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Downsian Model with Asymmetric Information: Possibility of Policy Divergence

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Abstract

This paper presents a model of Downsian political competition in which voters are imperfectly informed about economic fundamentals. In this setting, parties' choices of platforms influence voters' behavior not only through voters' preferences over policies, but also through formation of their expectation on the unknown fundamentals. We show that there exist pure-strategy equilibria in this political game with asymmetric information at which the two parties' policies diverge with positive probability. This result is in contrast with the well-known median voter theorem in the classical model of Downsian competition. We also study refinement of equilibria, and identify the perfect equilibria (Selten, 1975) and the strictly perfect equilibria (Okada, 1981). The Nash equilibria with the strongest asymmetry in the parties' strategies are proved to be strictly perfect.

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1 Introduction

The classical Downsian model of political competition has a well-known theoretical result called Median Voter Theorem (MVT), which states that under some natural assumptions, two office-seeking parties will announce the same platform: the median voter's ideal policy. Whereas the model is widely accepted, an inconsistency between the conclusion of MVT and real phenomena is often pointed out. In empirical studies, policy divergence, rather than convergence, between parties seems to be dominant. Therefore, it is important to construct an alternative model that can explain the real data.

With this basic motivation, this paper presents a Downsian electoral model with two policy alternatives in which voters have only incomplete information about the value of a "fundamentals" variable affecting the relative effectiveness of these policies. Two parties observe a realized value of the variable, and then simultaneously announce their platforms. Observing these platforms, voters choose a party to vote for. In this setting, parties' choices of platforms influence voters' behavior not only through voters' preferences over policies, but also through formation of their expectation on the fundamentals.

The assumption of incomplete information about the fundamentals on the side of voters reflects the idea that, in actual elections, some data necessary for evaluation of policies is often unfamiliar to voters, while parties have richer knowledge obtained perhaps through research activities. In such cases, voters seem to attribute observed political positions of parties to particular information of fundamentals which the parties have probably obtained prior to the determination of platforms.

For example, when redistributive policy is at issue, the fundamentals variable may summarize information about the extent to which taxation on income deteriorates the macroeconomic performances by lowering labor incentives. When there is a stable situation in which a party is "leftist", i.e., when this party is more likely to adopt a progressive tax policy than its opponent (the "rightist"), voters would expect higher average income elasticity of labor from observation of the leftist party's choice of the progressive tax than from observation of the rightist party's choice of the same policy. This paper is an attempt to explain how such interactions between strategies and expectation formation constitute an equilibrium in an election over the general issue.

In this political game with asymmetric information, we identify the pure-strategy Nash equilibria. We show that there exist Nash equilibria at which the two parties' policies diverge with positive probability. We then study refinement of equilibria, and identify the perfect equilibria (Selten, 1975) and the strictly perfect equilibria (Okada, 1981). The perfect equilibrium excludes Nash equilibria at which both parties are very likely to choose a policy that is unpopular among voters with the prior informationabout fundamentals. The Nash equilibria exhibiting the strongest asymmetry between parties' strategies are strictly perfect. The last result, in particular, is in marked contrast with the conclusion of MVT.

There are several studies related to the present paper either in concern with policy divergence or in focus on incomplete information of political games. Roemer (2001) shows that in a unidimensional Wittmanian electoral model, i.e., a political game with a unidimensional policy space in which parties are motivated to realize their ideal policies, introduction of parties' incomplete information on the side of parties about the distribution of voters' types generates an equilibrium with differentiated policies. Contrary to his hypothesis of voters' informational advantage over parties, we assume parties' advantage. The previous example of elections over redistributive policy illustrates a typical situation where our assumption fits. Furthermore, whereas parties' uncertainty in Roemer's model plays a subordinate role complementing the Wittmanian hypothesis, in our Downsian model, incomplete information for voters is the sole factor causing policy divergence.

Banks (1990, 1991) models voters' incomplete information about the candidates' true types, where a type of a candidate represents a policy that he will implement if elected. He shows that, if there exists a cost for each candidate which is increasing in the distance between his true type and his platform, then an equilibrium possesses some interval of types where the strategy is separating. A more recent work by Kartik and McAfee (2007) constructs a model with "character" of candidates. Each candidate either has the character or not; if he does, he commits to a platform, and if not, he strategically chooses a policy. A unique mixed equilibrium strategy of strategic candidates is explicitly constructed, and hence the equilibrium is symmetric, but different from that in the conclusion of MVT. The models in these studies share with ours the basic information structure in which candidates are advantageous. However, they both impose some additional assumption on candidates' action ex post or after election, while we have no such assumption. Also, an asymmetric equilibrium does not arise, or at least is not proved to exist, in either model, whereas it exists in our model and one such equilibrium is even strictly perfect.

The paper is organized as follows: in Section 2, we model political competition as a dynamic incomplete information game. Section 3 studies the weakly perfect Bayesian equilibria of incomplete infomation political games describing the conditions required for voters' beliefs which support the equilibria. Section 4 contrasts this equilibrium result with the complete information version of our political games. In Section 5, we examine the Nash equilibrium in perturbed games discussing the dependence of existence of equilibria on parties' error probability. Based on the observation obtained in Section 5, Sections 6 and 7 studies the perfect equilibrium and the strictly perfect equilibrium in incomplete information political games. Section 8 concludes.

2 Model

In this section, we construct a model of political competition. The model is defined as a dynamic game with incomplete information consisting of two parties and a continuum of voters in which the parties have informational advantage over voters.

We consider a society consisting of two political parties, A and B, and voters whose population is normalized to 1. There are two possible policies, 0 and 1. Let $I = \{A, B\}$ with generic element i, and $K = \{0, 1\}$ with generic element k. Variable $x \in X = [0, 1]$ describes "fundamentals" affecting the relative effectiveness of these policies. After the parties announce their policies, a majority voting determines one party as the winner. The winning party then carries out its platform.

Each individual's utility decreases (monotonically, in the weak sense) in the variable x, and his threshold for x is represented by his type. Specifically, each voter belongs to a type $\delta \in \Delta = [0, 1]$ distributed according to a distribution function F with median $\underline{\delta}$. His utility depends on the executed policy k, the variable x, and his type δ. For each type δ, we define the utility function $w_\delta : K \times X \to \mathbb{R}$ of a type δ voter by

$$w_\delta(k, x) = (\delta - x)k.$$

According to this definition, if the value of economic fundamentals is x, a voter of type δ prefers policy 1 if $\delta > x$, prefers policy 0 if $\delta < x$. Before the election, voters cannot observe the value of x. We model this uncertainty by a random variable θ with mean μ. Only the parties can observe the realized value of θ.

Let us provide some examples for the fundamentals variable x and individuals' utility functions. Suppose that there are two different rates of uniform income tax as the policy alternatives in a society: policy 1 represents the larger rate and policy 0 the smaller. The tax revenue will be transfered among voters. Suppose further that voters make decisions on their labor and consumption after the determination of tax policy. Thus adopting policy 1 will decrease the aggregate product in the economy compared with when policy 0 is adopted.

Let x be an index of this decrease in the aggregate product which takes values in $[0,1]$. Each voter has a threshold δ of the variable x so that he prefers policy 1 if and only if $x < \delta$. [1] However, voters only know the prior probability distribution, whereas the parties A and B know the extent to which levying the higher tax imposes a loss in the economy, perhaps through research.

As another example, consider a country J facing a diplomatic problem with a foreign country N. There is a suspicion against country N of possessing weapons of mass destruction (WMD). The probability that N has WMD is x. Citizens in country J, only know the prior probability distribution of x. Now, the country must take either a "hard-line" stance (policy 0) or a "soft-line" stance (policy 1) against country N. Thus every voter has a point δ such that as long as $x < \delta$, he support the soft-line policy.

We assume the following properties on the distribution functions of δ and θ.

Assumption 1.

(i) P is continuous and strictly increasing.

(ii) F has density function f such that $f(x) > 0$ for all $x \in (0,1)$.

Under Assumption 1, $0 < \overline{\delta}, \mu < 1$.

The timing of events is as follows: first, the parties observe the value x of economic fundamentals; second, the parties simultaneously announce their platforms; third, voters observe the announced policy pair; fourth, voters vote for the party with their preferred policy; and finally, the winning party carries out its policy. The parties thus can condition their decisions on the observed value x of θ. Voters, on the other hand, can condition their choices on the pair of announced platforms (k_A, k_B).

A *party's strategy* is a function $s : X \to \tilde{K}$, where $\tilde{K} = \{(q_0, q_1) \in \mathbb{R}^2_+ | q_0 + q_1 = 1\}$, which assigns for each possible value x of θ a pair $s(x) = (s_0(x), s_1(x))$. For each policy k, $s_k(x)$ represents the probability that party takes policy k conditional on $\theta = x$. We assume that $s_k : X \to [0,1]$ is Lebesgue measurable for $k = 0,1$. Denote by S the set of all strategies of a party:

$$S = \{s = (s_0, s_1) : X \to \tilde{K} | s_k \text{ is measurable, } k = 0,1\}.$$

[1] The types δ of voters in this example should be derived from their primitive data such as their utility functions or labor skills. This is true in general cases where we want to apply the model. However, through this paper, we assume that the distribution of δ in the population is given and known to the parties. We can imagine, for example, that given a political issue, the quantitative data of public opinion on this issue is provided by public or private surveys.

A *voter's strategy* is a function $t : K \times K \to \tilde{I}$, where $\tilde{I} = \{(q_A, q_B) \in \mathbb{R}^2_+ | q_A + q_B = 1\}$, which assigns to each policy pair (k_A, k_B), a pair $t_\delta(k_A, k_B) = (t_A(k_A, k_B), t_B(k_A, k_B))$. For each party $i \in I$, $t_i(k_A, k_B)$ represents the probability that the voter votes for party i after observing the pair of policy announcements (k_A, k_B). We assume that if the two parties announce the same policy, then he votes for each party with probability one half. Denote by T the set of all strategies of a voter:

$$T = \left\{t = (t_A, t_B) : K \times K \to \tilde{I} |\ t(k,k) = (\tfrac{1}{2}, \tfrac{1}{2}) \text{ for } k = 0,1\right\}$$

A profile of voting probabilities of the citizens, i.e., a family $(q_\delta)_{\delta\in\Delta} \in \prod_{\delta\in\Delta} \tilde{I}$, [2] completely determines the probability of electoral outcomes. We thus write $\pi_i\big((q_\delta)_{\delta\in\Delta}\big)$ for the winning probability of party i.

For each party i, define a function $U_i : S \times S \times \left(\prod_{\delta\in\Delta} T\right) \times X \to \mathbb{R}$ by

$$U_i(s_A, s_B, (t_\delta)_{\delta\in\Delta}, x) = \sum_{k_A\in K} \sum_{k_B\in K} s_{A,k_A}(x) s_{B,k_B}(x) \pi_i\big((t_\delta(k_A, k_B))_{\delta\in\Delta}\big).$$

$U_i(s_A, s_B, (t_\delta)_{\delta\in\Delta}, x)$ represents the expected utility of party i given the strategy profile $(s_A, s_B, (t_\delta)_{\delta\in\Delta})$ conditional on $\theta = x$.

For each $\delta \in \Delta$, define function $U_\delta : S \times S \times T \times X \to \mathbb{R}$ by

$$U_\delta(s_A, s_B, t_\delta, x) = \sum_{i\in I} \sum_{k_A\in K} \sum_{k_B\in K} s_{A,k_A}(x) s_{B,k_B}(x) t_{\delta,i}(k_A, k_B) w_\delta(k_i, x).$$

$U_\delta(s_A, s_B, t_\delta, x)$ then represents the expected utility of a type δ citizen given the strategy profile (s_A, s_B, t_δ) conditional on $\theta = x$.

Political competition in this society can be modeled by a dynamic game with incomplete information as follows.

Definition 1. An *incomplete information political game* is a tuple

$$\Gamma = \big((S, S, (T)_{\delta\in\Delta}), (U_A, U_B, (U_\delta)_{\delta\in\Delta}), F, P, \Lambda\big),$$

where S is the set of strategies of a party, T is the set of strategies of a voter, U_i is the conditional payoff function of party i, U_δ is the conditional payoff function of a type δ voter, F is the distribution function of citizens' types, P is the distribution function of θ, and Λ denotes the specific order of play and information structure: (i) the parties observes the value of θ and then simultaneously announce policies, and (ii) every voter cannot observe the value of θ, but observes the announced policies and then votes for a party.

[2] This notation implicitly assumes that all citizens of one type take the same action. Moreover, we will denote a strategy profile as $(t_\delta)_\delta \in \Delta$. In our setting, this causes no problem.

3 Nash equilibrium and beliefs of voters

In this section, we define the Nash equilibrium and the weakly perfect Bayesian equilibrium of an incomplete information political game. We then study the weakly perfect Bayesian equilibrium, paying attention to the relation between voters' beliefs on fundamentals and the parties' strategies. From the result obtained from this analysis, we derive a corollary on the Nash equilibria in terms of a newly-introduced function Q, which is more explicit in the locations of switching points of equilibrium strategies.

The Nash equilibrium in an incomplete information political game is defined as follows.

Definition 2. Let $\Gamma = \big((S, S, (T)_{\delta\in\Delta}), (U_A, U_B, (U_\delta)_{\delta\in\Delta}), F, P, \Lambda\big)$ be a political game. A strategy profile of the parties and the voters, $(s_A^*, s_B^*, (t_\delta^*)_{\delta\in\Delta})$, is a *Nash equilibrium* in Γ if

(i) $E[U_A(s_A^*, s_B^*, (t_\delta^*)_{\delta\in\Delta}, \theta)] = \max_{s_A\in S} E[U_A(s_A, s_B^*, (t_\delta^*)_{\delta\in\Delta}, \theta)]$,

(ii) $E[U_B(s_A^*, s_B^*, (t_\delta^*)_{\delta\in\Delta}, \theta)] = \max_{s_B\in S} E[U_B(s_A^*, s_B, (t_\delta^*)_{\delta\in\Delta}, \theta)]$, and

(iii) for every voter type δ, $E[U_\delta(s_A^*, s_B^*, t_\delta^*, \theta)] = \max_{t_\delta\in T} E[U_\delta(s_A^*, s_B^*, t_\delta, \theta)]$.

If $(s_A^*, s_B^*, (t_\delta^*)_{\delta\in\Delta})$ is a Nash equilibrium in Γ for some strategy profile of the voters $(t_\delta^*)_{\delta\in\Delta}$, we often simply say that (s_A^*, s_B^*) is a Nash equilibrium.

Using the specific information structure, Λ, of our political games, the above definition can be equivalently stated as follows.

Definition 3. Let Γ be a political game. A strategy profile of the parties and the voters, $(s_A^*, s_B^*, (t_\delta^*)_{\delta\in\Delta})$, is a *Nash equilibrium* in Γ if:

(i) For almost every $x \in X$, $U_A(s_A^*, s_B^*, (t_\delta^*)_{\delta\in\Delta}, x) = \max_{s_A\in S} U_A(s_A, s_B^*, (t_\delta^*)_{\delta\in\Delta}, x)$,

(ii) for almost every $x \in X$, $U_B(s_A^*, s_B^*, (t_\delta^*)_{\delta\in\Delta}, x) = \max_{s_B\in S} U_B(s_A^*, s_B, (t_\delta^*)_{\delta\in\Delta}, x)$, and

(iii) for every voter type δ and for every policy pair (k_A, k_B) such that $\int s_{A,k_A}^*(x) s_{B,k_B}^*(x) dP(x) > 0$,

$$\frac{\int U_\delta(s_A^*, s_B^*, t_\delta^*, x) s_{A,k_A}^*(x) s_{B,k_B}^*(x) dP(x)}{\int s_{A,k_A}^*(x) s_{B,k_B}^*(x) dP(x)} = \max_{t_\delta\in T} \frac{\int U_\delta(s_A^*, s_B^*, t_\delta, x) s_{A,k_A}^*(x) s_{B,k_B}^*(x) dP(x)}{\int s_{A,k_A}^*(x) s_{B,k_B}^*(x) dP(x)}.$$

The condition (iii) of Definition 3 clarifies that the notion of Nash equilibrium imposes no requirement on actions of the voters in out-of-equilibrium paths. Weakly perfect Bayesian equilibrium defined below requires that every voter's action at unreached moves be rational with respect to some "belief" about the conditional distribution of θ.

Definition 4. Let Γ be a political game.

(i) A *belief* of a voter is a family of probability measures on X, $\mathsf{b} = (\mathsf{b}_{k_A,k_B})_{(k_A,k_B)\in K\times K}$.

(ii) A belief b is *consistent* with a strategy pair (s_A, s_B) of the parties if for every policy pair (k_A, k_B) such that $\int s_{A,k_A}(x)s_{B,k_B}(x)dP(x) > 0$ and for every Borel set $Y \subset X$,

$$\mathsf{b}_{k_A,k_B}(Y) = \frac{\int_Y s_{A,k_A}(x)s_{B,k_B}(x)dP(x)}{\int s_{A,k_A(x)}s_{B,k_B}(x)dP(x)}. \tag{1}$$

The right hand side of (1) is exactly the conditional probability that $\theta \in Y$ given that the announced policy pair is (k_A, k_B) derived from the strategy pair (s_A, s_B). For any strategy pair (s_A, s_B) and any policy pair (k_A, k_B) reached with positive probability by (s_A, s_B), write $E_{s_A,s_B}(\theta|k_A, k_B)$ for the conditional expectation of θ given that (k_A, k_B) is announced, derived from (s_A, s_B):

$$E_{s_A,s_B}(\theta|k_A, k_B) = \frac{\int x s_{A,k_A}(x)s_{B,k_B}(x)dP(x)}{\int s_{A,k_A}(x)s_{B,k_B}(x)dP(x)}. \tag{2}$$

Definition 5. Let Γ be an incomplete information political game. A strategy profile $(s^*_A, s^*_B, (t^*_\delta)_{\delta\in\Delta})$ is a *weakly perfect Bayesian equilibrium* in Γ if

(i) it satisfies the conditions (i) and (ii) of Definition 3, and

(ii) for every voter, there exists a belief $\mathsf{b} = (\mathsf{b}_{k_A,k_B})_{(k_A,k_B)\in K\times K}$ consistent with (s^*_A, s^*_B), such that if he is of type $\delta \in \Delta$, then for every policy pair (k_A, k_B),

$$\int U_\delta(s^*_A, s^*_B, t^*_\delta, x)\mathsf{b}_{k_A,k_B}(dx) = \max_{t_\delta\in T} \int U_\delta(s^*_A, s^*_B, t_\delta, x)\mathsf{b}_{k_A,k_B}(dx).$$

The condition (ii) of Definition 5 requires that every voter's strategy be optimal conditional on any announced policy pair with some belief on θ consistent with the parties' strategies.

We proceed to derive the optimality condition of a voter's strategy given the parties' strategies, based on a belief consistent with them. For ease of notation, for any belief b and policy pair (k_A, k_B), write $E_{\mathsf{b}}(k_A, k_B)$ for the mean of b_{k_A,k_B}:

$$E_{\mathsf{b}}(k_A, k_B) = \int x \mathsf{b}_{k_A,k_B}(dx)$$

Given a strategy pair of the parties (s_A, s_B) and a belief b consistent with (s_A, s_B), if a type δ citizen observes the pair of announced policies $(k_A, k_B) = (1,0)$, then he should vote for party A, i.e., his strategy should give $t_\delta(1,0) = (1,0)$, if $\delta > E_{\mathsf{b}}(1,0)$. More generally, the optimal strategy of a type δ voter, t^*_δ, given the parties' strategy pair (s_A, s_B) and the voter's belief b consistent with it, must satisfy the following conditions.

$$t^*_{\delta,A}(1,0) = \begin{cases} 1 & \text{if } E_{\mathsf{b}}(1,0) < \delta \\ 0 & \text{if } E_{\mathsf{b}}(1,0) > \delta \end{cases}, \quad t^*_{\delta,A}(0,1) = \begin{cases} 1 & \text{if } E_{\mathsf{b}}(0,1) > \delta \\ 0 & \text{if } E_{\mathsf{b}}(0,1) < \delta \end{cases}, \tag{3}$$

and $t^*_\delta(0,0) = t^*_\delta(1,1) = \frac{1}{2}$ by our definition of the strategy set T.

We will concentrate on weakly perfect equilibria supported by an identical belief among voters. This may be justified since if we require some trembling hand stability of equilibria, then any stable equilibrium must be supported by such a *common beilief* of the voters as we will see in later sections.

From (3), given a pair of the parties' strategies $(s_A, s_B) \in S \times S$ and a common belief b of the voters consistent with (s_A, s_B), the fraction of citizens voting for party A having observed the pair of announced policies (1,0) is equal to $1 - F(E_{\mathsf{b}}(1,0))$. Noting the strict monotonicity of F in Assumption 1, the fraction of citizens voting for party A in this situation is therefore greater than or equal to one half if and only if $E_{\mathsf{b}}(1,0) \leq \bar{\delta}$. We assume that if the voting results in a tie, each party's winning probability is one half. Thus, the victory probability of the parties in an election when $(t^*_\delta)_{\delta \in \Delta}$ is a profile of voters' optimal strategies with a common belief b is given by the following formula.

$$\begin{aligned} \pi_A\big((t^*_\delta(1,0))_{\delta\in\Delta}\big) &= \begin{cases} 1 & \text{if } E_{\mathsf{b}}(1,0) < \bar{\delta} \\ \frac{1}{2} & \text{if } E_{\mathsf{b}}(1,0) = \bar{\delta} \\ 0 & \text{if } E_{\mathsf{b}}(1,0) > \bar{\delta} \end{cases}. \\ \pi_A\big((t^*_\delta(0,1))_{\delta\in\Delta}\big) &= \begin{cases} 1 & \text{if } E_{\mathsf{b}}(0,1) > \bar{\delta} \\ \frac{1}{2} & \text{if } E_{\mathsf{b}}(0,1) = \bar{\delta} \\ 0 & \text{if } E_{\mathsf{b}}(0,1) < \bar{\delta} \end{cases}. \\ \pi_A\big((t^*_\delta(0,0))_{\delta\in\Delta}\big) &= \pi_A\big((t^*_\delta(1,1))_{\delta\in\Delta}\big) = \tfrac{1}{2}. \end{aligned} \tag{4}$$

π_R is defined by $\pi_R = 1 - \pi_L$.

We will restrict our attention to those Nash equilibria in which each party takes a "cut-off strategy" defined as follows.

Definition 6. The *cut-off strategy* of a party switching around $x \in X$, denoted by $c_x = (c_{x,0}, c_{x,1})$, is a strategy defined by

$$c_{x,0}(y) = \begin{cases} 0 & \text{if } y \leq x \\ 1 & \text{if } y > x. \end{cases}$$

The following proposition specifies the set of all cut-off weakly perfect Bayesian equilibria of a political game in terms of the positions of switching points of the parties' strategies and the conditions on voters' beliefs consistent with those equilibria. The conditions are stated only for the strategy pairs with $x_A^* \geq x_B^*$. This is a sufficient way of description due to our symmetric modeling of the two parties: if $(c_{x_A^*}, c_{x_B^*})$ is an equilibrium for some equilibrium concept, then $(c_{x_B^*}, c_{x_A^*})$ is also an equilibrium.

Proposition 1. *Let* $\Gamma = \big((S, S, (T)_{\delta\in\Delta}), (U_A, U_B, (U_\delta)_{\delta\in\Delta}), F, P, \Lambda\big)$ *be an incomplete information political game, where the median of the distribution function* F *of voters' types is* $\overline{\delta}$*. Then, a profile of the parties' cut-off strategies* $(c_{x_A^*}, c_{x_B^*})$ *with* $x_A^* \geq x_B^*$ *is a weakly perfect Bayesian equilibrium supported by a common belief* b *of the voters consistent with* $(c_{x_A^*}, c_{x_B^*})$ *if and only if one of the following conditions is satisfied.*

(i) $0 < x_B^* \leq x_A^* < 1$ *and* $E_{\mathsf{b}}(1,0) = E_{\mathsf{b}}(0,1) = \overline{\delta}$.

(ii) $\mu > \overline{\delta}$, $0 = x_B^* < x_A^* < 1$, $E_{\mathsf{b}}(1,0) = \overline{\delta}$, *and* $E_{\mathsf{b}}(0,1) \geq \overline{\delta}$.

(iii) $\mu = \overline{\delta}$, $x_B^* = 0$, $x_A^* = 1$, *and* $E_{\mathsf{b}}(1,0) = \overline{\delta}$.

(iv) $\mu < \overline{\delta}$, $0 < x_B^* < x_A^* = 1$, $E_{\mathsf{b}}(1,0) = \overline{\delta}$, *and* $E_{\mathsf{b}}(0,1) \leq \overline{\delta}$.

(v) $x_A^* = x_B^* = 0$, $E_{\mathsf{b}}(1,0) \geq \overline{\delta}$, *and* $E_{\mathsf{b}}(0,1) \geq \overline{\delta}$.

(vi) $x_A^* = x_B^* = 1$, $E_{\mathsf{b}}(1,0) \leq \overline{\delta}$, *and* $E_{\mathsf{b}}(0,1) \leq \overline{\delta}$.

Proof. Condition (i). Suppose that $0 < x_B^* \leq x_A^* < 1$. By the assumption of monotonicity of P (the condition (i) of Assumption 1), this occurs if and only if both policy pairs $(1,1)$ and $(0,0)$ are announced with positive probability. By the formula (3), party A has no incentive to deviate from $(1,1)$ if and only if $E_{\mathsf{b}}(0,1) \leq \overline{\delta}$. Similarly, party B has no incentive to deviate from $(1,1)$ if and only if $E_{\mathsf{b}}(1,0) \le \overline{\delta}$. By the same reasoning, both parties cannot profitably deviate from the policy pair $(0,0)$ if and only if $E_{\mathsf{b}}(0,1) \ge \overline{\delta}$ and $E_{\mathsf{b}}(1,0) \ge \overline{\delta}$. Thus, $(c_{x_A^*}, c_{x_B^*})$ is a weakly perfect Bayesian equilibrium with common belief b if and only if $E_{\mathsf{b}}(1,0) = E_{\mathsf{b}}(0,1) = \overline{\delta}$.

Conditions (ii) and (iii). Suppose that $0 = x_B^* < x_A^* < 1$. This is equivalent to that the policy pairs $(1,0)$ and $(0,0)$ are announced with positive probabilities. Similar argument as in the preceding paragraph concludes that $(c_{x_A^*}, c_{x_B^*})$ is a weakly perfect Bayesian equilibrium with common belief b if and only if $E_{\mathsf{b}}(1,0) = \overline{\delta}$ and $E_{\mathsf{b}}(0,1) \ge \overline{\delta}$. But, by consistency of b,

$$E_{\mathsf{b}}(1,0) = E(\theta | 0 < \theta \le x_A^*) < \mu,$$

where the inequality follows again from (i) of Assumption 1. Hence, $\mu > \overline{\delta}$. The part (ii)-(vii) can be similarly verified.

Conditions (v) and (vi). Suppose $x_A^* = x_B^* = 0$. This is equivalent to that only the policy pair $(0,0)$ is announced with positive probability. Profitable deviation from $(0,0)$ by either party is impossible if and only if $E_{\mathsf{b}}(\theta|1,0) \ge \overline{\delta}$ and $E_{\mathsf{b}}(\theta|0,1) \ge \overline{\delta}$. The part (ix) can be similarly proved.

Since all possible locations of (x_A^*, x_B^*) have been checked, the proof is complete. □

Proposition 1 relates equilibrium strategy profiles of the parties to the conditional "expectations" of the voters with respect to their beliefs which support those strategy profiles. A remarkable feature is that any "interior" strategy profile of the parties, i.e., a strategy pair with switching points in the interior of X, is a weakly perfect Bayesian equilibrium if and only if it is supported by a common belief of the voters such that both conditional expectations of θ given policy pairs $(1,0)$ and $(0,1)$ are equal to the median type, while for "corner" strategy pairs, the corresponding conditions contain at most one equation for the two expectations.

For an interior strategy pair of the parties to be a weakly perfect Bayesian equilibrium, the winning probability conditional on distinct policies, $\pi_i\big((t_\delta^*(1,0))_{\delta\in\Delta}\big)$ and $\pi_i\big((t_\delta^*(0,1))_{\delta\in\Delta}\big)$, must be equal to one half because otherwise, either party can improve its expected payoff by deviating from the policy pair $(0,0)$ or $(1,1)$. With any corner strategy profile, one of these two pairs of convergent policy announcements does not occur, and hence the winning probability given this policy pair does not have to be exactly one-half. The difference in the equilibrium conditions in Proposition 1 reflects these facts and will be important in studying equilibrium refinement in later sections.

Proposition 1 can be restated in a form which is more explicit on the positions of equilibrium strategies by ignoring the constraints for the beliefs on out-of-equilibrium actions. To do this, we first define a function Q as follows.

Definition 7. Define a function $Q : X \times \Delta \to \mathbb{R}$ by

$$Q(x,\delta) = \int_0^x (u-\delta)dP(u). \tag{5}$$

The value $Q(x,\delta)$ represents the bias of θ from the type δ in terms of the distribution function P on the interval $[0,x]$. It serves as a measure of the distance between point x and type δ, but more detailed property of Q as a function depends on the property of distribution function P.

The properties of function Q described in the following lemma is derived directly from its definition.

Lemma 1. *For the function Q defined in Definition 7, the following statements hold under (i) of Assumption 1.*

(i) For each voter type δ, the function $Q(\cdot,\delta)$ is continuous, decreasing on $[0,\delta]$, increasing on $[\delta,1]$, and takes values $Q(0,\delta)=0$, $Q(1,\delta)=\mu-\delta$.

(ii) If $x_i, x_j \in X$ and $x_i < x_j$, then $E(\theta | x_i < \theta \leq x_j)$ is greater than, equal to, less than δ as $Q(x_j,\delta)$ is greater than, equal to, less than $Q(x_i,\delta)$, respectively.

The graph of $Q(\cdot,\bar{\delta})$ in a typical incomplete information political game in which $\mu > \bar{\delta}$ is illustrated in Figure 1.

If the parties select different cut-off points, the event that they announce different policies occurs with positive probability. The preference relation of a type-δ voter between the two policies is then equivalently described by the relation between the values of the $Q(\cdot,\delta)$ at these switching points: he prefers the policy of the party with smaller value of Q. Each of distinct policies thus yields one half of the total votes if and only if the values of function $Q(\cdot,\bar{\delta})$ at these cut-off points coincide. Due to the strict concavity of function $Q(\cdot,\bar{\delta})$ stated in (i) of Lemma 1, there are at most two distinct points at which the values of $Q(\cdot,\bar{\delta})$ are equal, such as x_A^* and x_B^* in Figure 1.

By (i) of Lemma 1 and the fact that $0 < \bar{\delta} < 1$ implied by Assumption 1, each of the two points defined in the following definition uniquely exists. These points determine the intervals in X where a point can always find a different point with equal value of $Q(\cdot,\bar{\delta})$.

Definition 8. *Points $\acute{x}$ and $\grave{x}$.* Under Assumption 1, if $\mu \geq \bar{\delta}$, we denote by $\acute{x}$ the unique point in the interval $(0,1]$ such that $Q(x,\bar{\delta}) = 0$. If $\mu \leq \bar{\delta}$, we denote by $\grave{x}$ the unique point in the interval $[0,1)$ such that $Q(x,\bar{\delta}) = \mu - \bar{\delta}$.

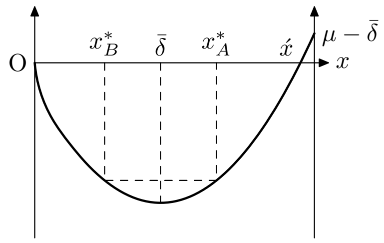

Figure 1: The function $Q(\cdot, \bar{\delta})$

According to this definition, it is clear that the only strategy pair satisfying the condition (ii) of Proposition 1 is $(c_{\acute{x}}, c_0)$ and that the only strategy pair satisfying (iv) of the proposition is $(c_1, c_{\grave{x}})$.

With these observations in hand, we translate Proposition 1 on weakly perfect Bayesian equilibria into the following corollary in terms of function Q on Nash equilibria, which, as a set of the parties' strategy profiles, coincide with weakly perfect Bayesian equilibria.

Corollary 1. *Let Γ be an incomplete information political game. Then, under Assumption 1, a cut-off strategy pair $(c_{x_A^*}, c_{x_B^*})$ such that $x_A^* \geq x_B^*$ is a Nash equilibrium of Γ if and only if one of the following condition is satisfied.*

(i) $\mu \geq \bar{\delta}$, $0 < x_B^* < x_A^* < \acute{x}$, *and* $Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta})$;

(ii) $\mu \geq \bar{\delta}$, $x_B^* = 0$ *and* $x_A^* = \acute{x}$;

(iii) $\mu \leq \bar{\delta}$, $\grave{x} < x_B^* < x_A^* < 1$, *and* $Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta})$;

(iv) $\mu \leq \bar{\delta}$, $x_B^* = \grave{x}$ *and* $x_A^* = 1$;

(v) $x_A^* = x_B^*$.

More simply, $(c_{x_A^*}, c_{x_B^*})$ is a Nash equilibrium if and only if

$$Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta}). \tag{6}$$

Corollary 1 suggests that the set of pairs of Nash equilibrium cut-off points, (x_A^*, x_B^*), is geometrically expressed as the union of two crossing curves in the unit square. Suppose, for example, $\mu > \bar{\delta}$. When Assumption 1 holds, by strict concavity of function $Q(\cdot, \bar{\delta})$, the set of points (x_A^*, x_B^*) satisfying (i) or (ii) is (if the point $(\bar{\delta}, \bar{\delta})$ is added) represented by a curve in the unit square region with negative gradient which is symmetric with respect to the 45-degree line,

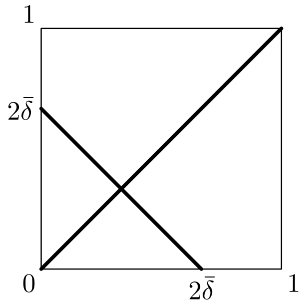

Figure 2: The Nash equilibrium switching point pairs (x_A^*, x_B^*) in Example 1

has interceptions with the sides of the square $\acute{x}$ or $\grave{x}$, and passes through $(\overline{\delta}, \overline{\delta})$. The points satisfying (v) constitute the 45-degree line in the unit square. These two parts constitute the set of pairs of Nash equilibrium cut-off points.

Example 1. Let Γ be a political game in which θ is uniformly distributed on the unit interval, that is, $P(x) = x$ for all $x \in X$. Moreover, assume that F is such that $\overline{\delta} < \frac{1}{2} = \mu$. In this game, the function $Q(\cdot, \overline{\delta})$ is given by

$$Q(x, \overline{\delta}) = \frac{x^2}{2} - \overline{\delta} x$$

for each $x \in X$. Therefore, by Corollary 1, the set of pairs of Nash equilibrium cut-off points in Γ is

$$\left\{(x_A, x_B) | \frac{x_A^2}{2} - \overline{\delta} x_A = \frac{x_B^2}{2} - \overline{\delta} x_B,\ 0 \le x_A, x_B \le 1\right\}.$$

This set is illustrated in Figure 2.

4 Comparison with complete information case

In this section, We briefly deviate from our main assumption of incomplete information, and check the fact that if we instead suppose the complete information, then the present model's version of Median Voter Theorem holds true.

We first define the complete information version of a political game as follows.

Definition 9. A *complete information political game* is a tuple

$$\Gamma' = \left((S, S, (T')_{\delta \in \Delta}), (U_A, U_B, (U_\delta)_{\delta \in \Delta}), F, P, \Lambda'\right),$$

where Λ' differs from Λ only in that the voters, as well as the parties, now can observe the value of θ, and the set of strategies of a voter is

$$T' = \left\{\tau : K \times K \times X \to \tilde{I} | \tau(1,1,x) = \tau(0,0,x) = (\tfrac{1}{2}, \tfrac{1}{2}) \text{ for all } x \in X\ \right\}.$$

The Nash equilibrium and the subgame perfect equilibrium in a complete information political game are defined in a standard way, and hence we omit the formal definition of these concepts.

We obtain the following results on Nash equilibria and subgame perfect equilibria in complete information political games.

Proposition 2. *Let Γ' be a complete information political game. Then,*

(i) a profile (s_A^*, s_B^*) of pure strategies is a Nash equilibrium of Γ' if and only if $s_A^*(x) = s_B^*(x)$ for almost every x with respect to P, and

(ii) a strategy profile (s_A^*, s_B^*) is a subgame perfect equilibrium in Γ' if and only if, for each party i,

$$s_{i,0}^*(x) = \begin{cases} 0 & \text{if } x < \overline{\delta} \\ 1 & \text{if } x > \overline{\delta} \end{cases}.$$

Proof. If $s_A^*(x) = s_B^*(x)$ for almost every x, then it is optimal for every voter to set for every x,

$$\tau^*(1,0,x) = \tau^*(0,1,x) = (\tfrac{1}{2}, \tfrac{1}{2})$$

since the policy pairs $(1,0)$ and $(0,1)$ are reached with probability zero. Such (s_A^*, s_B^*) are thus all Nash equilibria of Γ'. If $s_A^*(\theta) \neq s_B^*(\theta)$ with positive probability, then by our assumption that P is strictly increasing, the event that $s_A^*(\theta) \neq s_B^*(\theta)$ and $\theta \neq \overline{\delta}$ has positive probability, and hence a party loses with positive probability. This proves the first part of the proposition.

A strategy profile of voters, $(\tau_\delta^*)_{\delta\in\Delta}$, consists in a subgame perfect equilibrium of Γ' if and only if for each voter type δ,

$$\tau_{\delta,A}^*(1,0,x) = \begin{cases} 1 & \text{if } x < \delta \\ 0 & \text{if } x > \delta \end{cases}, \quad \tau_{\delta,A}^*(0,1,x) = \begin{cases} 1 & \text{if } x > \delta \\ 0 & \text{if } x < \delta \end{cases}.$$

This proves the last part of the proposition. □

The statement (ii) of the proposition is the version of Median Voter Theorem in our political game. It says that, given a value x of fundamentals, both parties will choose the ideal policy of voters who have the median type under the state x: policy 1 if $x < \overline{\delta}$, and policy 0 if $x > \overline{\delta}$. The subgame perfect equilibrium corresponds to the notion of political equilibrium in standard electoral models. The reason for the indeterminancy of Nash equilibria appearing in the statement (i) of the proposition is that, the definition of the Nash equilibrium in the present model allows arbitrariness of voters' actions off equilibrium.

Restricting these results to pairs of cut-off strategies, the following corollary may be more appropriate in comparison with the results for incomplete information games.

Corollary 2. *Let Γ' be a complete information political game.*

(i) Then, a cut-off strategy profile of the parties $(c_{x_A^*}, c_{x_B^*})$ is a Nash equilibrium of Γ' if and only if $x_A^* = x_B^*$, and

(ii) the unique cut-off subgame perfect equilibrium is $(c_{\bar{\delta}}, c_{\bar{\delta}})$.

By the statement (i) of Corollary 1 and the statement (i) of Corollary 2, the set of cut-off Nash equilibria in a complete information political game Γ' is a proper subset of that in the corresponding incomplete information political game Γ. Specifically, in any cut-off Nash equilibrium of Γ', the policies of the two parties coincide at every observed value of θ. Moreover, by Proposition 2, even if we allow the whole class of strategies of a party, the equilibrium policy convergence essentially remains true. In contrast, in a complete information game Γ, there are Nash equilibria in which policy divergence occurs with positive probability, i.e., the strategy profiles satisfying the conditions (i) and (ii) of Corollary 1.

By (ii) of Proposition 2, in a subgame perfect equilibrium of a complete information political game Γ', policy divergence is possible only at the parties' observation $\theta = \bar{\delta}$. This is because given that $\theta = \bar{\delta}$, the voters are divided into two groups prefering different policies. By looking at the conditions (i)-(iv) of Corollary 1, we understand that unobservability of θ by the voters in a political game expands the possibility of policy divergence from the point $\bar{\delta}$ in X to the various intervals keeping the conditional expectations of θ fixed at $\bar{\delta}$.

We recognize, however, that there is still a difficulty in interpreting this result because of the considerable multiplicity: there is a continuum of Nash equilibria in an incomplete information political game. Natural questions arise at this point: Can we refine the equilibria by some stability criterion? If so, which strategy pairs stated in Corollary 1 are stable? The following sections will study these problems.

5 Perturbed games

As the first step for equilibrium refinement, in this section we analyze perturbed games of incomplete information political games. We show the existence and important properties of the "critical type" of voters in a perturbed game, which plays essentially the same role as the median type in a non-perturbed game. Then we study the Nash equilibria in a perturbed game.

We define a perturbation in an incomplete information political game as follows.

Definition 10. Define $\mathscr{G} = \left\{g : \Delta \to (0, \frac{1}{2}) | g \text{ is continuous}\right\}$ and $E = (0, \frac{1}{2})$. A *perturbation* of an incomplete information political game is a triple $\rho = (\varepsilon_A, \varepsilon_B, g) \in E \times E \times \mathscr{G}$.

Then a perturbed game is defined as follows.

Definition 11. Given an incomplete information political game $\Gamma = \big((S, S, (T)_{\delta\in\Delta}), (U_A, U_B, (U_\delta)_{\delta\in\Delta}), F, P, \Lambda\big)$ and a perturbation $\rho = (\varepsilon_A, \varepsilon_B, g)$, a *perturbed game* of Γ with ρ is a political game

$$\hat{\Gamma}(\rho) = \big(\hat{S}(\varepsilon_A), \hat{S}(\varepsilon_B), (\hat{T}(g(\delta)))_{\delta\in\Delta}), (\hat{U}^\rho_A, \hat{U}^\rho_B, \hat{U}^\rho_\delta), F, P, \Lambda\big),$$

where

$$\hat{S}(\varepsilon) = \big\{s \in S | s(x) \in [\varepsilon, 1-\varepsilon] \times [\varepsilon, 1-\varepsilon] \text{ for all } x \in X\big\},$$

$$\hat{T}(\varepsilon) = \big\{t \in T | t(k_A, k_B) \in [\varepsilon, 1-\varepsilon] \times [\varepsilon, 1-\varepsilon] \text{ for all } (k_A, k_B) \in K \times K \ \big\},$$

and for each party i, $\hat{U}^\rho_i$ is the restriction of function U_i to $\hat{S}(\varepsilon_A) \times \hat{S}(\varepsilon_B) \times \prod_{\delta\in\Delta} \hat{T}(g(\delta))$; and for each voter type δ, $\hat{U}^\rho_\delta$ is the restriction of U_δ to $\hat{S}(\varepsilon_A) \times \hat{S}(\varepsilon_B) \times \hat{T}(g(\delta))$.

The analogue of cut-off strategy in a political game in Definition 6 is defined as follows.

Definition 12. The *cut-off strategy* switching around $x \in X$ of party i in a perturbed game $\hat{\Gamma}(\rho)$ with perturbation $\rho = (\varepsilon_A, \varepsilon_B, g)$, denoted by $c^{\varepsilon_i}_x = (c^{\varepsilon_i}_{x,0}, c^{\varepsilon_i}_{x,1})$, is a strategy defined by

$$c^{\varepsilon_i}_{x,0}(y) = \begin{cases} \varepsilon_i & \text{if } y \leq x \\ 1-\varepsilon_i & \text{if } y > x \end{cases}.$$

The Nash equilibrium of a perturbed game can be defined in the same way as in Definition 2 except that the original strategy sets of players are now replaced by those defined in Definition 11.

Recall the notation $E_{s_A,s_B}(\theta|k_A,k_B)$ in (2) for the conditional expectation of θ given that the announced policy pair is (k_A,k_B) derived from strategy pair (s_A,s_B). In any perturbed game $\hat{\Gamma}(\rho)$, this is defined for all policy pairs (k_A,k_B) since they are reached with positive probability, i.e., the denominator of the right-hand side of (2) is positive for all (k_A,k_B). Let $(s_A,s_B)=(c^{\varepsilon_A}_{x_A},c^{\varepsilon_B}_{x_B})$ and suppose $x_B \le x_A$. Then we have

$$\begin{aligned} E_{s_A,s_B}(\theta|1,0) &= \frac{(1-\varepsilon_A)\varepsilon_B\int_0^{x_B} x dP(x)+(1-\varepsilon_A)(1-\varepsilon_B)\int_{x_B}^{x_A} x dP(x)+\varepsilon_A(1-\varepsilon_B)\int_{x_A}^1 x dP(x)}{(1-\varepsilon_A)\varepsilon_B P(x_B)+(1-\varepsilon_A)(1-\varepsilon_B)[P(x_A)-P(x_B)]+\varepsilon_A(1-\varepsilon_B)[1-P(x_A)]}, \\ E_{s_A,s_B}(\theta|0,1) &= \frac{\varepsilon_A(1-\varepsilon_B)\int_0^{x_B} x dP(x)+\varepsilon_A\varepsilon_B\int_{x_B}^{x_A} x dP(x)+(1-\varepsilon_A)\varepsilon_B\int_{x_A}^1 x dP(x)}{\varepsilon_A(1-\varepsilon_B)P(x_B)+\varepsilon_A\varepsilon_B[P(x_A)-P(x_B)]+(1-\varepsilon_A)\varepsilon_B[1-P(x_A)]}. \end{aligned} \tag{7}$$

The formula (3) for a voter's optimal strategies given a belief consistent with the parties' strategy pair is therefore completely connected to the strategies of the parties in a perturbed game $\hat{\Gamma}(\rho)$ as follows: In a perturbed game $\hat{\Gamma}(\rho)$, given a strategy profile of the parties $(s_A,s_B)\in\hat{S}(\varepsilon_A)\times\hat{S}(\varepsilon_B)$, a strategy $t^*_\delta \in \hat{T}(g(\delta))$ is optimal for a type-δ voter if and only if

$$\begin{aligned} t^*_{\delta,A}(1,0) &= \begin{cases} 1 & \text{if } E_{s_A,s_B}(\theta|1,0)<\delta \\ 0 & \text{if } E_{s_L,s_R}(\theta|1,0)>\delta \end{cases}, \\ t^*_{\delta,A}(0,1) &= \begin{cases} 1 & \text{if } E_{s_A,s_B}(\theta|0,1)>\delta \\ 0 & \text{if } E_{s_A,s_B}(\theta|0,1)<\delta \end{cases}, \end{aligned} \tag{8}$$

and $t^*_{\delta,A}(0,0)=t^*_{\delta,A}(1,1)=\frac{1}{2}$.

Now, let $\hat{\Gamma}(\rho)$ be a perturbed game with perturbation $\rho=(\varepsilon_A,\varepsilon_B,g)$. Define a function $L_g : X \to [0,1]$ by

$$L_g(x)=\int_0^x g(u)f(u)du+\int_x^1[1-g(u)]f(u)du. \tag{9}$$

$L_g(x)$ is the population of voters who vote for the party announcing policy 1 given $\theta=x$ and the two parties take different policies after observing x. The first term in the right-hand side of (9) is equal to the fraction of voters who prefer policy 0 in the original game Γ but vote for the party announcing policy 1 due to perturbation. The second term is the fraction of voters who prefers policy 1 in the original game minus the fraction of voters within this group who vote for the party announcing policy 0 due to perturbation.

Using function L_g, we define the notion of critical type in a perturbed game, which plays the same role as the median type $\bar{\delta}$ in a non-perturbed game.

Definition 13. Let $\hat{\Gamma}(\rho)$ be a perturbed game with perturbation $\rho = (\varepsilon_A, \varepsilon_B, g)$. A voter type $\hat{\delta}(g)$ is called the *critical type* of $\hat{\Gamma}(\rho)$ if $L_g(\hat{\delta}(g)) = \frac{1}{2}$ and for any $x, y \in X$,

$$x < \hat{\delta}(g) < y \Longrightarrow L_g(x) < \tfrac{1}{2} < L_g(y).$$

We then have the following lemma for the properties of the critical type of a perturbed game.

Lemma 2. *Let* Γ *be an incomplete information political game.*

(i) In any perturbed game $\hat{\Gamma}(\rho)$*, there exists the critical type* $\hat{\delta}(g)$*.*

(ii) If $(g^n)_{n=1}^{\infty}$ *is a sequence of voters' perturbations converging pointwise to the constantly 0-valued function, then* $\lim_{n\to\infty} \hat{\delta}(g^n) = \bar{\delta}$*.*

(iii) There exist sequences $(g_r^n)_{n=1}^{\infty}$*,* $r = 1, 2, 3$*, of the voters' perturbations, each of which converges pointwise to* 0 *as* $n \to \infty$*, such that* $\hat{\delta}(g_1^n) < \bar{\delta} = \hat{\delta}(g_2^n) < \hat{\delta}(g_3^n)$ *for every n.*

Proof. See Appendix. □

The statement (i) in Lemma 2 guarantees the existence of the critical type in every perturbed game. It allows us to study stability of Nash equilibria in non-perturbed games by comparing the conditional expectations of θ and the critical types in perturbed games. Moreover, by the statement (ii), the critical type in a perturbed game converges to the median type in the original game as voters' perturbations go to zero. The statement (iii) says that the direction of the convergence of critical types to the median type depends on the manner of convergence of voters' perturbations. This fact will be important particularly in considering the strictly perfect equilibrium where we have to take into account all kinds of perturbations which converge to zero.

Let $\Gamma(\rho)$ be a perturbed game with $\rho = (\varepsilon_A, \varepsilon_B, g)$. By Definition 13 of the critical type and the condition (8) for the voters' optimal strategies, we then obtain the following formula for the winning probability of the parties in the perturbed game, which is analogous to (3): If $(t_\delta^*)_{\delta\in\Delta} \in \prod_{\delta\in\Delta} \hat{T}(g(\delta))$ is a profile of voters' optimal strategies in the perturbed game $\Gamma(\rho)$ given a pair of the parties' strategies $(s_A, s_B) \in \hat{S}(\varepsilon_A) \times \hat{S}(\varepsilon_B)$, then

$$\pi_A\big((t^*_\delta(1,0))_{\delta\in\Delta}\big) = \begin{cases} 1 & \text{if } E_{s_A,s_B}(\theta|1,0) < \hat{\delta}(g) \\ \frac{1}{2} & \text{if } E_{s_A,s_B}(\theta|1,0) = \hat{\delta}(g) \\ 0 & \text{if } E_{s_A,s_B}(\theta|1,0) > \hat{\delta}(g) \end{cases},$$

$$\pi_A\big((t^*_\delta(0,1))_{\delta\in\Delta}\big) = \begin{cases} 1 & \text{if } E_{s_A,s_B}(\theta|0,1) > \hat{\delta}(g) \\ \frac{1}{2} & \text{if } E_{s_A,s_B}(\theta|0,1) = \hat{\delta}(g) \\ 0 & \text{if } E_{s_A,s_B}(\theta|0,1) < \hat{\delta}(g) \end{cases}, \tag{10}$$

$$\pi_A\big((t^*_\delta(0,0))_{\delta\in\Delta}\big) = \pi_A\big((t^*_\delta(1,1))_{\delta\in\Delta}\big) = \tfrac{1}{2}.$$

As the following proposition will show, the necessary and sufficient conditions for Nash equilibrium in a perturbed game is almost the same as the conditions of the weakly perfect Bayesian equilibria described in Proposition 1. However, since all policy pairs are reached with positive probability in a perturbed game, the conditions are now completely based on the strategies of the parties.

Proposition 3. *Let $\hat{\Gamma}(\rho)$ be a perturbed game with perturbation $\rho = (\varepsilon_A, \varepsilon_B, g)$ Then, a pair of the parties' cut-off strategies $(s_A, s_B) = (c^{\varepsilon_A}_{x_A}, c^{\varepsilon_B}_{x_B})$ in $\Gamma(\rho)$ such that $x_A \geq x_B$ is a Nash equilibrium of $\Gamma(\rho)$ if and only if one of the following conditions is satisfied.*

(i) $0 < x_B \leq x_A < 1$ *and* $E_{s_A,s_B}(\theta|1,0) = E_{s_A,s_B}(\theta|0,1) = \hat{\delta}(g)$.

(ii) $\mu > \hat{\delta}(g)$, $0 = x_B < x_A < 1$,
$E_{s_A,s_B}(\theta|1,0) = \hat{\delta}(g)$, *and* $E_{s_A,s_B}(\theta|0,1) \geq \hat{\delta}(g)$.

(iii) $\mu = \hat{\delta}(g)$, *and* $(x_A, x_B) = (0,0)$ *or* $(x_A, x_B) = (1,0)$ *or* $(x_A, x_B) = (1,1)$.

(iv) $\mu < \hat{\delta}(g)$, $0 < x_B < x_A = 1$,
$E_{s_A,s_B}(\theta|1,0) = \hat{\delta}(g)$, *and* $E_{s_A,s_B}(\theta|0,1) \leq \hat{\delta}(g)$.

The conditional expectations $E_{s_A,s_B}(\theta|1,0)$ and $E_{s_A,s_B}(\theta|0,1)$ is given by (7).

Proof. Conditions (i), (ii), and (iv). The gain or the loss for a party i from deviating from an equilibrium probability pair $(s_A(x), s_B(x))$ to the "opposite" strategy, for example, party A deviating from $(\varepsilon_A, \varepsilon_B)$ to $(1-\varepsilon_A, \varepsilon_B)$, is always ε_i less than the corresponding deviation without error in the original game Γ. Thus, this does not alter the essential argument for possibility of a party's deviation in the proof of Proposition 1, except that now the median type δ must be replaced with the critical type $\hat{\delta}(g)$ and a belief-based expectation $E_{\mathsf{b}}(k_A, k_B)$ with the strategy-based expectation $E_{s_A,s_B}(\theta|k_A, k_B)$, by (10).

Condition (iii). With any of the three strategy pairs in the condition (iii) of the proposition, the conditional expectations given policy pairs (1,0) and (0,1) are equal to μ. Thus, it is a Nash equilibrium if and only if $\mu = \hat{\delta}(g)$. □

Corollary 3 below restates Proposition 3 using function Q, more explicitly describing the locations of equilibrium cut-off points. For notational ease, we preliminarily define functions φ and ψ from $\left[(E \times E) \setminus \left\{(\varepsilon_A, \varepsilon_B) | \varepsilon_A = \varepsilon_B\right\}\right] \times \Delta$ to $\mathbb{R}$ and functions $\tilde{\varphi}$ and $\tilde{\psi}$ from $E \times \Delta$ to $\mathbb{R}$ by, given mean μ of θ,

$$\varphi(\varepsilon_A, \varepsilon_B, \delta) = \frac{[(\varepsilon_A)^2(1-\varepsilon_B)+(1-\varepsilon_A)^2\varepsilon_B](\mu-\delta)}{(\varepsilon_B-\varepsilon_A)(1-2\varepsilon_A)}, \;\; \psi(\varepsilon_A, \varepsilon_B, \delta) = \frac{\varepsilon_B(1-\varepsilon_B)(\mu-\delta)}{(\varepsilon_B-\varepsilon_A)(1-2\varepsilon_B)},$$
$$\tilde{\varphi}(\varepsilon, \delta) = -\frac{\varepsilon(\mu-\delta)}{1-2\varepsilon}, \;\; \tilde{\psi}(\varepsilon, \delta) = \frac{(1-\varepsilon)(\mu-\delta)}{1-2\varepsilon}. \tag{11}$$

Corollary 3. *Let $\hat{\Gamma}(\rho)$ be a perturbed game with perturbation $\rho = (\varepsilon_A, \varepsilon_B, g)$ Then, a pair of the parties' cut-off strategies $(s_A, s_B) = (c_{x_A}^{\varepsilon_A}, c_{x_B}^{\varepsilon_B})$ in $\Gamma(\rho)$ such that $x_A \geq x_B$ is a Nash equilibrium of $\Gamma(\rho)$ if and only if one of the following conditions is satisfied.*

(i) $\mu > \hat{\delta}(g)$, $\varepsilon_B < \varepsilon_A$, $0 < x_B < x_A < 1$, $x_B < \hat{\delta}(g)$,
$Q(x_A, \hat{\delta}(g)) = \varphi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g))$, *and* $Q(x_B, \hat{\delta}(g)) = \psi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g))$.

(ii) $\mu > \hat{\delta}(g)$, $0 = x_B < x_A < 1$, *and* $Q(x_A, \hat{\delta}(g)) = \tilde{\varphi}(\varepsilon_A, \hat{\delta}(g))$.

(iii) $\mu = \hat{\delta}(g)$, *and* $(x_A, x_B) = (0,0)$ *or* $(x_A, x_B) = (1,0)$ *or* $(x_A, x_B) = (1,1)$.

(iv) $\mu = \hat{\delta}(g)$, $\varepsilon_A = \varepsilon_B$, $0 < x_B \leq x_A < 1$, *and* $Q(x_A, \hat{\delta}(g)) = Q(x_B, \hat{\delta}(g))$.

(v) $\mu < \hat{\delta}(g)$, $\varepsilon_A < \varepsilon_B$, $0 < x_B < x_A < 1$, $x_A > \hat{\delta}(g)$,
$Q(x_A, \hat{\delta}(g)) = \varphi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g))$, *and* $Q(x_B, \hat{\delta}(g)) = \psi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g))$.

(vi) $\mu < \hat{\delta}(g)$, $0 < x_B < x_A = 1$, *and* $Q(x_B, \hat{\delta}(g)) = \tilde{\psi}(\varepsilon_B, \hat{\delta}(g))$.

Proof. See Appendix. □

Based on Corollary 3, we can outline the set of Nash equilibria of a perturbed game in terms of cut-off point pairs as in Figure 3 in which the graphs are illustrated for cases of a uniform θ, where the dashed lines represent the set of Nash equilibria in the original game. For example, consider a perturbed game in which $\mu > \hat{\delta}(g)$ and $\varepsilon_A > \varepsilon_B$. There are at most two pairs of cut-off points satisfying the condition (i) of the corollary such as (x_A, x_B) and (x'_A, x_B) in the left-hand graph of Figure 6 in Appendix. Note that, by symmetry between the parties, the corollary implies that there exists no interior Nash equilibrium at

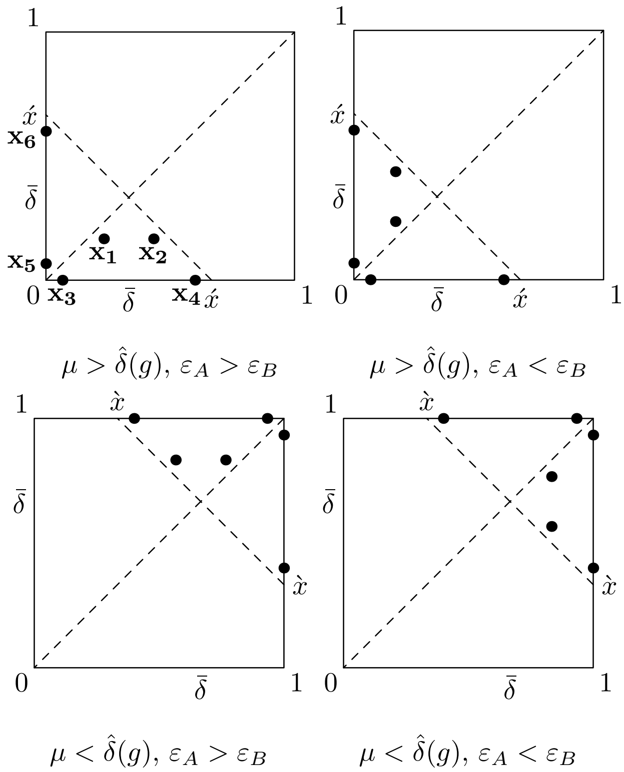

Figure 3: Nash equilibria in perturbed games

which party B's switching point is greater than or equal to party A's. Also, there are at most two pairs of cut-off points satisfying the condition (ii) such as $(y_A, 0)$ and $(y'_A, 0)$ in the left-hand graph of Figure 7. But, since this condition imposes no requirement on the relation between ε_A and ε_B, cut-off pairs such as $(0, z_B)$ and $(0, z'_B)$, where z_B and z'_B are the two points whose values of function $Q(\cdot, \hat{\delta}(g))$ are equal to $\tilde{\varphi}(\varepsilon_B, \hat{\delta}(g))$, are also equilibrium pairs of cut-off points. Therefore, there exist at most six Nash equilibria in this perturbed game, whose cut-off point pairs are illustrated as $\mathbf{x_1}, \cdots, \mathbf{x_6}$ in the left-top square of Figure 3. The relation between the rest of Figure 3 and Corollary 3 can be similarly explained.

As shown in Figure 3, Corollary 3 suggests that, for any interior Nash equilibrium (s^*_A, s^*_B) in a non-perturbed game, in which parties are more likely to announce the ex ante popular policy in this game (i.e., policy 0 if $\mu > \overline{\delta}$, and policy 1 if $\mu < \overline{\delta}$), there exists *some* slight perturbation that possesses an interior Nash equilibrium near (s^*_A, s^*_B) like the strategy pair with cut-off point pair $\mathbf{x_1}$ or $\mathbf{x_2}$ in Figure 3. It also suggests that near each of the corner Nash equilibria in the original game, there always exists a corner Nash equilibrium in *any* slightly perturbed game like the strategy pairs with cut-off point pairs $\mathbf{x_3}, \cdots, \mathbf{x_6}$. We derive these conjectural claims from the fact that, by (iii) of Lemma 2, the relation between the prior mean of fundamentals μ and the median type $\overline{\delta}$ in a political game implies the same relation between μ and the critical type $\hat{\delta}(g)$ in any slightly perturbed game, while either relation between ε_A and ε_B can happen given only that the perturbations are small.

In particular, there exists no interior Nash equilibrium in a perturbed game in which the party with smaller perturbation is more likely to choose the ex ante unpopular policy in the perturbed game. This simply reflects that a pary can be less populist in an equilibrium as long as it is publicly believed to make more mistakes.

6 Perfect equilibrium

In the preceding section, we have analyzed the Nash equilibria of perturbed games. Using those results, we now proceed to examine the stability of Nash equilibria of incomplete information political games. In this section, we study the perfect equilibrium of Selten (1975).

We define the perfect equilibrium of a political game as follows. We apply the equilibrium concept only to strategy pairs in which the parties take cut-off strategies.

Definition 14. A strategy profile $(c_{x^*_A}, c_{x^*_B}, (t^*_\delta)_{\delta \in \Delta})$ is a *perfect equilibrium* in a political game Γ if there exists a sequence of perturbations $(\rho^n)_{n=1}^{\infty} = (\varepsilon_A^n, \varepsilon_B^n, g^n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} \rho^n = (0,0,0)$, [3] and a number N such that there exists a sequence of strategy profiles in the perturbed games $\hat{\Gamma}(\rho^n)$ in which the parties take cut-off strategies, $(c_{x_A^n}^{\varepsilon_A^n}, c_{x_B^n}^{\varepsilon_B^n}, (t_\delta^n)_{\delta\in\Delta})_{n\geq N}$, satisfying the following conditions:

(i) For every $n \geq N$, $(c_{x_A^n}^{\varepsilon_A^n}, c_{x_B^n}^{\varepsilon_B^n}, (t_\delta^n)_{\delta\in\Delta})$ is a Nash equilibrium of $\hat{\Gamma}(\rho^n)$, and

(ii) $\lim_{n\to\infty} (x_A^n, x_B^n, (t_\delta^n)_{\delta\in\Delta}) = (x_A^*, x_B^*, (t_\delta^*)_{\delta\in\Delta})$.

Remark . A perfect equilibrium by this definition is always a perfect equilibrium by the original definition of Selten (1975), except that our political games are not finite, but the converse may not hold because our definition involves the requirement that Nash equilibrium strategies of parties in perturbed games be cut-off strategies.

As the following proposition will state, if $\mu > \overline{\delta}$ ($\mu < \overline{\delta}$), the perfect equilibrium only excludes Nash equilibria such that both parties choose the same cut-off point larger (smaller) than the median type. Proving that these Nash equilibria are not perfect equilibria is easy: recall that in the previous section we have seen that there is no Nash equilibrium with both parties switching at points larger (smaller) than the critical type in a perturbed game with $\mu > \hat{\delta}(g)$ ($\mu < \hat{\delta}(g)$). Note also that the critical type $\hat{\delta}(g)$ converges to the median type $\overline{\delta}$. For any of the above-mentioned Nash equilibria of the original political game and for any slight perturbation, therefore, there is no Nash equilibrium in the perturbed game near that Nash equilibrium of the original game. Those Nash equilibria are therefore not perfect equilibria.

For any of the remaining Nash equilibria, on the other hand, *some* proportion between the two parties' perturbations exists so that keeping this proportion, the values of functions φ and ψ in (11) converge to the level of Q-distance of the Nash equilibrium strategies from the median type as perturbations go to zero. This fact, together with the continuity of Q and the convergence of the critical type to the median type, implies that those Nash equilibria are perfect equilibria. This part of the claim is proved more formally below.

Proposition 4. *Let $\Gamma = \big((S, \underline{S}, (T)_{\delta\in\Delta}), (U_A, U_B, (U_\delta)_{\delta\in\Delta}), F, P, \Lambda\big)$ be a political game with median type $\overline{\delta}$. Then, a cut-off strategy of the parties $(c_{x_A^*}, c_{x_B^*})$ with $x_A^* \geq x_B^*$ is a perfect equilibrium if and only if one of the following conditions is satisfied:*

[3] Here the convergence of the sequence of functions (g^n) to 0 is in the sense that it converges to the constantly 0-valued function from Δ.

(i) $\mu > \bar{\delta}$, $0 \leq x_B^* < x_A^* \leq \acute{x}$, *and* $Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta})$*;*

(ii) $\mu > \bar{\delta}$ *and* $x_A^* = x_B^* \leq \bar{\delta}$*;*

(iii) $\mu = \bar{\delta}$ *and* $Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta})$*;*

(iv) $\mu < \bar{\delta}$, $\grave{x} \leq x_B^* < x_A^* \leq 1$, *and* $Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta})$*; and*

(v) $\mu < \bar{\delta}$ *and* $x_A^* = x_B^* \geq \bar{\delta}$*.*

Proof. The proof for the excluded Nash equilibria has been already done in the above text. It only remains to show that there is a proportion of the parties' perturbation against which any of remaining Nash equilibria is stable. Suppose $\mu > \bar{\delta}$. Note that for any $\beta \in (0,1) \cup (1,\infty)$,

$$\lim_{(\varepsilon,g)\to(0,0)} \varphi(\beta\varepsilon, \varepsilon, \hat{\delta}(g)) = \lim_{(\varepsilon,g)\to(0,0)} \psi(\beta\varepsilon, \varepsilon, \hat{\delta}(g)) = \tfrac{\mu-\bar{\delta}}{1-\beta}. \tag{12}$$

The right-hand side has range $(-\infty, 0)$ for $\beta > 1$. Recall that the perturbed game has a Nash equilibrium such that $1 > x_A > x_B > 0$ only if $\beta > 1$. But the range of $Q(\cdot, \bar{\delta})$ for the remaining Nash equilibria is contained in $(-\infty, 0]$. By continuity of Q, therefore, Nash equilibria $(c_{x_A^*}, c_{x_B^*})$ in (i) and (ii) of the proposition except (c_0, c_0), whose $Q(\cdot, \bar{\delta})$-value is 0, are perfect equilibria stable against the perturbation with proportion β such that $Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta}) = (\mu - \bar{\delta})/(1-\beta)$. Finally,

$$\lim_{(\varepsilon,g)\to(0,0)} \varphi(\varepsilon^2, \varepsilon, \hat{\delta}(g)) = \lim_{(\varepsilon,g)\to(0,0)} \psi(\varepsilon^2, \varepsilon, \hat{\delta}(g)) = 0.$$

This proves that $([0],[0])$ is a perfect equilibrium. The case $\mu < \bar{\delta}$ is similar. If $\mu = \bar{\delta}$, by (iii) and (iv) of Corollary 3, if we can choose a sequence $(\rho^n)_{n=1}^{\infty} = (\varepsilon_A^n, \varepsilon_B^n, g^n)_{n=1}^{\infty}$ converging to $(0,0,0)$ such that for all n, $\hat{\delta}(g^n) = \mu$ and $\varepsilon_A^n = \varepsilon_B^n$, the statement of the proposition is proved. This is indeed possible by (iii) of Lemma 1. □

The set of cut-off point pairs of perfect equilibria in a political game with a uniform θ is illustrated in Figure 4, where the dashed line represent the set of non-perfect Nash equilibria.

Proposition 4 states that if the distribution of random variable θ is biased toward the right (left) relative to the median type, Nash equilibria in which both parties switch around points greater (less) than the median type are not perfect equilibria. This is a direct consequence of Corollary 3 describing the relation between the relative ex ante popularity of policies in the original political game and the Nash equilibria in a perturbed game. Thus these non-perfect

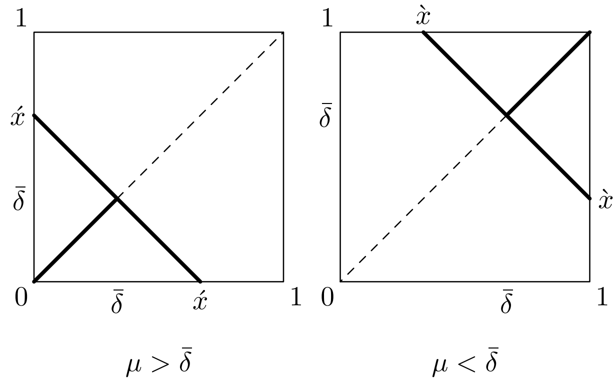

Figure 4: Perfect equilibria

Nash equilibria rely on the condition that one of policy pairs cannot occur with positive probability. In the proof of the proposition, however, we argued that there exists *some* proportion of the parties' error probabilities against which a Nash equilibrium is stable. This suggests that we must further examine whether there are perfect equilibria that are stable against *any* direction of perturbation.

7 Strictly perfect equilibrium

In this section, we study the problem of complete stability raised at the end of the previous seciton through analysis of the strictly perfect equilibrium of Okada (1982). After defining an appropriate notion of strictly perfect equilibrium in our setting, we will show that the only strictly perfect only the corner perfect equilibria are strictly perfect.

We define the strictly perfect equilibrium in a political game as follows focusing, again, only on strategy pairs in which parties choose cut-off strategies.

Definition 15. A strategy profile $(c_{x_A^*}, c_{x_B^*}, (t_\delta^*)_{\delta\in\Delta}) \in S \times S \times \prod_{\delta\in\Delta} T$ is a *strictly perfect equilibrium* in a political game Γ if for every sequence of perturbations $(\rho^n)_{n=1}^\infty = (\varepsilon_A^n, \varepsilon_B^n, g^n)_{n=1}^\infty$ with $\lim_{n\to\infty}\rho^n = (0,0,0)$, and for some number N, there is a sequence of strategy profiles in the perturbed games $\hat{\Gamma}(\rho^n)$ in which the parties take cut-off strategies, $(c^{\varepsilon_A^n}_{x_A^n}, c^{\varepsilon_B^n}_{x_B^n}, (t_\delta^n)_{\delta\in\Delta})_{n\geq N} \in \prod_{n\geq N}\left[\hat{S}(\varepsilon_A^n) \times \hat{S}(\varepsilon_B^n) \times \prod_{\delta\in\Delta}\hat{T}(g^n(\delta))\right]$, satisfying the following conditions:

(i) For every $n \geq N$, $(c^{\varepsilon_A^n}_{x_A^n}, c^{\varepsilon_B^n}_{x_B^n}, (t_\delta^n)_{\delta\in\Delta})$ is a Nash equilibrium of $\hat{\Gamma}(\rho^n)$, and

(ii) $\lim\limits_{n\to\infty}(x_A^n, x_B^n, (t_\delta^n)_{\delta\in\Delta}) = (x_A^*, x_B^*, (t_\delta^*)_{\delta\in\Delta})$.

Remark . If we seek to follow the original definition of strictly perfect equilibrium rigorously, even if we put aside that our political games are not finite, we have to let the set of perturbations for party i include all *functions* $\varepsilon_i : X \to [0,1]$ whose value $\varepsilon_i(x)$ represents the error probability of i at $x \in X$. We thus have made a restriction on the set of possible perturbations , which may bring about some weakening on the requirement of strict perfectness. Similarly, continuity assumption of g may be some weakening. It thus remains open whether a strictly perfect equilibrium by Definition 15 satisfies the same conditions in the definition if we allow for a more general set of perturbations. On the other hand, since we restricted equilibrium strategies in perturbed games to the class of cut-off strategies, the existence of a sequence of Nash equilibria in perturbed games in Definition 15 is a stronger condition than allowing all strategies to consist in Nash equilibria of perturbed games. It is therefore still unclear whether the above definition is a necessary or it is a sufficient condition for a strategy profile to be a strictly perfect equilibrium according to the original definition.

By Corollary 3 and several facts used in the proof for Proposition 4, we first obtain the following result.

Corollary 4. *In any incomplete information political game* Γ*, there is no strictly perfect equilibrium* $(c_{x_A^*}, c_{x_B^*})$ *such that* $0 < x_A^*, x_B^* < 1$.

Proof. Suppose that $\mu \neq \bar{\delta}$. Then, by (i) and (v) of Corollary 3, the equations (12), and continuity of Q, for any interior Nash equilibrium $(c_{x_A^*}, c_{x_B^*})$ of a political game, there exists a particular proportion β of parties' perturbations such that the value of $Q(\cdot, \hat{\delta}(g))$ of any Nash equilibrium in any perturbed game with this proportion β converges to the value of $Q(\cdot, \bar{\delta})$ of x_I^*, $I = A, B$, as perturbations goes to zero. (Recall that $(c_{x_A^*}, c_{x_B^*})$ is a Nash equilibrium if and only if $Q(x_A^*, \bar{\delta}) = Q(x_B^*, \bar{\delta})$) as shown in Corollary 1.) Thus by continuity of Q, for any interior perfect equilibrium in the original game, there exists some β such that any sequence of Nash equilibria of perturbed games with this β converges to another Nash equilibrium of the original game. Therefore, no interior perfect equilibrium is strictly perfect. □

The following proposition shows that only (part of, when $\mu = \bar{\delta}$) the corner perfect equilibria are strictly perfect. As we have stated before, in perturbed games, the corner Nash equilibria always exist. Moreover, as Figure 7 suggests, these equilibria converge to the corner perfect equilibria as perturbation goes to zero. This is the main idea of the proof.

Proposition 5. *Let* $\Gamma = \big((S, S, (T)_{\delta\in\Delta}), (U_A, U_B, (U_\delta)_{\delta\in\Delta}), F, P, \Lambda\big)$ *be a political game, where the mean of* θ *is* μ *and the median type with respect to* F *is* $\bar{\delta}$.

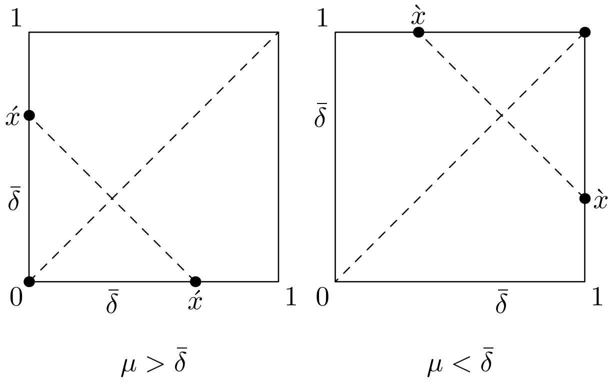

Figure 5: Strictly perfect equilibria

Then, under Assumption 1, the strictly perfect equilibria of Γ *are*

(i) (c_0, c_0), $(c_{\acute{x}}, c_0)$, *and* $(c_0, c_{\acute{x}})$ *if* $\mu > \overline{\delta}$*;*

(ii) (c_1, c_0) *and* (c_0, c_1) *if* $\mu = \overline{\delta}$*;*

(iii) (c_1, c_1), $(c_{\grave{x}}, c_1)$, *and* $(c_1, c_{\grave{x}})$ *if* $\mu < \overline{\delta}$*.*

Proof. See Appendix. □

Figure 5 illustrates the set of strictly perfect equilibria of incomplete information political games in which θ is a uniform random variable, and $\mu \neq \overline{\delta}$, where the dashed lines represent the set of Nash equilibria.

When a corner strategy pair is chosen by the parties, at least one party sticks to announcing one particular policy independent of what value of θ it has observed. Proposition 5 states that such strategy pairs are strictly perfect, and which policy is constantly chosen by a party depends on the relative ex ante popularity of policies in the political game. As noted earlier, the main reason for the robustness of such a corner Nash equilibrium is that for any slight perturbation, there exists a *corner* Nash equilibrium of the perturbed game near that equilibrium. This implies that an inelastic party in the original Nash equilibrium keeps on choosing one policy with probability as large as possible even after perturbation is introduced. Thus the notion of strictly perfect equilibrium in our model requires at least one party not only to take a constant strategy in the original political game but also to do so even if small imperfection in its rationality is introduced.

There is also a remarkable feature in the voters' beliefs about θ consistent with strictly perfect equilibria. Consider, for example, a political game with

$\mu > \overline{\delta}$ and a strictly perfect equilibrium $(s_A, s_B) = (c_{\acute{x}}, c_0)$ in this game. For this strategy pair, $E_{s_A,s_B}(\theta|1,0) = \overline{\delta}$, while $E_{s_A,s_B}(\theta|0,1)$ is not defined since the policy pair $(0,1)$ is out of equilibrium in this case. If, however, $(s_A^n, s_B^n)_{n=1}^{\infty} = (c_{x_A^n}^{\varepsilon_A}, c_0^{\varepsilon_B})_{n=1}^{\infty}$ is a sequence of Nash equilibria in perturbed games converging to (s_A, s_B) as perturbation goes to zero, then

$$\lim_{n\to\infty} E_{s_A^n, s_B^n}(\theta|0,1) = E(\theta|\theta > \acute{x}) > \overline{\delta}.$$

That is, any belief of voters consistent with (s_A, s_B) would expect, on average, θ to be higher than $\overline{\delta}$ if (0,1) were observed, while $E_{s_A,s_B}(\theta|1,0) = \overline{\delta}$. Since a larger level of θ implies that the policy 1 is unpopular among the voters, it then prevents party B to deviate from taking policy 0. Such public image on the policy pair (0,1) is derived only from the equilibrium strategy of party A, due to the stubbornness of party B. Having observed announcement (0,1), the voters consider that even party A takes policy 0 and therefore it is probable that the value of θ is considerably large.

8 Concluding remarks

We have constructed incomplete information political games with Downs type parties in which the parties have informational advantage over voters. We have shown the existence of multiple Nash equilibria and perfect equilibria, and proved that Nash equilibria with strongest asymmetry in the parties' strategies are strictly perfect. Possibility of policy divergence in Nash equilibrium depends on the effect of voters' beliefs on fundamentals that is consistent with the strategies of the parties. A policy that is unpopular among citizens according to the prior information about fundamentals is relatively unlikely to be chosen in perfect equilibria. We have also shown that in any strictly perfect equilibrium, at least one party adopts the sticky strategy selecting an ex ante popular policy independent of observed fundamentals.

The results concerning the Nash equilibrium can be extended to more standard settings where policy spaces have uncountable cardinality. Indeed, if for any level of fundamentals variable, there are two policies with supporters of equal masses, then multiple Nash equilibria including policy divergence exist. In such a model, however, equilibrium refinement would be much more difficult than the present model.

We may also generalize the uncertainty environment to include incomplete information of parties as well as voters, where both parties and voters receive private signals of fundamentals prior to elections. In such a case, on one hand, voters have two sources of information about fundamentals variable: their signals and parties' policy announcements. Each party, on the other hand, infers from its private signal what signals its opponent and voters have received. Bernhardt et al. (2007) develop, somewhat relatedly, a model of political parties with private information about the distribution of voters' preferences, where the set of signals for each party is finite, while assuming complete information on the side of voters. These extensions and generalizations are left open for future research.

9 Appendix: proofs of propositions

Proof of Lemma 2 . Statement (i). By continuity of f and g, L_g is continuous for every $g \in \mathscr{G}$. Since $g(\delta) < \frac{1}{2}$ for all $\delta \in \Delta$, $L_g(0) < \frac{1}{2} < L_g(1)$. Also, $L'_g(x) = -[1-2g(x)]f(x) < 0$ for all $x \in (0,1)$. Thus there exists the critical type $\hat{\delta}(g) = (L_g)^{-1}(\frac{1}{2})$.

Statement (ii). Note that the convergence of a sequence of continuous function to a continuous function on a compact domain is uniform. Let $(g^n)_{n=1}^{\infty}$ be a sequence in $\mathscr{G}$ converging pointwise to the constantly 0-valued function. Then the convergence is uniform. Take any $\alpha \in (0, \frac{1}{2})$. Then there is N such that for all $n > N$ and $\delta \in \Delta$, $g^n(\delta) < \alpha$. Fix such a number N. Then for all $n > N$, $L_{g^n}(0) > 1-\alpha$ and $L_{g^n}(1) < \alpha$.

By the proof of statement (i), the inverse function $(L_g)^{-1}$ exists for every $g \in \mathscr{G}$. For every $n > N$, the domain of $(L_{g^n})^{-1}$ contains interval $A = (\alpha, 1-\alpha)$, which includes $\frac{1}{2}$. So $((L_{g^n})^{-1}|_A)_{n>N}$ converges pointwise to $F^{-1}|_A$. Therefore, $\lim_{n\to\infty} \hat{\delta}(g^n) = \lim_{n\to\infty}(L_{g^n})^{-1}|_A(\frac{1}{2}) = F^{-1}|_A(\frac{1}{2}) = \underline{\delta}$.

Statement (iii). $\hat{\delta}(g) = \overline{\delta}$ if and only if $L_g(\overline{\delta}) = \frac{1}{2}$, which is equivalent to that $A_g = \int_0^{\overline{\delta}} g(u)f(u)du$ equals $B_g = \int_{\overline{\delta}}^1 g(u)f(u)du$. $\hat{\delta}(g) < \overline{\delta}$ if and only if $A_g < B_g$. A sequence $(g_1^n)_{n=1}^{\infty}$ satisfying the properties in the statement can be obtained as follows: for $n > 2$, define g_1^n by

$$g_1^n(\delta) = \begin{cases} \frac{1}{n} & \text{if } \delta \leq \overline{\delta} \\ \frac{\frac{1}{2n}-\frac{1}{n}}{1-\overline{\delta}}(\delta - \overline{\delta}) + \frac{1}{n} & \text{if } \delta > \overline{\delta}. \end{cases}$$

Let $g_1^1 = g_1^2 = g_1^3$. Then, $g_1^n(\delta) < \frac{1}{n}$ for all $\delta > \overline{\delta}$ and all n. Thus $A_{g_1^n} = \frac{1}{2n} > B_{g_1^n}$ for all n, and $\lim_{n\to\infty} g_1^n(\delta) = 0$ for each $\delta \in \Delta$. A sequence $(g_3^n)_{n=1}^{\infty}$ satisfying the properties is constructed by reversing the definitions of $g_1^n(\delta)$ over $[0, \underline{\delta}]$ and $(\underline{\delta}, 1]$ for each n. Finally, for each n, define g_2^n by $g_2^n(\delta) = \frac{1}{n+2}$ for all $\delta \in \Delta$. Then, $A_{g_2^n} = B_{g_2^n} = \frac{1}{2(n+2)}$ for all n, and $\lim_{n\to\infty} g_2^n(\delta) = 0$ for each $\delta \in \Delta$.

Proof of Corollary 3 . Conditions (i) and (v). Suppose $\mu \neq \hat{\delta}(g)$, $\varepsilon_A \neq \varepsilon_B$, and $0 < x_B \leq x_A < 1$. Then, substituting (7) into the equation system

$$E_{s_A,s_B}(\theta|1,0) = E_{s_A,s_B}(\theta|1,0) = \hat{\delta}(g) \tag{13}$$

in (i) of Proposition 3, where $(s_A, s_B) = (c_{x_A}^{\varepsilon_A}, c_{x_B}^{\varepsilon_B})$, and solving for $Q(x_i, \hat{\delta}(g))$, $i = A, B$, yield the two equations

$$Q(x_A, \hat{\delta}(g)) = \varphi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)), Q(x_B, \hat{\delta}(g)) = \psi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)) \tag{14}$$

in the conditions (i) and (v) of Corollary 3. (14) has no solution such that $x_A = x_B$ under the above assumptions. Note that

$$\begin{aligned} &\mu > \hat{\delta}(g),\ \varepsilon_A < \varepsilon_B \Longrightarrow \varphi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)) > \mu - \hat{\delta}(g) = \max_{x \in X} Q(x, \hat{\delta}(g)), \\ &\mu < \hat{\delta}(g),\ \varepsilon_A > \varepsilon_B \Longrightarrow \varphi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)) > 0 = \max_{x \in X} Q(x, \hat{\delta}(g)). \end{aligned} \tag{15}$$

Thus, if $\mu > \hat{\delta}(g), \varepsilon_A < \varepsilon_B$, or if $\mu < \hat{\delta}(g), \varepsilon_A > \varepsilon_B$, then there exists no solution to (14) such that $x_B \leq x_A$. Also, note that

$$\begin{aligned} &\varepsilon_A > \varepsilon_B \Longrightarrow \varphi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)) > \psi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)), \\ &\varepsilon_A < \varepsilon_B \Longrightarrow \varphi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)) < \psi(\varepsilon_A, \varepsilon_B, \hat{\delta}(g)). \end{aligned} \tag{16}$$

Since, by the property (i) in Lemma 1, function $Q(\cdot, \hat{\delta}(g))$ is decreasing on $[0, \hat{\delta}(g)]$ and increasing on $[\hat{\delta}(g), 1]$, there are at most two solutions to (14) such that $0 < x_B \leq x_A < 1$ in each possible case: if $\mu > \hat{\delta}(g)$ and $\varepsilon_A > \varepsilon_B$, strategy pairs with pairs of cut-off points such as (x_A, x_B) and (x'_A, x_B) in the left-hand graph of Figure 6; if $\mu < \hat{\delta}(g)$ and $\varepsilon_A < \varepsilon_B$, strategy pairs with pairs of cut-off points such as (x'_A, x_B) and (x'_A, x'_B) in the right-hand graph of Figure 6. Therefore, in particular, if $\mu > \hat{\delta}(g)$, there exists no Nash equilibrium such that $\hat{\delta}(g) \leq x_B \leq x_A$; if $\mu < \hat{\delta}(g)$, there exists no Nash equilibrium such that $x_B \leq x_A \leq \hat{\delta}(g)$.

Conditions (ii) and (vi). Substituting (7) and $x_B = 0$ into the equation $E_{s_A,s_B}(\theta|1,0) = \hat{\delta}(g)$ in (ii) of Proposition 3 and solving for $Q(x_A, \hat{\delta}(g))$ yield the equation in (ii) of Corollary 3. If this equation is satisfied, then the inequality in (ii) of Proposition 3 is necessarily satisfied. Thus, the condition (ii) in Corollary 3 is equivalent to the condition (ii) in Proposition 3. The equivalence result between the condition (vi) in the corollary and the condition (iv) can be similarly proved.

Condition (iv). If $\varepsilon_A = \varepsilon_B$, the two equations in (i) of Proposition 3, when seen as equations for two variables $Q(x_i, \hat{\delta}(g))$, $i = A, B$, are linearly dependent, and have a solution if and only if $\mu = \hat{\delta}(g)$. Also, the solutions in this case are all $(Q(x_A, \hat{\delta}(g)), Q(x_B, \hat{\delta}(g)))$ such that $Q(x_A, \hat{\delta}(g)) = Q(x_B, \hat{\delta}(g))$. $\square$

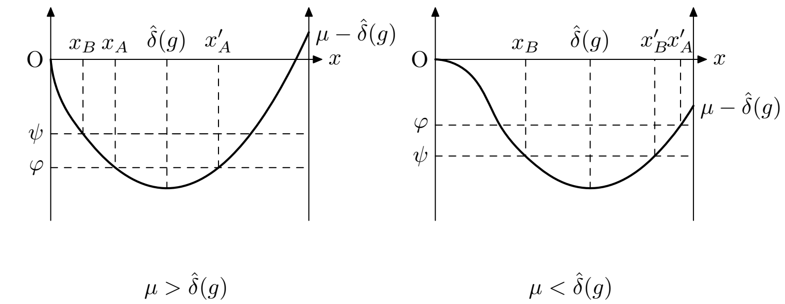

Figure 6: Function Q and interior NE in perturbed games with $\varepsilon_A > \varepsilon_B$

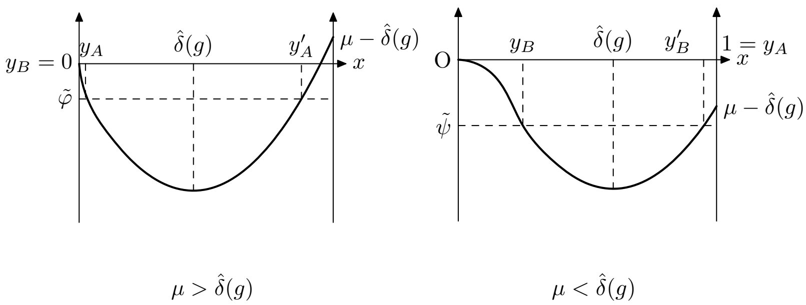

Figure 7: Function Q and corner NE in perturbed games with $\varepsilon_A > \varepsilon_B$

Proof of Proposition 5 . By Corollary 4, the possible strictly perfect equilibria of Γ are the eight strategy pairs which appear in the conditions (i)-(iii) of Proposition 5.

We preliminarily define a function $\tilde{Q} : X \times (\Delta \setminus \{\mu\}) \to \mathbb{R}$ by

$$\tilde{Q}(x,\delta) = \tfrac{Q(x,\delta)}{\mu-\delta} \tag{17}$$

for each $(x,\delta) \in X \times (\Delta \setminus \{\mu\})$.

Then, properties that correspond to the properties (i) and (ii) of function Q stated in Lemma 1 follows:

(i) For each type $\delta \in [0,\mu)$, the function $\tilde{Q}(\cdot,\delta)$ is continuous, decreasing on $[0,\delta]$, increasing on $[\delta,1]$, and takes values $\tilde{Q}(0,\delta) = 0$, $\tilde{Q}(1,\delta) = 1$.

(ii) For each $\delta \in (\mu,1]$, $\tilde{Q}(\cdot,\delta)$ is continuous, decreasing on $[0,\delta]$, increasing on $[\delta,1]$, and takes values $\tilde{Q}(0,\delta) = 0$, $\tilde{Q}(1,\delta) = 1$.

Statement (i). Suppose that $\mu > \bar{\delta}$. Let $\hat{\Gamma}(\rho)$ be a perturbed game with g close enough to 0 so that $\hat{\delta}(g) < \mu$. Then, by Corollary 4, $(s_A,s_B) = (c_{x_A}^{\varepsilon_A}, c_{x_B}^{\varepsilon_B})$ can be a Nash equilibrium of $\hat{\Gamma}(\rho)$ for all $(\varepsilon_A,\varepsilon_B) \in (E\times E)\cap V$ for some neighborhood V of $(0,0)$ in $\mathbb{R}^2$, only if (s_A,s_B) satisfies the condition (ii) or (iv) of Proposition 3.

Consider the condition (ii) of Proposition 3. The constraints for the conditional expectations in (ii) are rewritten as

$$\tilde{Q}(x_A,\hat{\delta}(g)) = -\tfrac{\varepsilon_A}{1-2\varepsilon_A} \text{ and } \tilde{Q}(x_A,\hat{\delta}(g)) \leq \tfrac{1-\varepsilon_A}{1-2\varepsilon_A}. \tag{18}$$

If $\mu > \hat{\delta}(g)$, the inequality in (18) is satisfied for any $\varepsilon_A \in E$.

Note that the convergence $\lim\limits_{\delta<\mu,\delta\to\bar{\delta}} \tilde{Q}(\cdot,\delta) = \tilde{Q}(\cdot,\bar{\delta})$ is uniform since $\tilde{Q}(\cdot,\delta)$, $\delta \in \Delta$, and $\tilde{Q}(\cdot,\bar{\delta})$ are continuous and their domain X is compact. Hence, by the statement about the function $\tilde{Q}(\cdot,\delta)$ with $\delta < \mu$ in (ii) of Lemma 1, there exist $\alpha,\beta > 0$ such that for all $\delta \in B_\alpha(\bar{\delta}) = (\bar{\delta}-\alpha,\bar{\delta}+\alpha)$, the function $\tilde{Q}(\cdot,\delta)$ is decreasing on $[0,\bar{\delta}-\beta]$ and increasing on $[\bar{\delta}+\beta,1]$. Thus, for such α, there exist $\gamma > 0$ such that for all $\delta \in B_\alpha(\bar{\delta})$, functions $\xi_\delta : [-\gamma,0] \to \mathbb{R}$ and $\eta_\delta : [-\gamma,1] \to \mathbb{R}$ given by

$$\text{for each } z \in [-\gamma,0],\ \tilde{Q}(\xi_\delta(z),\delta) = z, \text{ and for each } z \in [-\gamma,1],\ \tilde{Q}(\eta_\delta(z),\delta) = z \tag{19}$$

are well-defined. If we define two other functions $\xi : [-\gamma,0] \to \mathbb{R}$ and $\eta : [-\gamma,1] \to \mathbb{R}$ by

$$\text{for each } z \in [-\gamma,0],\ \tilde{Q}(\xi(z),\bar{\delta}) = z, \text{ and for each } z \in [-\gamma,1],\ \tilde{Q}(\eta(z),\bar{\delta}) = z, \tag{20}$$

the families of functions $(\xi_\delta)_{\delta\in B_\alpha(\bar{\delta})}$ and $(\eta_\delta)_{\delta\in B_\alpha(\bar{\delta})}$ uniformly converge to ξ and η as $\delta \to \bar{\delta}$ again by continuity of the functions and compactness of their domains. Note that $\xi(0) = 0$ and $\eta(0) = \acute{x}$. Therefore, by the same reasoning as in the proof for the statement (ii) of Lemma 2,

$$\lim_{(\varepsilon,\delta)\to(0,\bar{\delta})} \xi_\delta\left(-\tfrac{\varepsilon}{1-2\varepsilon}\right) = 0, \text{ and } \lim_{(\varepsilon,\delta)\to(0,\bar{\delta})} \eta_\delta\left(-\tfrac{\varepsilon}{1-2\varepsilon}\right) = \acute{x}.$$

Since $\left(\xi_{\hat{\delta}(g)}\left(-\frac{\varepsilon}{1-2\varepsilon}\right), 0\right)$ and $\left(\eta_{\hat{\delta}(g)}\left(-\frac{\varepsilon}{1-2\varepsilon}\right), 0\right)$ are pairs of Nash equilibrium cut-off points in $\hat{\Gamma}(\rho)$ and $\hat{\delta}(g) \to \bar{\delta}$ as g goes to the constantly 0-valued function by the statement (ii) of Lemma 2, it has been verified that (c_0, c_0) and $(c_{\acute{x}}, c_0)$ are strictly perfect equilibria of Γ. By symmetry between the parties, $(c_0, c_{\acute{x}})$ is also a strictly perfect equilibrium.

Statement (iii). Similarly, using the condition (vi) of Proposition 3 and the statement (ii) of Lemma 1, (iii) of Proposition 5 can be proved.

Statement (ii). Suppose that $\mu = \bar{\delta}$. In this case, $\acute{x} = 1$ and $\grave{x} = 0$ by definition. By the condition (iii) of Lemma 2, there exists two sequences $(g_1^n)_{n=1}^\infty$ and $(g_3^n)_{n=1}^\infty$ of the voters' perturbations, each converging to 0, such that $\hat{\delta}(g_1^n) < \bar{\delta} = \mu$ and $\hat{\delta}(g_3^n) > \bar{\delta} = \mu$ for all n. Thus, from the proofs of the statements (i) and (iii), the possible strictly perfect equilibria are (c_1, c_0) and (c_0, c_1). Moreover, for any sequences $(g_1^n)_{n=1}^\infty$ and $(g_3^n)_{n=1}^\infty$ with the above properties, and for any sequence of the parties' perturbations $(\varepsilon_A^n, \varepsilon_B^n)_{n=1}^\infty$ converging to $(0,0)$, there exist sequences $(c_{a_A^n}^{\varepsilon_A^n}, c_{a_B^n}^{\varepsilon_B^n})_{n=1}^\infty$ and $(c_{b_A^n}^{\varepsilon_A^n}, c_{b_B^n}^{\varepsilon_B^n})_{n=1}^\infty$ of Nash equilibria in perturbed games $\hat{\Gamma}(\varepsilon_A^n, \varepsilon_B^n, g_1^n), n = 1, 2, \cdots$, such that

$$\lim_{n\to\infty}(a_A^n, a_B^n) = (1,0) \text{ and } \lim_{n\to\infty}(b_A^n, b_B^n) = (0,1) \tag{21}$$

and sequences $(c_{d_A^n}^{\varepsilon_A^n}, c_{d_B^n}^{\varepsilon_B^n})_{n=1}^\infty$ and $(c_{e_A^n}^{\varepsilon_A^n}, c_{e_B^n}^{\varepsilon_B^n})_{n=1}^\infty$ of Nash equilibria in perturbed games $\hat{\Gamma}(\varepsilon_A^n, \varepsilon_B^n, g_3^n), n = 1, 2, \cdots$, such that

$$\lim_{n\to\infty}(d_A^n, d_B^n) = (1,0) \text{ and } \lim_{n\to\infty}(e_A^n, e_B^n) = (0,1) \tag{22}$$

Furthermore, by the condition (iii) of Corollary 3, if $\mu = \hat{\delta}(g_2)$, $(c_1^{\varepsilon_A}, c_0^{\varepsilon_B})$ and $(c_0^{\varepsilon_A}, c_1^{\varepsilon_B})$ are Nash equilibria of any perturbed game $\hat{\Gamma}(\rho)$ with the voters' perturbation g_2.

Thus, if $\mu = \bar{\delta}$, for any sequence of perturbation $(\rho^n)_{n=1}^\infty = (\varepsilon_A^n, \varepsilon_B^n, g^n)_{n=1}^\infty$ converging to $(0,0,0)$, construct two sequences $(x_A^n, x_B^n)_{n=1}^\infty$ and $(y_A^n, y_B^n)_{n=1}^\infty$ as follows: if $\mu > \hat{\delta}(g^n)$, let $(x_A^n, x_B^n) = (a_A^n, a_B^n)$ and $(y_A^n, y_B^n) = (b_A^n, b_B^n)$; if $\mu < \hat{\delta}(g^n)$, let $(x_A^n, x_B^n) = (d_A^n, d_B^n)$ and $(y_A^n, y_B^n) = (e_A^n, e_B^n)$; and if $\mu = \hat{\delta}(g^n)$, let $(x_A^n, x_B^n) = (1,0)$ and $(y_A^n, y_B^n) = (0,1)$. Then, by (21) and (22), $\lim_{n\to\infty}(x_A^n, x_B^n) =$

$(1,0)$ and $\lim_{n\to\infty}(y_A^n, y_B^n) = (0,1)$. Therefore, $([1],[0])$ and $([0],[1])$ are indeed strictly perfect equilibria. This completes the proof for the statement (ii).$\square$

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