Interaction Between Financial Markets

--The limit of the option pricing theory

and arbitrage trading strategy--

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Abstract

We shall investigate the interaction between the Stock Market and the Stock Option Market. When the prediction of the future distribution of the stock prices is different between these markets, there is a chance that the option pricing method of Black and Scholes with its application to an arbitrage trading strategy may lead us to a big market turmoil. We shall investigate the conditions of market stability.

§ 1 Introduction.

Ever since the pioneering work by Black and Sholes (1973), the price of European type option has been calculated from the price and volatility of underlying asset and the risk free interest rate under the assumption of no arbitrage opportunities in the market. For ease of presentation, we will assume throughout the paper that the underlying asset is common stock. Also for the sake of simplicity, we will assume only the stock and option markets comprise the whole economy.

The word "derivative" comes from the fact that option values depends solely on the underlying asset. Nearly all results from arbitrage pricing theory for derivative securities implicitly assume that prices of the derivatives are functions of prices of the underlying assets and their future volatilities. For example, see Black and Scholes (1973), Merton (1973), Harrison and Kreps (1979), and Harrison and Pliska (1981).

In the early history of derivative securities, the total trading volume of the derivatives was very much smaller than that of stocks. Thus, the influence of the option market to the stock market might have been overlooked. However, since the mid 1980's the trading volume of derivatives has experienced a tremendous increase and the derivative markets have begun to significantly influence stock markets. A theoretical investigation into this fact has become necessary. We will consider a symmetric relation between markets where the effect of the present "theoretical" stock price determined in the option market on the stock market is investigated, similar to the influence of the theoretical option price on the "real" option price. In later sections of this paper we will show that if both markets possess the same information concerning the distributions of the future stock price processes, then the arbitrage trading strategy will lead to a stable state in both markets. To be more explicit, we will define what is meant by the "complete information" below and defer the definition of "stable state" to a later section.

Definition 1. If both stock and option markets possess the same prospect of the future distribution of the stock price processes, we say that these markets possess complete information in the strong sense. In this case, these markets are said to be in the state of complete information in the strong sense. Otherwise they are said to be in the state of the incomplete information in the strong sense. //

Remarks. In the next section we are interested in determining the theoretical prices of options and stocks in the absence of arbitrage opportunities in the markets. Here we will assume binomial models for stock movements and hence, having set the binomial model, we can disregard the probabilities of each state occurring at stage 1. The role of "real" probabilities are found when we relate the volatilities in the binomial models to the general continuous time state models. In this situation we may disregard the probabilities attached to each state from definition 1 and the condition of the complete information in the strong sense may be slightly weakened in the binomial models. However, we will assume the strong completeness in the next section in order to maintain the point that the binomial model is an approximation to the more general continuous time model. We shall discuss a weaker version in Section 3 where we briefly discuss a continuous time model. //

In the standard Black-Sholes option pricing theory, the volatility of the underlying assets plays a central role. Here the volatility which will be used to price the option is not the past nor the present volatility; it is the volatility between the present and the maturity day of the options. Therefore, the volatilities to be used must somehow be predicted by the traders in the market. (They can be estimated from past data only if the assumption of constant volatility holds). In addition, the perspectives of the traders strongly influence the determinations of the predicted volatilities in the real markets.

In the following sections we will provide a heuristic discussion about the conditions under which the incomplete information will create market instability. The use of the so-called "Computer Trading" may amplify the problems created by the arbitrage trading strategy with incomplete information.

Many papers discussed the asymmetry of the information within a single market; for example Milgrom and Stokey (1982) consider the effect of private information on the Parato optimal market. However, in this paper, we will consider the asymmetric information between markets. The simple binomial model is discussed in Section 2 and a brief discussion of the Black-Scholes model follows in Section 3.

§ 2 Binomial Model.

In this section, we shall consider a simple binomial with two period model for both the stock and the option markets, where time 0 and time 1 are the present day and the future day respectively. And in the stock market, we are interested in determining the time 0 theoretical price C_0 of the European call option of the stock maturing at $t = 1$ with exercise price K ($K > 0$). We also assume that the risk free interest rate is r ($\geqq 0$).

In the stock market, we will let S_0 be the time $t = 0$ ***actual stock price***. At the date $t = 1$, we will assume that the stock price becomes either $S_1^{(u)}$ with probability $0 < p < 1$, or $S_1^{(d)}$ with probability $1-p$ ($S_1^{(u)} > S_1^{(d)}$). And at $t = 1$, it follows that the value of the option is either $C_1^{(u)} = \max\{S_1^{(u)} - K, 0\}$, if the stock price goes up to $S_1^{(u)}$ or $C_1^{(d)} = \max\{S_1^{(d)} - K, 0\}$, otherwise. The risk free interest rate r is assumed to satisfy the following inequality so as to avoid the arbitrage opportunity;

$$(1) \qquad d = S_1^{(d)} / S_0 \leqq 1 + r \leqq S_1^{(u)} / S_0 = u \quad \text{(Say)}$$

The possible values of the random variable $S_1^{(\cdot)}$ also reflect the market's prediction of the future volatility of the stock price. For example, the larger the difference $S_1^{(u)} - S_1^{(d)}$, the larger the market's predicted volatility. The present stock price is determined by the traders in the stock market. And the predicted future option values in this market may be determined by the predicted future stock prices at the stock market. The theoretical option price C_0 will be obtained from these predicted future stock prices (See (2) below).

On the other hand, the time $t = 0$ actual option prices C^*_0 are determined by the traders in the option market. These traders in the option market will determine the price of the options from their predicted future stock prices, or their future volatilities. Their prediction may be the following: At the future date $t = 1$, the predicted stock price becomes either $S^*_1{}^{(u)}$ with probability $0 < p^* < 1$, or $S^*_1{}^{(d)}$ with probability $1-p^*$. It follows that time $t = 1$ value of the option is either $C^*_1{}^{(u)} = \max\{S^*_1{}^{(u)} - K, 0\}$, when the stock price goes up to $S^*_1{}^{(u)}$, or $C^{*}_{1}{}^{(d)} = \max\{S^{*}_{1}{}^{(d)} - K, 0\}$, otherwise. Then, given this information, just like in the stock market, the theoretical value of the present stock price $\tilde{S}^{*}_{0}$ will be determined in this market. And this kind of symmetry is the key to our analysis.

Now, if the trading of stocks and options are taken place in the different markets, there is no guarantee that they share the same information. And thus they don't have the same perspectives toward future stock prices. Or it will take at least few microsecond to exchange the information. So, in general, we are unable to assume that $S_{1}^{(u)} = S^{*}_{1}{}^{(u)}$, $S_{1}^{(d)} = S^{*}_{1}{}^{(d)}$, and $p = p^{*}$. It follows that the predicted future volatilities in these markets may be different.

In the stock market, the theoretical option price (arbitrage free option price) is obtained by considering the equivalent portfolio and it is given by:

(2) $$\tilde{C}_{0} = (1/1+r)\{qC_{1}^{(u)} + (1-q)C_{1}^{(d)}\},$$

where $q = [(1+r)S_{0} - S_{1}^{(d)}]/[S_{1}^{(u)} - S_{1}^{(d)}] = [(1+r) - d]/[u - d]$ is the Martingale measure (See [Picture 1]). By the same argument leading to (2), the theoretical stock price in the option market will be given by

(3) $$\tilde{S}^{*}_{0} = (1/1+r)\{q^{*} S^{*}_{1}{}^{(u)} + (1-q^{*})S^{*}_{1}{}^{(d)}\}$$

where, $q^{*} = [(1+r)C^{*}_{0} - C^{*}_{1}{}^{(d)}]/[C^{*}_{1}{}^{(u)} - C^{*}_{1}{}^{(d)}]$ is the Martingale measure at the option market (See [Picture 2]).

[Picture1]

[Picture2]

If, in the stock market, the theoretical option price $\hat{C}_0$ is higher than the market price C^*_0, then the investor may create the arbitrage opportunity by taking the following trading strategy: Buy one unit of option at the price C^*_0. And form the portfolio P_s: Short α unit of stock and rent B yen at the rate r, where

$$(4) \qquad \alpha = [C_1{}^{(u)} - C_1{}^{(d)}] / [S_1{}^{(u)} - S_1{}^{(d)}],$$

$$(5) \qquad B = [C_1{}^{(d)} S_1{}^{(u)} - C_1{}^{(u)} S_1{}^{(d)}] / [(1+r)(S_1{}^{(u)} - S_1{}^{(d)})]$$

Then the time 0 value of this portfolio P_s is $\hat{C}_0$, and gives the trader the immediate positive profit of $\hat{C}_0 - C^*_0$. And at day t = 1, the value of the portfolio P_s is exactly equal to the option value, so that the payoff for the trader is zero. Thus there is an arbitrage opportunity. Now, this trading strategy consists of buying the option and selling the stock, and this may result in pushing the price of the option higher and lower the stock price.

Next, we turn to the option market. Given the time 0 option price C^*_0, the arbitrage free stock price in the option market will be $\hat{S}^*_0$. And as before, we shall suppose that $\hat{C}_0 > C^*$ in the stock market. If $\hat{S}^*_0$ is smaller than the real stock price, then the trading strategy in this market will be that of selling the stock and buying the option, thus the price adjusting process here will be in the same direction as in the stock market. (Here, we may sell a share of stock and buy a portfolio with α^* unit of option and B^* yen of saving at the risk free interest rate r, where

$$(4') \qquad \alpha^* = [S^*_1{}^{(u)} - S^*_1{}^{(d)})] / [C^*_1{}^{(u)} - C^*_1{}^{(d)})]$$

(5') $B^* = [C^*_1{}^{(d)} S^*_1{}^{(u)} - C^*_1{}^{(u)} S^*_1{}^{(d)}] / [(1+r)(C^*_1{}^{(u)} - C^*_1{}^{(d)})]$).

The trouble may occur if the theoretical stock price $\tilde{S}^*_0$ becomes higher than the real stock price. In this case, the traders in the option market take the trading strategy of buying the stock and selling the option, thus getting an arbitrage opportunity within the market. This trading strategy may push up the stock price and may push down the option price which are contradictory to the movement of the stock market. If the trading volume of either market is small enough so that there is a dominant market, we may expect the movement in the dominant market to make the stock and option price move toward the equilibrium state. But, if neither is dominant, the two markets may crash and the movement of the stock price or its volatility becomes wilder and wilder. We are now ready to introduce the concept of market stabilization:

Definition 2. If in the option market and the stock market, $\tilde{C}_0 \geqq C^*_0$ occurs if and only if $S_0 \geqq \tilde{S}^*_0$, then the markets are said to be in a *stable state* or simply *stable*.

We will now investigate the conditions under which the markets satisfy the stability condition. The following lemma gives us the most stringent conditions.

Lemma 1. We will suppose that the stock price processes are two period binomial process. Suppose that both the stock and the option market are in the state of complete information in the strong sense. Then, the markets are stable. Namely, the price adjusting process by means of the arbitrage trading strategies in both markets are consistent.

Proof. If the assumption of complete information is satisfied, we can assume that $S^*_1{}^{(u)} = S_1{}^{(u)} = uS_0$, $S^*_1{}^{(d)} = S_1{}^{(d)} = dS_0$. It follows that $C^*_1{}^{(u)} = C_1{}^{(u)}$, $C^*_1{}^{(d)} = C_1{}^{(d)}$, and

(6) $$\tilde{S}^*_0 = \{q^* uS_0 + (1-q^*) dS_0\} / [1+r],$$

where $q^* = [(1+r)C^*_0 - C^*_1{}^{(d)}] / [C^*_1{}^{(u)} - C^*_1{}^{(d)}]$.
If $\tilde{C}_0 \geqq C^*_0$, then

(7) $$q^* \leqq [(1+r)C_0 - C_1^{(d)}]/[C_1^{(u)} - C_1^{(d)}] = q.$$

Thus, we have,

(8) $$S^*_0 \leqq \{quS_0 + (1-q)dS_0\}/[1+r] = S_0.//$$

The assumption of the complete information is far too strong, we will therefore investigate weaker conditions under which the markets are stable. Of course, if there is no complete information, the perspective of the future distribution of the stock price in both markets may be different. We will consider several cases through numerical examples below.

To fix the idea, suppose in the option market $S^*_0 > S_0$ holds. Since the option price is an increasing function of the present stock price, we may expect that $C^*_0 > C_0$. But at the same time the option price is an increasing function of the volatility. So, if the expected volatility in the stock market is much bigger than that of the stock market, the reverse relation ($C^*_0 < C_0$) may hold. We will illustrate the situation by the following artificial example.

Example. Suppose the riskfree interest rate is $r = 0.1$. We will consider the call option on the stock maturing at day 1 with the exercise price $K = 150$ yen. In the stock market, we suppose the actual stock price is 180 yen per share, and we expect the next day stock price is either 270 yen or 90 yen. It follows that the Martingale measure is given by $q = (180 \times 1.1 - 90)/(270 - 90) = 0.6$. Hence $C_0 = (1/1.1)(0.6 \times 120) = 65.45$ (yen) ([Picture 3]).

[Picture3]

Next, we will consider the pricing problems in the option market. In case 1, we will assume the predicted volatility in the option market is slightly smaller than that of stock market, where as in case 2, the volatility is far much smaller.

[Case 1] Suppose that the trader's anticipation of the stock price of the day 1 is 263.25yen and 92.25yen. We also suppose that the option price in the market is 63yen, which is less than the theoretical value at the stock market C_0 = (65.45). Here, we have the Martingale measure : q^* = (63x1.1-0) / (113.25-0) = 0.612 and thus have S_0^* = (1/1.1){263.25x0.612 + 92.25x0.388) = 178.90 (this is less than 180). And in this case the price adjusting processes may work properly (See [Picture 4]).

[Picture 4]

$$S^*_0 = (178.90) \quad \begin{cases} S^{*(u)}_1 = 263.25 \\ S^{*(d)}_1 = 92.25 \end{cases} \qquad C^*_0 = 63 \quad \begin{cases} C^{*(u)}_1 = 113.25 \\ C^{*(d)}_1 = 0 \end{cases}$$

[Case 2] If in the option market, the trader's anticipation of the next day stock price is either 243yen or 99yen, where the volatility is substantially smaller than the case 1. Then the option prices will be either 93yen or 0yen. If the market price of the option is 63yen as above, then the Martingale measure is given by q^* = (63x1.1-0) / (93-0) = 0.745, and therefore S_0^* = (1/1.1){243x0.745+99x0.255} = 187.54 (> 180). And in this case the price adjusting processes may not work properly ([Picture 5]).

[Picture5]

$$S^*_0 = (187.54) \quad \begin{cases} S^{*(u)}_1 = 243 \\ S^{*(d)}_1 = 99 \end{cases} \qquad C^*_0 = 63 \quad \begin{cases} C^{*(u)}_1 = 93 \\ C^{*(d)}_1 = 0 \end{cases} \quad //$$

It is easily seen from (3) and by some algebra that the arbitrage free stock price S^*_0 is given as the solution of the following equation,

(9) $\quad C^*_0 = (1+r)^{-1}\{q^* C_1^{*(u)} + (1-q^*) C_1^{*(d)}\}$

where $q^* = [(1+r)S^*_0 - S^{*(d)}_1] / [S^{*(u)}_1 - S^{*(d)}_1]$. Namely, S^*_0 is an implied stock price given r and the option price C^*_0. And this fact is the key step to extend our results to the continuous time and state models in the next section.

It is not difficult to see that if $S_1^{(u)} - S_1^{(d)}$ is much bigger than $S_1^{*(u)} - S_1^{*(d)}$, then there is a chance that $\tilde{C}_0 > C^*_0$ and $S_0 < \tilde{S}^*_0$ occurs. To see this, we shall suppose $r = 0$ for simplicity. It follows from (2) and (9) that

(10) $S_0 = \{[S_1^{(u)} - S_1^{(d)}]\tilde{C}_0 + S_1^{(d)}C_1^{(u)} - S_1^{(u)}C_1^{(d)}\}/\{C_1^{(u)} - C_1^{(d)}\}$, and

(11) $\tilde{S}^*_0 = \{[S^{*}_1{}^{(u)} - S^{*}_1{}^{(d)}]C^*_0 + S^{*}_1{}^{(d)}C^{*}_1{}^{(u)} - S^{*}_1{}^{(u)}C^{*}_1{}^{(d)}\}/\{C^{*}_1{}^{(u)} - C^{*}_1{}^{(d)}\}$

For some $\varepsilon > 0$ and $\eta > 0$, let us suppose $A = [S_1^{(u)} - S_1^{(d)}] = [S^{*}_1{}^{(u)} - S^{*}_1{}^{(d)}] - \varepsilon$, and $a = \{C_1^{(u)} - C_1^{(d)}\} = \{C^{*}_1{}^{(u)} - C^{*}_1{}^{(d)}\} - \eta$, (say). Then, $S_0 < \tilde{S}^*_0$ if and only if

$$\text{(12)}\quad \{A\tilde{C}_0 + S_1^{(d)}C_1^{(u)} - S_1^{(u)}C_1^{(d)}\}/a < \{(A-\varepsilon)C^*_0 + S^{*}_1{}^{(d)}C^{*}_1{}^{(u)} - S^{*}_1{}^{(u)}C^{*}_1{}^{(d)}\}/(a-\eta)$$

It is not difficult to see that even if $\tilde{C}_0 > C^*_0$, with sufficiently large ε (and therefore large η), $[S_1^{(d)}C_1^{(u)} - S_1^{(u)}C_1^{(d)}]$ may be sufficiently smaller than $[S^{*}_1{}^{(d)}C^{*}_1{}^{(u)} - S^{*}_1{}^{(u)}C^{*}_1{}^{(d)}]$ so that the equation (12) holds. The magnitude of the ε corresponds to the difference of the volatility forecast.

We have proved heuristically that the difference of the volatility forecast between markets may cause the turmoil in the financial markets. To obtain the qualitative conditions under which the market stabilization is difficult and messy even in this simple model.

To close this section, we will answer to the possible criticism against our formulation. One of the criticism is that since the state of natures at the stock market and the option market are different in our formulation, there will be no equivalent measure applied to both markets. Therefore these markets cannot have the same transaction to these assets and the model may not be reasonable. In this paper, we will simply mention the following reason which answers the question. We will claim that the binomial models are the approximations to the general continuostime state models which are discussed in the next section. And in the original model, the possible states are assumed to be the subset of the real line and the both markets can have the same transactions. Also we still be able to assume the different volatilities in these markets.

§ 3. Continuous time models.

In this section we will assume Black-Scholes model for the stock price processes. Suppose that (μ, σ) and (μ^*, σ^*) are the predicted drift and volatility at the stock and the option market respectively. Then the predicted stock price process in each markets are given as follows:

$$(13) \qquad dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 = s \qquad \text{(Stock Market)}$$

$$(14) \qquad dS^*_t = \mu^* S^*_t dt + \sigma^* S^*_t dW_t, \qquad S^*_0 = s \qquad \text{(Option Market)}$$

We will consider the European call option on this stock with maturity T, and exercise price K. And we shall suppose the riskfree interest rate is r and we will use a continuously compounding interest rate.

In the stock market, given the time t stock price S_t, the time t theoretical option price $\mathbb{C}_t$ is given by the famous Black-Scholes formula,

$$(15) \qquad \mathbb{C}_t = S_t \Phi(h) - e^{-r(T-t)} K \Phi(h - \sigma (T-t)^{1/2}),$$

where Φ denotes the standard normal distribution function, and

$$(16) \qquad h = \{\log(S_t / Ke^{-r(T-t)}) + (\sigma^2/2)(T-t)\} / \{\sigma (T-t)^{1/2}\}.$$

As in the previous section, if the actual option price C^*_t is different from $\mathbb{C}_t$, there is an arbitrage opportunity. For example, if $\mathbb{C}_t > C^*_t$, then the strategy of selling the stock and buying the option will gives us an arbitrage opportunity.

In the option market we are given the market price of the option C_t^*. Then, the time t theoretical stock price $\mathbb{S}^*_t$ may be obtained implicitly by solving the following equations with respect to $\mathbb{S}^*_t$:

$$(17) \qquad C^*_t = \mathbb{S}^*_t \Phi(h^*) - e^{-r(T-t)} K \Phi(h^* - \sigma^* (T-t)^{1/2}),$$

where

$$(18) \qquad h^* = \{\log(\mathbb{S}^*_t / Ke^{-r(T-t)}) + (\sigma^{*2}/2)(T-t)\} / \{\sigma^* (T-t)^{1/2}\}.$$

In this market, the traders may be interested in observing the difference between the theoretical stock price $\tilde{S}^*_t$ and the actual stock price S_t. And, if $S_t > \tilde{S}^*_t$, then the strategy of selling the stock and buying the option will give us an arbitrage profit. Here, if we have $S_t > \tilde{S}^*_t$ and $\tilde{C}_t < C^*_t$, as in the previous section, we may face with the problem of the market instability (See Definition 2).

Of course, if the information is complete in the strong sense of Definition 1, then the both market share the same future distribution of the stock price. Therefore $S_t = \tilde{S}^*_t$ and $\tilde{C}_t = C^*_t$ must hold and the markets are in the stable state. We will see that the stable state may be achieved even under the weaker assumption. And for this purpose we will state the following,

Definition 3. If both stock and option markets possess the same prospect of the future volatilities of the stock price processes, we say that these markets possess the complete information in the weak sense. In this case, these markets are said to be in the state of complete information in the weak sense.//

If the market are in the state of complete information in the weak sense, then both market use the same volatility value to calculate the Black-Scholes' formula; $\sigma^2 = \sigma^{*2}$. Since the option price is the increasing function of the stock price for fixed volatility values, $\tilde{C}_t \geqq C^*_t$ holds if and only if $S_t \geqq \tilde{S}^*_t$. And these markets are in the stable state. We have thus trivially proved the following,

Lemma 2. Suppose the stock price processes are given by (13) and (14). Suppose also that the stock and option market are in the state of complete information in the weak sense. Then, the markets are stable.//

If the assumption of the complete information in the weak sense does not hold, the trouble may occur if $\tilde{C}_t > C^*_t$ and $\sigma^2 > \sigma^{*2}$ or $\tilde{C}_t < C^*_t$ and $\sigma^2 < \sigma^{*2}$. In the former case, there is a chance to have $S_t < \tilde{S}^*_t$ (and $\tilde{C}_t > C^*_t$), and in the latter case, we may have $S_t > \hat{S}^*_t$ (and $\hat{C}_t < C^*_t$). In both cases, the markets will be in the unstable state and the no arbitrage pricing mechanism will break down. Analytically obtaining the area of $(S_t, \hat{S}^*_t, \sigma^2, \sigma^{*2})$ where we have stable state may be difficult, but it may be obtained numerically in each case.

§ 4. Concluding Remarks.

This paper is inconclusive at least in two respects. The first one is that we have not obtained the analytical conditions under which the markets are stable. The second problem is that we did not conduct any empirical justification. For the empirical study of this problem, we need to collect the trader's perspective of the future volatility values which may have been used to set up the basic parameters in the computer trading. Or the use of the implied volatilities may give us some hint. We will continue to work on these issues and will present the result elsewhere.

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