T\eta']\). Using Lemma 3, we have

\[
T\eta'(\tilde{\gamma}_1, \tilde{\tau}_1) \overset{d}{\longrightarrow} [Y', 0] = Y'S_1,
\]

where \(S_1 = [I_{n-r-1}, 0]\), and then, using \([\tilde{\gamma}_1, \tilde{\tau}_1] \overset{p}{\longrightarrow} [\gamma_1, \tau_1] = \beta_{\perp,1}\),

\[
H'(\tilde{\gamma}_1, \tilde{\tau}_1)[\hat{\Psi}[\tilde{\gamma}_1, \tilde{\tau}_1]]H' \overset{d}{\longrightarrow} \left[ \begin{array}{ccc} \beta'_{\perp,1}\beta_{\perp,1} & \beta'_{\perp,1}\beta_{\perp,1} & \beta'_{\perp,1}\beta_{\perp,1} \\ Y'S_1\hat{\Psi}[\beta'_{\perp,1}\beta_{\perp,1}] & Y'S_1\hat{\Psi}[\beta'_{\perp,1}\beta_{\perp,1}] & Y'S_1\hat{\Psi}[\beta'_{\perp,1}\beta_{\perp,1}] \end{array} \right].
\]

On the other hand, because \(\beta'_{\perp,1}\tilde{\gamma}_1 \overset{p}{\longrightarrow} \beta'_{\perp,1}\gamma_1, \beta'_{\perp,1}\tilde{\tau}_1 \overset{p}{\longrightarrow} \beta'_{\perp,1}\tau_1, \beta'_{\perp,1}\beta_1 \overset{p}{\longrightarrow} \beta'_{\perp,1}\beta_1, T\eta'\tilde{\gamma}_1 = O_p(1), T\eta'\tilde{\tau}_1 = o_p(1), \) and \(T\eta'\tilde{\beta}_1 = T\eta'\beta_1 + o_p(T)\) by Lemma 3, \(\hat{\mu}H'\hat{\Phi}H\) is asymptotically equivalent to

\[
T^2\hat{\mu} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta'\beta_1(\beta)'^{-1}(\alpha'\Sigma^{-1}\alpha)^{-1}(\beta)'^{-1}\beta_1\eta' \end{bmatrix}.
\]

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Then, for large values of $T$, the determinant equation (31) can be seen as

$$
\begin{vmatrix}
\beta_{1,1}' \beta_{1,1} \tilde{\psi} \beta_{1,1}' \beta_{1,1}'
\end{vmatrix}
\times
\left| Y'S_1 \{ \tilde{\psi} - \tilde{\psi} \beta_{1,1}' \beta_{1,1} \beta_{1,1}' \beta_{1,1}' \beta_{1,1}' \beta_{1,1}' \beta_{1,1}' \beta_{1,1}' \right| S'Y

- T^2 \mu \eta' \beta_1 (\beta' \beta)^{-1} (\alpha' \Sigma^{-1} \alpha)^{-1} (\beta' \beta)^{-1} \beta_1' \eta

\propto
\left| Y'S_1 JJ' S_1 Y - T^2 \mu \eta' \beta_1 (\beta' \beta)^{-1} (\alpha' \Sigma^{-1} \alpha)^{-1} (\beta' \beta)^{-1} \beta_1' \eta \right|

= 0,

(43)

where an $(n-r) \times (n-r-g)$ matrix $J$ satisfies $J' \tilde{\psi}^{-1} J = I_{n-r-g}$. Noting that the conditional variance of $Y'S_1 J$ is given by

$$
\eta' \beta_1 (\beta' \beta)^{-1} (\alpha' \Sigma^{-1} \alpha)^{-1} (\beta' \beta)^{-1} \beta_1' \eta \otimes J'S_1' \Omega^{11} S_1 J,
$$

the test statistic conditioned on $G(\cdot)$ converges in distribution to

$$
tr(Y'Y + J'S_1' \Omega^{11} S_1 JY) = tr(\eta' \beta_1 (\beta' \beta)^{-1} (\alpha' \Sigma^{-1} \alpha)^{-1} (\beta' \beta)^{-1} \beta_1' \eta \otimes J'S_1' \Omega^{11} S_1 J),
$$

(44)

where $vec(Y) \sim N(0, I_{n_1-g} \otimes I_{n-r-g})$ and $J = [J_1', J_2']'$. Since

$$
J' \tilde{\psi}^{-1} J = J_1' \Omega^{11} J_1 + J_2' J_2 = I_{n-r-g},
$$

the limiting distribution (44) is bounded above by

$$
tr(Y'Y) \leq tr(Y'Y + J_1' \Omega^{11} J_1 + J_2' J_2 Y) = tr(Y'Y) \sim X^2_{(n_1-g)(n-r-g)}.
$$

This proves the statement of Theorem 6.$\square$

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References


Table 1. Critical values of the $T^2\hat{\lambda}_p$ statistic

<table>
<thead>
<tr>
<th>$n_1 - f$</th>
<th>1</th>
<th>2</th>
<th>$r - f$</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>$1.54 \times 10^{-1}$</td>
<td>0.00381</td>
<td>0.0156</td>
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<td></td>
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<td>2</td>
<td>0.0197</td>
<td>$6.27 \times 10^{-5}$</td>
<td>0.103</td>
<td>0.00157</td>
<td>0.210</td>
<td>0.00638</td>
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<tr>
<td>3</td>
<td>0.114</td>
<td>0.0100</td>
<td>$3.97 \times 10^{-5}$</td>
<td>0.348</td>
<td>0.0510</td>
<td>9.81 $\times 10^{-4}$</td>
</tr>
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<td></td>
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<td>0.105</td>
<td>0.00396</td>
<td></td>
<td></td>
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<td>$0.00669$</td>
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<td>0.0704</td>
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<td></td>
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<td>5</td>
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<td>0.0455</td>
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Table 2. Rejection frequencies of the tests

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<th>$rk(\beta_1) = 1$</th>
<th>$rk(\beta_{\perp}, 1) = 0$</th>
<th>$rk(\beta_{\perp}, 1) = 1$</th>
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<tr>
<td><strong>Non-trending data with $d = 0$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>DGP1 $T = 100$</td>
<td>1.000</td>
<td>0.088</td>
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<tr>
<td>$T = 200$</td>
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<td>0.068</td>
<td>1.000</td>
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<tr>
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<td>0.077</td>
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<tr>
<td>DGP3 $T = 100$</td>
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<td>0.998</td>
<td>1.000</td>
<td>0.994</td>
</tr>
<tr>
<td>$T = 200$</td>
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<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td><strong>Non-trending data with $d = \alpha_{p_0}$</strong></td>
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<td></td>
<td></td>
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<tr>
<td>DGP1 $T = 100$</td>
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<td>$T = 200$</td>
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<td>0.071</td>
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<td><strong>Trending data</strong></td>
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### Table 3. Powers of the tests

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<td>(T)</td>
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<td>200</td>
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<td>(c_1) or (c_2)</td>
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<td>0.095</td>
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<td>0.994</td>
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<td>(c_1) or (c_2)</td>
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