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RISK SHARING IN A NETWORK OF TRANSACTIONS
– A PUBLIC INFORMATION CASE –

by

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Abstract

In this paper we present a model of a network or an arrangement of transactions that involve a risky transfer of assets. Transactions are generated endogenously. There is a risk in asset transfers and we are concerned with the question of optimal risk management in such a network. Assets in this paper may well be usual commodities and not limited to financial assets.

If there is some risk of failure in a transfer from one party to another, should the transfer be done through that arrangement? If so, then what considerations are relevant to determining whether third parties ought to share that risk? Are there conditions under which the general public or the government (in the case of a private arrangement) ought to bear some risk and, if so, what level of compensation would it be appropriate for them to receive? In the present paper, we address these questions by analyzing a schematic, formal, model of a network of transactions.

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1 Introduction

In this paper we present a model of a network or an arrangement of transactions that involve a risky transfer of assets. Transactions are generated endogenously. There is a risk in asset transfers and we are concerned with the question of optimal risk management in such a network. Although this paper is largely motivated by the consideration of risk management in a network of financial assets transfers, assets in this paper may well be usual commodities and not limited to financial assets.

Some specific questions regarding risk management in a network of transactions are the following. If there is some risk of failure in a transfer from one party to another, should the transfer be done through that arrangement? If so, then what considerations are relevant to determining whether third parties ought to share that risk? Are there conditions under which the general public or the government (in the case of a private arrangement) ought to bear some risk and, if so, what level of compensation would it be appropriate for them to receive? If a third party possesses private information that would be of value in determining how best to make a transfer, how does the exposure of that party to the transfer risk affect the quality of information that the party chooses to provide? In the present paper, we address the first three questions by analyzing a schematic, formal, model of a network of transactions. [1]

In a sense, a network of transactions generates rents for its participants, both the service providers and their customers, in most states of the world. In exceptional states where there are catastrophic losses, though, these losses can be transmitted to many or all of the participants through transfer failures. Thus, since the network participants bear this risk that is highly correlated among them, it is efficient for them as a group to make an insurance contract with the group of non-participants. In our model environment, it is efficient for network participants normally to distribute part of their rents to non-participants, in return for which the non-participants will indemnify part of the participants' losses in the catastrophic event. Such a loss-sharing arrangement between the group of the network participants and the non-participants is the theoretical counterpart of "safety-net" arrangements in actual economies.

This explanation has a general-equilibrium character. That is, it depends on reasoning about the welfare and incentives of all of the participants in the network — and even of non-participants — rather than of service providers (such as banks) alone. For that reason, it cannot be formalized within traditional models of policy towards network risk. Those models have been partial-equilibrium models that analyze the profitability and decision-making of traders as endogenous, but that represent the traders' customers only implicitly, via a stream of transfer orders that is specified to be exogenous. Because the gist of our argument is that wise network risk policy enhances welfare largely through its indirect effect on the equilibrium decisions of the makers and receivers of a transfer, rather than through its direct effect on service providers alone, this argument has to be formulated in a new, general-equilibrium, model.

In order to make such a general-equilibrium model analytically tractable, we have chosen to abstract from two features of transactions' network in modern economies. We disregard the distinction between transfer-service providers and their customers who are

[1] The last question will be analyzed by our joint paper. For a very preliminary result, see F. Fujiki, E. Green, and A. Yamazaki [2].

the ultimate makers and receivers of transfers. Instead, in our model environment, some traders play a dual role as providers of transfer services to others and as customers of others. Also, we disregard the fact that transactions are settled by transfers of money rather than goods. Strictly speaking, ours is a model of circular trade with risky delivery of goods.

Transfer-arrangement designers, managers, and policy makers are well aware that the rules governing an arrangement can affect users' decisions about which transactions to make through the arrangement. Thus, to set the rules of an arrangement is implicitly to decide which transactions will be done through it, and which transactions people will decide to do in alternative ways. (In fact, in case of a network of transactions of financial assets, rules governing an arrangement that lacks stringent risk controls are sometimes designed deliberately to make the arrangement infeasible or unattractive for use in making very large-value payments.) By modeling the cooperative setting of rules by participants in a transfer arrangement, and by participants in the economy as a whole, from this perspective, we are able to analyze welfare questions in a conceptually satisfactory way. Rather than taking that approach of specifying transactions exogenously as previous researchers have typically done, what we take to be exogenous are traders' utility functions, which we specify in a way that provides scope for welfare-improving transactions among some of the traders to occur. We also specify a transfer technology that imputes risks and costs to those potential transactions. Having specified the model in these terms, we are able to characterize the efficient patterns of transactions that the traders would choose to make.

This approach provides answers, for the class of model economies that we study, to the questions posed above. Not surprisingly, risk considerations play a role in determining which transactions ought to be made. The specifics of that role can be quite interesting. For instance, under plausible conditions, even the general public (that is, traders who would not have transactions with the members of the transfer arrangement if risk were not present) ought to share transfer risk, as can happen in practice in a settlement arrangement of financial assets, when a central bank serves as guarantor of the settlement arrangement. While the results obtained about a schematic model economy are far from constituting definitive advice regarding actual transfer arrangements, we hope that our analysis may at least provide a helpful framework within which to think in an organized way about the issues involved in practical cases.

2 Model of a transaction network

Our first task is to formulate a model of a transaction that involves a risky asset transfer. The model should be rich enough to describe such a transaction recognizably, but simple enough to be analytically tractable.

Consider what sort of model could satisfy both the requirements of richness and simplicity. A transaction is a related set of asset transfers between traders. The assets involved might be either commodities or financial assets. An asset transfer involves two traders, the donor and the recipient, but a transaction can generally involve more than two traders. Therefore, at the very least, a model of a transaction involving a risky transfer should include three traders, so that a distinction can be drawn between a participant in the broad transaction and a participant (that is, the donor or the recipient) in the specific transfer where the risk occurs. In order for the third-party participant in the transaction — that is, the participant who is neither the donor nor the recipient of the risky transfer — to be essential to making a mutually beneficial transaction, there should be no "double coincidence of wants" between the donor and the receiver. This consideration suggests modeling the three participants as a "Wicksell triangle."

There is a distinction between two types of third party (or potential third party) that a good model ought to capture. A third party to risky transfer in a Wicksell triangle might be intrinsically necessary in the sense that the donor and recipient of the risky transfer would have no double coincidence of wants, even if the transfer did not involve risk (that is, if the recipient would receive the expected value of the transfer with certainty). Alternatively, the riskiness of the transfer might impair a double coincidence of wants that would exist under certainty between the donor and the recipient, and the third party might be needed solely to restore that double coincidence by serving as a guarantor or insurer of the transfer. For characterizing the differences between the roles of these two types of third parties, a four-trader model (including both an intrinsic third party and a trader whose only involvement would be to share risk) can be useful. On the basis of these considerations, we will specify the set of traders to be $\{1, 2, \ldots, N\}$ with $N = 4$. We will assume that trader 1 is essential to a mutually beneficial transaction but that trader 2 is the donor and trader 3 is the recipient of the risky transfer. The attributes of trader 4 will be specified in such a way that trader 4 can only participate in a risk-sharing capacity.

2.1 Probability structure

The risky transfer will be formalized in terms of a state space, Ω. An algebra of events (that is, subsets of Ω) is assumed to exist, and a probability measure Pr is defined on the algebra.

There is a distinguished event $S \subset \Omega$, with $0 < \Pr(S) < 1$. Assume that the risky transfer from trader 2 to trader 3 succeeds in S, and that it fails in the complementary event $F = \Omega \backslash S$. When we say that the transfer succeeds, we mean that trader 3 receives the entire quantity of the asset that is transferred. When we say that the transfer fails, we mean that the quantity of the asset that was intended to be transferred disappears irretrievably from the economy.[2] [3] To simplify the notation we write $\sigma = \Pr(S)$, and we assume

$$1/2 < \sigma < 1$$

.

2.2 Commodities, endowments, and preferences

Assume that each trader i has an endowment consisting solely a type of commodity that only he possesses. We denote that type of commodity also by i. Intuitively, trader i is

[2]Failure of an actual transfer seldom involves such an irretrievable loss, although there are some contemporary examples and many historical examples of that type of failure.

[3]To analyze incentive issues, we will specify that trader 1 privately observes an event that is statistically relevant to the outcome of that risky transfer. This private information will be formalized in a separate paper in terms of events in the probability space Ω. See H. Fujiki, E. Green, and A. Yamazaki [2].

endowed with one unit of commodity of type i with certainty. In order to discuss state-contingent trade of these endowments, we adopt Arrow's convention that each type of commodity is a class of state-contingent commodities, one for each state in Ω. Thus the set of all commodities is $N \times \Omega$. Each trader i is endowed with one unit of commodity (i, ω) for every $\omega \in \Omega$.

A commodity bundle is represented by a measurable function $\gamma: N \times \Omega \to \mathsf{R}_+$. This definition is conventional, since $N \times \Omega$ is the commodity space.

Each trader's preference between commodity bundles conforms to expected utility. Trader i has a von Neumann-Morgenstern utility function $U^i: \mathsf{R}_+^N \to \mathsf{R} \cup \{-\infty\}$. Trader i's expected utility of consuming a commodity bundle γ is the expectation of the random variable $U^i(\vec{\gamma})$, where $\vec{\gamma}: \Omega \to \mathsf{R}_+^N$ is defined by

$$\forall \omega \in \Omega \quad \vec{\gamma}(\omega) = (\gamma(1, \omega), \ldots, \gamma(N, \omega)) \tag{1}$$

2.3 Structure of information and economic activity

The sequence of economic activities in this economy is as follows.

Initially, before knowing whether the actual state of nature is in S or F, traders make an agreement for transfers of goods among them. The agreement among the traders is binding.

With one exception, the transfers are safe. That is, everything sent out reaches its intended recipient in its entirety and with certainty. The exception is the transfer of trader 2's endowment to trader 3. Recall that this transfer reaches trader 3 in its entirety in event S, but is completely and irretrievably lost in event F.

The traders also agree *ex ante* on a second round of transfers, to be made after the first round transfers have been completed and the result of the risky transfer has become known. Thus the transfer to be made in the second round can be made contingent on which of the events S and F has occurred.[4] All second-round transfers, including the one from trader 2 to trader 3, are nonstochastic. However, second-round transfers are costly. Only a proportion $\rho < 1$ of the goods that a trader sends in the second round are received.[5]

The two rounds of contractually specified transfers are then made. Traders consume their stocks of goods after these two rounds of transfers have been completed.

[4]Strictly speaking, this sentence describes a different information structure from the preceding one. If traders can only distinguish between events S and F on the basis of observing the success or failure of a transfer, then they can not make any distinction unless a (non-zero) transfer has been attempted. To assume that they can make a state-contingent transfer in the second round even if no first-round transfer from 2 to 3 has been attempted neglects this limitation of their opportunity for inference. In the case where there is no private information, this ambiguity is harmless because risk-averse traders would not cooperatively choose to make a state-contingent transfer in the second round unless they had exposed themselves to transfer risk in the first round. How the ambiguity is resolved is important in the private-information case, though, and we will discuss this issue further when we analyze private information in a separate paper.

[5]This assumption, sometimes called "iceberg cost," can be viewed as a crude way of reflecting various intuitive considerations including time preference and exposure to business loss due to delayed availability of transferred funds.

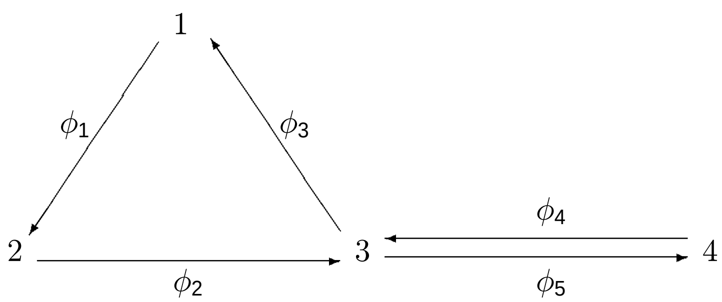
2.4 Consumption determined by a transaction

To simplify the characterization of traders' consumption resulting from settlement, we make two assumptions: that a trader is able to transfer only his own endowment good, and that only a few of the possible flows of those goods are feasible. Specifically a *round of transfers* is a vector $\phi \in \mathsf{R}^5_+$. The components of ϕ are interpreted as follows.

1. ϕ_1 is the amount sent from trader 1 to trader 2;
2. ϕ_2 is the amount sent from trader 2 to trader 3;
3. ϕ_3 is the amount sent from trader 3 to trader 1;
4. ϕ_4 is the amount sent from trader 4 to trader 3;
5. ϕ_5 is the amount sent from trader 3 to trader 4.

As described above, either all, a proportion ρ, or none of the goods sent may be received. A *transaction* is a sequence $\tau = (\tau^1, \tau^S, \tau^F)$ of rounds of transfers. The elements τ^1, τ^S, and τ^F specify the initial round of transfers, and the round of transfers in event S and in event F, respectively.

Figure 1: Round of Transfers



A transaction is feasible if no trader is ever required to send a cumulative amount that would exceed his endowment. That is, transaction τ is feasible if

$$\forall i \;\; \tau_i^1 + \max\{\tau_i^S, \tau_i^F\} \leq 1. \tag{2}$$

Let $\mathcal{T}$ denote the set of feasible transactions.[6]

Now we provide an explicit definition of traders' consumptions resulting from a transaction. The following table specifies vectors z^1–z^5 in terms of which these consumptions are defined.

z^1	$(1,0,0,0)$
z^2	$(0,1,0,0)$
z^3	$(0,0,1,0)$
z^4	$(0,0,0,1)$
z^5	$(0,0,1,0)$

Furthermore, let χ_S and χ_F denote the indicator functions of S and F respectively, and define $\tau^\chi(\omega) = \chi_S(\omega)\tau^S + \chi_F(\omega)\tau^F$.

The consumption vector $c^i(\tau,\omega)$ that trader i receives in state ω as a consequence of transaction τ is as follows.

$$\begin{aligned}
c^1(\tau,\omega) &= (1-(\tau_1^1+\tau_1^\chi(\omega))z^1 + (\tau_3^1+\rho\tau_3^\chi(\omega))z^3 \\
c^2(\tau,\omega) &= (1-(\tau_2^1+\tau_2^\chi(\omega))z^2 + (\tau_1^1+\rho\tau_1^\chi(\omega))z^1 \\
c^3(\tau,\omega) &= (1-(\tau_3^1+\tau_3^\chi(\omega))z^3 - (\tau_5^1+\tau_5^\chi(\omega))z^5) \\
&\quad + (\chi_S(\omega)\tau_2^1+\rho\tau_2^\chi(\omega))z^2 + (\tau_4^1+\rho\tau_4^\chi(\omega))z^4 \\
c^4(\tau,\omega) &= (1-(\tau_4^1+\tau_4^\chi(\omega))z^4 + (\tau_5^1+\rho\tau_5^\chi(\omega))z^5
\end{aligned} \tag{3}$$

3 The core

We modify the core of an exchange economy to serve as the solution concept to characterize the set of transactions to which the traders might agree. A core allocation is one that can be obtained (according to (3)) by a feasible transaction, and such that no coalition of traders can implement another allocation that its members unanimously prefer — with at least one of them having a strict preference — by using an alternative transaction that is feasible for its members. Define a core transaction to be a feasible transaction from which a core allocation is obtained via (3).

To formalize the notion of unanimous preference within a coalition, for each nonempty $C \subseteq \{1,\ldots,N\}$, define $\theta \in \mathcal{T}$ to *C-dominate* $\tau \in \mathcal{T}$ if

$$\begin{aligned}
&\forall i{\in}C \quad E[U^i(c^i(\tau,\omega))] \le E[U^i(c^i(\theta,\omega))] \quad \text{and} \\
&\exists i{\in}C \quad E[U^i(c^i(\tau,\omega))] < E[U^i(c^i(\theta,\omega))].
\end{aligned} \tag{4}$$

Also define $\theta \in \mathcal{T}$ to be *feasible for C* if

$$\forall i{\notin}C \quad \forall\omega \quad c^i(\theta,\omega) = z^i \qquad \text{(No participation of other traders is required).} \tag{5}$$

Finally, define $\tau \in \mathcal{T}$ to be a *core transaction* if there exist no $C \subseteq N$ and $\theta \in \mathcal{T}$ such that θ is feasible for C and θ C-dominates τ. Let us say that transaction τ is

[6] As noted in the footnote above, the informational constraint that, if $\tau_2^1 = 0$, then $\tau^{\mathsf{S}} = \tau^{\mathsf{F}}$, may or not be added to the definition of feasibility for a transaction. If all traders are risk averse, then the constraint is never binding when traders have common information.

individually rational if it is weakly preferred to autarky by every $i \in N$, and that τ is *Pareto-undominated* if it is undominated for N.

4 Analysis of a public information environment

We will carry through our analysis using specific utility functions to show why the preference (and private information) do matter in a network of transactions.[7] To this end, we study core transactions in some parametric versions of the economic environment defined above. $N = \{1, 2, 3, 4\}$ where 1, 2 and 3 are the essential participants and 4 is the stand-by party to a transaction. We specify the traders' utilities as follows.

$$\begin{aligned} U^1(c) &= \ln(c_1 + \beta c_3) \\ U^2(c) &= \ln(c_2 + \beta c_1) \\ U^3(c) &= \ln(c_3 + \beta c_2 + \gamma c_4) \\ U^4(c) &= \ln(c_4 + \varphi c_5) \\ & \quad \text{with } \beta > \max\{\sigma^{-1}, \rho^{-1}\}, 0 < \varphi\gamma < 1. \end{aligned} \tag{6}$$

Here, goods received in trade are "better" substitutes for endowment goods for essential participants 1,2,3. Trader 4 considers trader 3's good to be a "worse" substitute for his own endowment good, and trader 3 considers 4's good to be a "worse" substitute for trader 2's good or even for his own endowment good. We assume that the transfer technology to satisfy $0 < \rho \leq \sigma < 1$ and $\sigma > 1/2$. In the present paper we assume that there is no private information.

By an abuse of notation, we write $EU^i(\mu)$ instead of $EU^i(\Gamma_i(\mu, \omega))$ for the expected utility of trader i for a given transaction μ. Hence,

$$\begin{aligned} EU^i(\mu) &= \sigma \ln\left(1 - \mu_i^1 - \mu_i^S + a_{i-1}(\mu_{i-1}^1 + \rho\mu_{i-1}^S)\right) \\ &\quad +(1-\sigma)\ln\left(1 - \mu_i^1 - \mu_i^F + a_{i-1}(\mu_{i-1}^1 + \rho\mu_{i-1}^F)\right) \end{aligned} \tag{7}$$

for i =1,2,4 where $i-1 = 3$ for $i = 1$ and $i-1 = 5$ for $i = 4$, and $a_i = \beta$ for i =1,2,3, $a_4 = \gamma$, and $a_5 = \varphi$. For $i = 3$, we have

$$\begin{aligned} EU^3(\mu) &= \sigma \ln\left(1 - \mu_3^1 - \mu_3^S + \beta(\mu_2^1 + \rho\mu_2^S) - \mu_5^1 - \mu_5^S + \gamma(\mu_4^1 + \rho\mu_4^S)\right) \\ &\quad +(1-\sigma)\ln\left(1 - \mu_3^1 - \mu_3^F + \beta\rho\mu_2^F - \mu_5^1 - \mu_5^F + \gamma(\mu_4^1 + \rho\mu_4^F)\right). \end{aligned} \tag{8}$$

When a transaction μ is clear from the context, we may write for simplicity

$$\begin{aligned} C_i^S &= 1 - \mu_i^1 - \mu_i^S + a_{i-1}(\mu_{i-1}^1 + \rho\mu_{i-1}^S), \\ C_i^F &= 1 - \mu_i^1 - \mu_i^F + a_{i-1}(\mu_{i-1}^1 + \rho\mu_{i-1}^F) \end{aligned} \tag{9}$$

for $i = 1, 2, 4$ and

[7] Analysis in an economic environment with private information will be carried out in a separate paper.

$$
\begin{aligned}
C_3^S &= 1-\mu_3^1-\mu_3^S+\beta(\mu_2^1+\rho\mu_2^S)-\mu_5^1-\mu_5^S+\gamma(\mu_4^1+\rho\mu_4^S),\\
C_3^F &= 1-\mu_3^1-\mu_3^F+\beta\rho\mu_2^F-\mu_5^1-\mu_5^F+\gamma(\mu_4^1+\rho\mu_4^F).
\end{aligned} \tag{10}
$$

C_i^A is interpreted as "real" consumption level of trader i in event A in the sense that it directly determines i's utility level in event A.

We begin by establishing a series of properties to be satisfied by core transactions that simplify the characterization of core transactions.

We first want to show that a transaction is in fact endogenously generated in our model. In other words we shall show that if $\mu = (\mu^1, \mu^S, \mu^F)$ is a transaction which is identically equal to zero, then μ is dominated. Thus, the following lemma will guarantee the nontriviality of considering endogenous transactions using the utility functions given by (6).

Lemma 1 *Let* $\mu = (\mu^1, \mu^S, \mu^F)$ *be a feasible transaction such that* $\min\{\epsilon_i \mid i = 1, \cdots, 4\} > 0$ *where*

$$
\begin{aligned}
\epsilon_i &= 1-\mu_i^1-\max\{\mu_i^S,\mu_i^F\} \text{ for } i = 1,2,4\\
\epsilon_3 &= 1-\mu_3^1-\mu_5^1-\max\{\mu_3^S+\mu_5^S,\mu_3^F+\mu_5^F\}.
\end{aligned}
$$

Assume that $\beta\rho \geq \sqrt[3]{2}$. *Then,* μ *is {1,2,3}-dominated. In particular, if* $\mu = 0$, *then* μ *is {1,2,3}-dominated. In other words, a transaction* μ *which is not {1,2,3}-dominated generates a transaction. Furthermore, at least one trader among the essential participants to transaction must be sending all of his endowment to others, i.e.,* $\min\{\epsilon_i \mid i = 1, 2, 3\} = 0$.

If the cost of a second round transfer $1-\rho$ is no greater than the cost of risk $1-\sigma$ of a second round transfer so that $\rho \geq \sigma$, then it is immediate that second round transfers dominate first round transfers. Now, assume that the second round transfer cost is higher than the cost of risk in transfer from 2 to 3 so that $0 < \rho < \sigma$. Then, it is clear that among those who do not face direct transfer risk, *i.e.*, i= 1,2,4,5, [8] second round transfers, should they be effected, involve transfers in the event F or S but not in both events.

Lemma 2 *Assume* $0 < \rho < \sigma$. *Let* μ *be a feasible transaction such that* $\mu_i^S \mu_i^F > 0$ *for some* $i \in \{1, 2, 4, 5\}$. *Then,* μ *is* $\{i, i+1\}$*-dominated* [9] *where, for* $i = 5, i+1$ *is understood to be* 4.

By the basic lemma 1 of a transaction, at least one trader among the essential participants {1,2,3} to transactions must be sending all of his endowment if the transaction is undominated. We shall show next that regardless of who is sending his entire endowment to other participants, trader 2 must be sending at least some of his endowment to trader 3.

Lemma 3 *Let* μ *be a feasible transaction which is individually rational and {1,2,3}-undominated. Then, trader 2 must be sending some of its endowments to trader 3, i.e.,*

$$\max\{\mu_2^1, \mu_2^S, \mu_2^F\} > 0.$$

[8] It is convenient to refer to trader 3 as trader $i = 5$ when we look at his transfer to trader 4.

[9] Here again, {4,5}-dominated means {4,3}-dominated.

We now wish to check some of the further details of undominated transactions. For this purpose we like to derive conditions under which a change in transfer from a trader or traders induces another feasible transaction that can dominate a given transaction.

Let $\epsilon, \eta_i^S, \eta_i^F$ be real numbers in a neighborhood of zero. For $i = 1, \cdots, 5$, and a real number t in a neighborhood of zero, define

$$\mu_{ti}^1 = \mu_i^1 + t\epsilon, \mu_{ti}^S = \mu_i^S + t\eta_i^S, \mu_{ti}^F = \mu_i^F + t\eta_i^F. \tag{11}$$

Let μ_t^i be a transaction such that [10]

$$\begin{aligned} \mu_{ti}^i &= (\mu_{ti}^1, \mu_{ti}^S, \mu_{ti}^F) \quad \text{and} \\ \mu_{tj}^i &= (\mu_j^1, \mu_j^S, \mu_j^F) \quad \text{for} \quad j \neq i \end{aligned} \tag{12}$$

so that in μ_t^i only trader i's transfers are changed.

For any ϵ (could be either positive or negative) in a neighborhood of zero, we set either η_i^S or η_i^F or both so as to make

$$\frac{d}{dt} EU^i(\mu_t^i)(0) = 0. \tag{13}$$

We then look at a change in expected utility of trader $i+1$ resulting from a shift in transfer induced by ϵ, η_i^S, and η_i^F, where $i+1$ is 1 for $i = 3$, and 4 for $i = 5$. In other words, we change a transfer or transfers of a trader so as to keep his expected utility unchanged, and see under what conditions the receiver's expected utility is increased. If that happens, the donor and the receiver can dominate the original transaction. We will check these conditions trader by trader.

4.1 Effects of a change in transfers of trader $i = 2$

We first establish properties concerning changes in transfers of trader $i = 2$.

Lemma 4 *Let μ be a feasible transaction. Then, for a change in transfers of trader 2 as defined by (11), (12), and (13), we have the following.*

1. *If we consider a simultaneous change in second round transfers in both events F and S by a same amount so that $\eta = \eta_2^F = \eta_2^S$, then:*

$$sgn\left[\frac{d}{dt} EU^3(\mu_t^2)(0)\right] = sgn[\epsilon] sgn\left[\frac{C_3^F}{C_3^S} - r^{\sigma\rho}\right] \tag{14}$$

where

$$r^{\sigma\rho} = \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\rho}{1-\rho}\right). \tag{15}$$

2. *If the second round transfer in event F, μ_2^F, is held constant so that $\eta_2^F = 0$, then:*

$$sgn\left[\frac{d}{dt} EU^3(\mu_t^2)(0)\right] = sgn[\epsilon] sgn\left[\frac{C_2^F}{C_2^S} - r^{\sigma\rho}\right]. \tag{16}$$

[10]Here, there is an abuse of notation again since, for i=1, we have $\mu_{t1}^1 = (\mu_{t1}^1, \mu_{t1}^S, \mu_{t1}^F)$. But no confusion should arise in the context of our arguments below.

3. If the second round transfer in event S, μ_2^S, is held constant so that $\eta_2^S = 0$, then:

$$\begin{aligned} sgn\left[\frac{d}{dt}EU^3(\mu_t^2)(0)\right] &= sgn[\epsilon]sgn\left[\frac{C_3^F}{C_3^S} - \rho\left(\frac{1-\sigma}{\sigma}\right) - \rho\frac{C_2^F}{C_2^S}\right] \\ &= sgn[\epsilon]sgn\left[\frac{C_3^F}{C_3^S} - r_2^{\sigma\rho}\right] \end{aligned} \tag{17}$$

where

$$r_i^{\sigma\rho} = \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\rho}{1-\rho R_i}\right), \tag{18}$$

$$R_i = \frac{C_i^F/C_i^S}{C_{i+1}^F/C_{i+1}^S}. \tag{19}$$

4. If the first round transfer, μ_2^1, is held constant so that $\epsilon = 0$, then:

$$\begin{aligned} sgn\left[\frac{d}{dt}EU^3(\mu_t^2)(0)\right] &= sgn[\eta_2^S]sgn[1-R_2] \\ &= sgn[\eta_2^F]sgn[R_2-1]\ . \end{aligned} \tag{20}$$

One may note that

$$\frac{\sigma C_3^F}{(1-\sigma)C_3^S} = \frac{\sigma(1/C_3^S)}{(1-\sigma)(1/C_3^F)}$$

is trader 3's expected marginal rate of substitution of consumption in event F for consumption in event S. Thus, $\rho/(1-\rho)$ is playing the role of relative price of consumption in event S, and (14) says for example that if trader 3's expected marginal rate of substitution of consumption in event F for consumption in event S exceeds the relative cost of consumption in event S, then an increase in first round transfer from 2 to 3 accompanied by a decrease in second round transfer by the same amount both in event S and F so as to keep trader 2's expected utility unchanged would increase trader 3's expected utility.

It is interesting to note that whether the first round transfer from trader 2 to trader 3 should be increased or not depends on trader 3's expected marginal rate of substitution of consumption in event F for consumption in event S whereas, by (16), the second round transfer in event S depends on the expected marginal rate of substitution of trader 2. On the other hand, by (17), the second round transfer in event F depends on the ratio of the expected marginal rates of substitution of both traders. By (20), these marginal rates are typically identical when there are positive second round transfers from 2 to 3 in both events.

One may summarize the results of lemma 4 in terms of real consumptions as follows.

Lemma 5 *Let μ be a feasible transaction which is not $\{2,3\}$-dominated. Then, the transfer of trader 2, $\mu_2 = (\mu_2^1, \mu_2^S, \mu_2^F)$ must satisfy the following properties.*

1. When one has $\mu_2^1, \mu_2^S, \mu_2^F > 0$, μ satisfies

$$\frac{C_2^F}{C_2^S} = \frac{C_3^F}{C_3^S} = r^{\sigma\rho}. \tag{21}$$

2. When one has $\mu_2^1 > 0, \mu_2^F > 0, \mu_2^S = 0, \mu$ satisfies

$$r^{\sigma\rho} \leq \frac{C_3^F}{C_3^S} \leq \frac{C_2^F}{C_2^S} \ , \ and \quad \frac{C_3^F}{C_3^S} \leq r_2^{\sigma\rho}, \tag{22}$$

where in the last weak inequality, equality holds if $\mu_2^1 + \mu_2^F < 1$, in which case we have

$$\frac{C_2^F}{C_2^S} - \frac{C_3^F}{C_3^S} = \left(\frac{1-\rho}{\rho}\right)\left(\frac{C_3^F}{C_3^S} - r^{\sigma\rho}\right). \tag{23}$$

In particular, if $\frac{C_3^F}{C_3^S} = r^{\sigma\rho}$ holds, then

$$\frac{C_2^F}{C_2^S} = \frac{C_3^F}{C_3^S}.$$

3. When one has $\mu_2^1 > 0, \mu_2^S > 0, \mu_2^F = 0, \mu$ satisfies

$$\frac{C_2^F}{C_2^S} \leq r^{\sigma\rho} \leq \frac{C_3^F}{C_3^S}. \tag{24}$$

4. When one has $\mu_2^F > 0, \mu_2^S > 0, \mu_2^1 = 0, \mu$ satisfies the following.

   (a) If $\max\{\mu_2^F, \mu_2^S\} < 1$, then

   $$\frac{C_2^F}{C_2^S} = \frac{C_3^F}{C_3^S} \leq r^{\sigma\rho}.$$

   (b) If $\mu_2^F < 1$, then

   $$\frac{C_2^F}{C_2^S} \leq \frac{C_3^F}{C_3^S} \leq r^{\sigma\rho}.$$

   (c) If $\mu_2^S < 1$, then

   $$\frac{C_3^F}{C_3^S} \leq \frac{C_2^F}{C_2^S} \leq r^{\sigma\rho}.$$

4.2 Effects of a change in transfers of trader $i = 1$ or 3

We now proceed to check the properties of transfers from trader i=1 or 3.

Lemma 6 *Let μ be a feasible transaction. Then, for a change in transfers of trader i=1 or 3 as defined by (11), (12), and (13), we have the following.*

1. If the second round transfer in event F, μ_i^F, is held constant so that $\eta_i^F = 0$, then:

$$\begin{aligned} sgn\left[\frac{d}{dt}EU^{i+1}(\mu_t^i)(0)\right] &= sgn[\epsilon]sgn\left[\left(\frac{\sigma}{1-\sigma}\right)\left(\frac{1-\rho}{\rho}\right) - \left(\frac{C_i^S}{C_i^F} - \left(\frac{1}{\rho}\right)\frac{C_{i+1}^S}{C_{i+1}^F}\right)\right] \\ &= sgn[\epsilon]sgn\left[\frac{C_{i+1}^F}{C_{i+1}^S} - r^{\sigma\rho}\left(\frac{1}{R_i} - \frac{1}{\rho}\right)\right] \ . \end{aligned} \tag{25}$$

2. If the second round transfer in event S, μ_i^S, is held constant so that $\eta_i^S = 0$, then:

$$
\begin{aligned}
sgn\left[\frac{d}{dt}EU^{i+1}(\mu_t^i)(0)\right] &= sgn[\epsilon]sgn\left[\left(\frac{1-\sigma}{\sigma}\right)\left(\frac{1-\rho}{\rho}\right) - \left(\frac{C_i^F}{C_i^S} - \left(\frac{1}{\rho}\right)\frac{C_{i+1}^F}{C_{i+1}^S}\right)\right] \\
&= sgn[\epsilon]sgn\left[\left(\frac{1-\sigma}{\sigma}\right)(1-\rho) - \frac{C_{i+1}^F}{C_{i+1}^S}(\rho R_i - 1)\right] . \qquad (26)
\end{aligned}
$$

3. If the first round transfer, μ_i^1, is held constant so that $\epsilon = 0$, then:

$$
\begin{aligned}
sgn\left[\frac{d}{dt}EU^{i+1}(\mu_t^2)(0)\right] &= sgn[\eta_i^F]sgn[R_i - 1] \\
&= sgn[\eta_i^S]sgn[1 - R_i] . \qquad (27)
\end{aligned}
$$

There are two major differences between the results obtained in lemma 6 and lemma 4. One is that whether a change in the second round transfer in event F or S increases the expected utility of the recipient or not, each depends on the ratio of the expected marginal rates of substitution of both the donor and the recipient when the donor is $i = 1$ or 3, whereas in the previous case of donor being i=2, whether a change in the second round transfer in event S increases the expected utility of the recipient or not depends on the the expected marginal rate of substitution of the donor only. The other is that because of lemma 2 it does not make sense to consider a simultaneous change in the second round transfer in events F and S for traders i=1,3.

4.3 Effects of a change in transfers between traders $i = 3$ and 4

We now come to a consideration of transfers between traders 3 and 4. Here, the trader 3 is an essential participant to a transaction and is regarded to represent a tie between the settlement system and the outside party. The trader 4 is the stand-by party to a transaction. It might function as the central bank depending upon whether a transaction μ requires the trader 4 to effect transfer to 3 in event F.

Lemma 7 *Let μ be a feasible transaction and consider a change in transfers between traders 3 and 4 as defined by a transaction μ_t satisfying*

$$\mu_{ti}^1 = \mu_i^1 + t\epsilon_i, \mu_{ti}^F = \mu_i^F + t\eta_i^F, \mu_{ti}^S = \mu_i^S \qquad (28)$$

where $\epsilon_i = \eta_i^F = 0$ for i=1,2,3. Set ϵ_i and η_i^F for i=4,5 so that the expected utility of trader 4 remains unchanged at t=0. Then:

1. If $\epsilon_i \neq 0$ and $\eta_i^F = 0$ for i=4,5, we have

$$\frac{d}{dt}EU^3(\mu_t)(0) = \left(\frac{\sigma}{C_4^F} + \frac{1-\sigma}{C_4^S}\right)(\varphi\gamma - 1)\,\epsilon_5 . \qquad (29)$$

2. *If $\eta_4^F \neq 0, \epsilon_5 \neq 0$, and $\epsilon_4 = \eta_5^F = 0$, we have*

$$sgn\left[\frac{d}{dt}EU^3(\mu_t)(0)\right] = sgn[\epsilon_5]sgn\left[\varphi\gamma\rho\left(\frac{\sigma}{C_4^S}+\frac{1-\sigma}{C_4^F}\right)-\frac{1-\sigma}{C_4^F}\left(1+\left(\frac{\sigma}{1-\sigma}\right)\frac{C_3^F}{C_3^S}\right)\right]. \tag{30}$$

Let us evaluate the terms inside the above bracket assuming that μ is a feasible transaction such that $\mu_i^1 = \mu_i^F = \mu_i^S = 0$ for $i = 4, 5$. Certainly one has $C_4^F = C_4^S = 1$. When μ is individually rational and not {2,3}-dominated, with $\mu_2^1, \mu_2^F, \mu_2^S$ all strictly positive, then by lemma 5 we have

$$\frac{C_3^F}{C_3^S} = \frac{C_2^F}{C_2^S} = r^{\sigma\rho} = \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\rho}{1-\rho}\right).$$

It thus follows that

$$sgn\left[\frac{d}{dt}EU^3(\mu_t)(0)\right] = sgn[\epsilon_5]sgn\left[\varphi\gamma - \frac{1-\sigma}{\rho(1-\rho)}\right].$$

Let us note the term $\dfrac{1-\sigma}{\rho(1-\rho)}$ in the above bracket. If $\sigma = \rho$, it is equal to $1/\rho$, and since we have $\varphi\gamma < 1$, a transaction μ is not dominated even if $\mu_i^1 = \mu_i^F = \mu_i^S = 0$ for $i = 4, 5$. However, note that the denominator $\rho(1-\rho)$ of the term achieves its maximum at $\rho = 1/2$ with the maximum value $1/4$. Therefore, one has

$$\min_\rho\left\{\frac{1-\sigma}{\rho(1-\rho)}\right\} = 4(1-\sigma).$$

It means that if the value of σ is close to 1, for example if $\sigma = 0.9$, then it is small enough, i.e. 0.4, and inside the bracket tends to become positive, in which case a transaction μ must specify $\mu_5^1 > 0$ and $\mu_4^F > 0$ if it is not {2,3}-dominated. The following lemma summarizes the above arguments.

Lemma 8 *Let μ be a feasible transaction which is not {2,3}-dominated.*

1. *If μ specifies a state non-contingent trade between traders 3 and 4, i.e., $\mu_5^1 > 0$ and $\mu_4^1 > 0$, then μ is {3,4}-dominated.*

2. *Assume $\varphi\gamma > (1-\sigma)/\rho(1-\rho)$. If it is feasible to increase the state non-contingent transfer from 3 to 4 as well as the state contingent transfer from 4 to 3 in event F, then μ is {3,4}-dominated.*

4.4 Properties of undominated transactions

Using the properties we have derived so far, we will see how undominated transactions induce state contingent transfers and consumptions. For this purpose, we start from a transaction that has state non-contingent transfers and show how it needs to be changed in order for the transaction to be undominated.

We begin by a consideration of a feasible and individually rational transaction μ which is not state contingent, so that all the transfers are done at first round. In this case we have the following.

Proposition 1 *Let μ be a feasible individually rational transaction such that:*

1. *μ effects first round transfers only.*
2. *Traders 1,2, and 3 make positive transfers.*
3. *Traders 3 and 4 do not make transfers between them.*

Then:

1. *Even if trader 1 initiates a state contingent transfer by reducing state non-contingent transfer so as to keep his own expected utility level unchanged, trader 2 will never gain in expected utility.* [11]
2. *If trader 2 initiates a second round transfer in event F by reducing state non-contingent transfer so as to keep his own expected utility level unchanged, trader 3 can gain in expected utility provided that the sum of the first round transfers from trader 2 to trader 3 and trader 3 to trader 1 is large enough.*
3. *If trader 3 initiates a second round transfer in event S by reducing state non-contingent transfer so as to keep his own expected utility level unchanged, trader 1 will gain in expected utility level provided that the sum of the first round transfers from trader 2 to 3 and trader 3 to 1 are sufficiently large.*
4. *Trader 3 will gain in expected utility if trader 3 makes a first round transfer to trader 4 while trader 4 makes a second round transfer to trader 3 in event F in such a manner to keep trader 4's expected utility unchanged, provided that the sum of the first round transfers from trader 2 to trader 3 and trader 3 to trader 1 is large enough, and if expected marginal rate of substitution between endowment goods of traders 3 and 4 is not too small relative to the iceberg-cost adjusted cost of risk so that $\varphi\gamma > (1-\sigma)/\rho$.*

Thus, according to the proposition 1, if we start from a transaction which has only first round state non-contingent transfers, then there will be no incentives for traders 1 and 2 to initiate state contingent transfers between the two. But as regards to traders 2 and 3, they have incentives to initiate a state contingent transfer in event F from 2 to 3 provided the sum of the first round transfers from trader 2 to trader 3 and trader 3 to trader 1 is large enough to satisfy the following inequality.[12]

$$\mu_3^1 + \left(\frac{\beta\rho}{\sigma-\rho}\right)\mu_2^1 > 1 \tag{31}$$

For example, for parameter values $\sigma = 0.9, \rho = 0.8$, and $\beta = 1.6$, the coefficient of μ_2^1 in the inequality is 12.8 so that if the transfer from trader 2 exceeds $\frac{5}{64}$, regardless of the

[11] This is equivalent to stating that "Trader 1 will never gain in expected utility by initiating a state contingent transfer while keeping trader 2's expected utility level unchanged." Similar remarks apply to the subsequent statements of this proposition.

[12] For this inequality as well as other inequalities below, see the proof of the proposition 1 in the appendix.

amount of transfer from trader 3 the inequality is satisfied. Thus, all those transactions satisfying (31) will be {2,3}-dominated by increasing the transfer from trader 2 in event F and decreasing the first round transfer from 2.

As to transfers from 3 to 1, when the given transaction is entirely state non-contingent, they have incentives to initiate a state contingent transfer in event S from 3 to 1 provided the sum of the first round transfers from trader 2 to 3 and trader 3 trader to 1 is large enough to satisfy the following inequality:

$$\mu_3^1 + \beta(1-\sigma)\left(\frac{1-\rho}{\rho}\right)\mu_2^1 > 1 \ ,$$

because μ will be {1,3}-dominated by increasing μ_3^S and decreasing μ_3^1 under this condition. For the parameter values we specified earlier, the condition above is met, for example, if $\mu_2^1 \geq 0.02$ regardless of the value of μ_3^1. Thus, we may expect at this stage that a core transaction μ to specify $\mu_3^S > 0$. [13] One might think at first that it is counter-intuitive to have $\mu_3^S > 0$. An economic intuition behind this is the following: as the first round transfer μ_2^1 fails in event F, the consumption of trader 3 in event F, C_3^F, is below the level of his consumption in event S, C_3^S. If the excess of trader 3's expected marginal rate of substitution of consumption in F for consumption in S over the iceberg-cost-adjusted trader 1's expected marginal rate of substitution of consumption in F for consumption in S exceeds relative cost of consumption in event S and F, then it is mutually desirable for traders 3 and 1 to increase the transfer μ_3^S by decreasing the first round transfer μ_3^1. This adjustment cannot be done by changing the level of the first round transfer μ_3^1 alone or by changing the second round transfer in event F, μ_3^F, because one is required to increase C_3^F and μ_3^F cannot be decreased beyond zero to achieve this. Of course, it would be a different story if a given transaction μ has state contingent transfers. For example, if trader 2 sends a part of his endowment to 3 in event F and/or if trader 3 sends a part of his endowment in event F, then it is possible that μ would be {3,1}-dominated unless 3 does send a part of his endowment to 1 in event F.

For transfers between traders 3 and 4, we have seen in lemma 8 that if the transaction μ is not {2,3}-dominated, then there are incentives to initiate a state non-contingent transfer from 3 to 4 and a state contingent transfer from 4 to 3 in event F. Note, however, that in the statement of the proposition 1 above, a given transaction μ may be {2,3}-dominated. But even so, it turns out that there are incentives to initiate a state non-contingent transfer from 3 to 4 and a state contingent transfer from 4 to 3 in event F, provided that the sum of the first round transfers from trader 2 to trader 3 and trader 3 trader to 1 is large enough to satisfy the following inequality.

$$\frac{\beta\left(\varphi\gamma - \frac{(1-\sigma)}{\rho}\right)}{(1/\rho) - \varphi\gamma}\mu_2^1 + \mu_3^1 > 1$$

In fact, traders 3 and 4 can dominate the state non-contingent transaction μ by initiating a state contingent transfer from 4 to 3 in event F together with a state non-contingent transfer μ_5^1 from 3 to 4 despite the fact that traders 3 and 4 do not mutually gain from trades in general. This shows that the trader 4 will participate in transaction only in risk-sharing capacity.

[13] Later on we shall show that this need not be the case when traders do make state contingent transfers.

It may be of interest to note that although transfers are done in one direction only from a trader to another among essential participants, two rounds of transfers function as if there is an explicit means of payment or are barter trades between any two participants in the sense that an increase of a transfer in one round can be matched to a decrease of another transfer in the other round. This gives another sense in which the model in this paper can be said to represent a settlement network.

Given a transaction μ, a *net transfer gap* of trader i is defined to be

$$g_i(\mu) = 1 - \mu_i^1 - \chi_{\{3\}}\mu_5^1 - \max\{\mu_i^S + \chi_{\{3\}}\mu_5^S, \mu_i^F + \chi_{\{3\}}\mu_5^F\},$$

Net transfer gap $g_i(\mu)$ among the essential participants shows the maximal amount that trader i can further transfer to others, given a transaction μ. We also define *transfer gap* $\bar{g}_i(\mu)$ (among essential participants) by $\bar{g}_i(\mu) = g_i(\mu)$ for $i = 1, 2$ and

$$\bar{g}_3(\mu) = 1 - \mu_i^1 - \max\{\mu_3^S, \mu_3^F\} \ .$$

Finally, we give two statements concerning core transactions, assuming parameter values to satisfy

$$\begin{aligned} \varphi\gamma &> \left(\frac{1-\sigma}{\rho}\right)\frac{1}{1-\rho} \ , \\ \beta\rho &> \sqrt[3]{2} \ , \ \rho > \sqrt[3]{2}/2 \ , \end{aligned} \tag{32}$$

where the first inequality is satisfied, for example, if $\varphi\gamma > 0.63$ when $\sigma = 0.9$ and $\rho = 0.8$. It is also satisfied whenever we have $\varphi\gamma > r^{\sigma\rho}$. The second and the third inequalities are to ensure that second round transfers are not too costly so that traders are induced to make such transfers.

Proposition 2 *A core transaction μ always specifies state contingent transfers. A typical core transaction μ specifies transfers such that:*

$$\begin{aligned} &\mu_i^1 > 0 \quad \text{for } i = 1, 2, 3, 5, \ \mu_4^1 = 0 \ , \\ &\mu_1^F > 0, \mu_2^F > 0, \mu_3^F > 0, \mu_4^F > 0, \mu_5^F = 0 \ , \\ &\mu_1^S = \mu_3^S = \mu_4^S = \mu_5^S = 0, \mu_2^S > 0 \ , \end{aligned}$$

in which case consumptions associated with the transaction are given by:

$$\begin{aligned} \frac{C_2^F}{C_2^S} &= \frac{C_3^F}{C_3^S} = r^{\sigma\rho} \\ \frac{C_1^F}{C_1^S} &\geq \left(\frac{1-\sigma}{\sigma}\right)\frac{1}{1-\rho} > r^{\sigma\rho} \\ &\quad \text{with equality holding when } g_1(\mu) > 0 \\ \frac{C_4^F}{C_4^S} &= \frac{1-\sigma}{\sigma}\left(\frac{1}{\varphi\gamma\rho(1-\rho)} - 1\right) > \left(\frac{1-\sigma}{\sigma}\right)\frac{1}{1-\rho} \ . \end{aligned}$$

We like to note the extent to which traders' consumptions that a typical core transaction induces are state contingent. Given a typical core transaction as in the beginning of the statement of the proposition above, for trader 2 and trader 3 the consumption level in event F relative to that in event S is $r^{\sigma\rho}$, which is less than 1 but approaches 1 as the value of ρ becomes closer to σ. This may be interpreted to say that the failure of receipt by trader 3 is compensated by other traders by the factor of $(\rho/(1-\rho))-1$. Trader 2 is as responsible as trader 3 for the loss as his relative consumption level in event F is reduced to the level of trader 3. Trader 1 in turn compensates trader 2 but extent to which he joins in the compensation is less than that of trader 2 so that his relative consumption in event F exceeds $r^{\sigma\rho}$. It is very instructive to note that trader 4 also participates in this compensation scheme but extent to which he does compensate trader 3 is much less than those of other traders in the sense that his relative consumption level in event F is higher than those of all the essential participants.

Proposition 3 *Let μ be a core transaction. Then:*

1. *At least one essential participant must be sending all his endowment to other traders in some event. That is,*
$$(\exists i \in \{1,2,3\}) g_i(\mu) = 0.$$
2. *Suppose that trader 3 is not sending all of his endowment to other essential participants so that his transfer gap is positive, i.e., $\bar{g}_3(\mu) > 0$. Then, the transaction μ is a core transaction if and only if trader 3 is making a transfer to trader 4 either by the amount of his transfer gap or by the amount of "feasibility bound" given by*
$$v(\varphi\gamma, \sigma, \rho, \beta) = \frac{(1-\sigma)\,(1-\varphi\gamma\rho(1-\rho))}{\beta\,[\varphi\gamma\rho(1-\rho)-(1-\sigma)]},$$
whichever is smaller, i.e.,
$$\mu_5^1 = \min\{\bar{g}_3(\mu), v(\varphi\gamma, \sigma, \rho, \beta)\},$$
and trader 4 in turn is making a state contingent transfer in event F at most the amount given by
$$\mu_4^F = \left(\frac{\varphi\gamma\rho(1-\rho)-(1-\sigma)}{\sigma\varphi\gamma\rho(1-\rho)}\right)(1+\beta\mu_5^1)\ . \tag{33}$$

The first part of the proposition 3 is a direct consequence of the lemma 1 and is due to our specification of preferences of essential participants that they prefer the endowment of another trader to his own. The second part comes from two things. One is that a core transaction in general specifies positive second round transfers in both events F and S as well as a positive first round state non-contingent transfer from trader 2 to trader 3. By (21) of lemma 5, this ensures trader 3's consumption in event F relative to that in event S, C_3^F/C_3^S, to be given by $r^{\sigma\rho}$. Second is that under this circumstance, given (32), the expected utility of both of the traders 3 and 4 can be increased whenever first round state non-contingent transfer from 3 to 4 and second round state contingent transfer in event F from 4 to 3 can be increased. For a first round state non-contingent transfer from 3 to 4, μ_5^1, the maximal amount that trader 4 would be just willing to send to 3 is given by the amount shown in (33).

5 Conclusion

At the outset of the present paper we posed the following three questions concerning a given network of transactions that involve a risky transfer of assets.

1. If there is some risk of failure in a transfer from one party to another, should the transfer be done through that arrangement?

2. If so, then what considerations are relevant to determining whether third parties ought to share that risk?

3. Are there conditions under which the general public or the government (in the case of a private arrangement) ought to bear some risk and, if so, what level of compensation would it be appropriate for them to receive?

To answer these questions we carried through our analysis using a schematic model economy with four traders. A risky transfer is made between trader 2 and trader 3, trader 2 being the donor and trader 3 the recipient. There are two types of third party participants. Trader 1 is one of essential participants along with traders 2 and 3 in the sense that he is needed in the network of transactions to complete a round of transaction. The attributes of trader 4 is specified in such a way that he can only participate in a risk-sharing capacity. Trader 4 is added to a model of three traders in order to answer the third question listed above.

We answer the first question affirmatively provided that $\beta\sigma > 1$. In other words, if the risk of transfer is not too high relative to the asset preference of traders, then transfers will be made through the network. Traders' expected marginal utility of obtaining other trader's endowments must exceed that of their own endowment goods to the extent that it can overcome a risk of transfer.

The second and the third questions are answered by the proposition 2 and the proposition 3. An efficient transaction typically requires that a transfer loss incurred by trader 3 to be shared not only among the essential participants but also by unessential third party participant such as the government or the public under certain conditions. However, trader 3's loss is not entirely indemnified by others. The efficiency requires that his loss is indemnified to the extent that his expected marginal rate of substitution of consumption in event F for consumption in event S is equated to the cost of consumption in event S relative to that in event F. It is interesting to note that the directly concerned participant, i.e. trader 2, needs to indemnify trader 3 to the extent that his consumption in event F is decreased so that his expected marginal rate of substitution of consumption in event F for consumption in event S becomes identical to that of trader 3. This may be interpreted to say that the directly concerned participant, the donor, is as responsible as the intended recipient for the lost transfer.

There are two types of third party participants. Trader 1 is the third party who is an integral part of the transfer network and is an essential participant. He is required to participate in the loss-sharing by making a transfer to trader 2 in event F. However, the extent to which he shares the loss of trader 3 is less that of the directly concerned participant, trader 2, in the sense that his relative consumption in event F exceeds that of trader 2 and 3 at least by the factor of cost of the second round transfer.

On the other hand, trader 4 is the third party who is not, in a narrow sense, an integral part of the transfer network and is not an essential participant. An intended interpretation of a role of trader 4 is the general public or the government (in the case of a private arrangement) because trader 4 as well as trader 3 regards the other's endowment goods as a worse substitute for his own endowment goods. A typically efficient arrangement requires that the general public or the government ought to bear some risk. However, the extent to which the unessential third party participant bear the risk is much less than that of the essential participants in the sense that his relative consumption in event F far exceeds those of the essential participants as exactly shown in the proposition 2. Of course, we must be aware of the fact that this result does not say that the government or the public should bear the risk of the transfer arrangement unconditionally. What it does say is that the level of compensation must depends upon state non-contingent transfer from the network participants to the government or the public.

These results appear very intuitive on economic grounds. While the results obtained about a schematic model economy are far from constituting definitive advice regarding actual transfer arrangements, we hope that our analysis may at least provide a helpful framework within which to think in an organized way about the issues involved in practical cases.

References

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Appendix

Proof of the lemma 1

Let ϵ be a positive number satisfying

$$\epsilon < \frac{1}{2}\min\{\epsilon_i \mid i = 1, \cdots, 4\}.$$

Now, for real numbers t in a neighborhood of zero, define for $i = 1, 2, 3$,

$$\mu_{ti}^1 = \mu_i^1 + t\epsilon, \mu_{ti}^F = \mu_i^F + t\eta_i,$$

and for $i = 4, 5$

$$\mu_{ti}^1 = \mu_i^1, \mu_t^S = \mu_i^S, \mu_{ti}^F = \mu_i^F.$$

Since ϵ_i is positive for all i , transaction $\mu_t = \{(\mu_{ti}^1, \mu_{ti}^S, \mu_{ti}^F) \mid i = 1, \cdots, 5\}$ is feasible for $t \geq 0$ in a neighborhood of 0. By differentiation of expected utilities with respect to t , one obtains

$$\frac{d}{dt}EU^i(\mu_t)(0) = \frac{\sigma}{C_i^S}(\beta - 1)\epsilon + \frac{1-\sigma}{C_i^F}[(\beta - 1)\epsilon - \eta_i + \beta\rho\eta_{i-1}] \tag{34}$$

for $i = 1, 2$, and

$$\frac{d}{dt}EU^3(\mu_t)(0) = \frac{\sigma}{C_3^S}(\beta - 1)\epsilon + \frac{1-\sigma}{C_3^F}[-\epsilon - \eta_3 + \beta\rho\eta_2]. \tag{35}$$

Here, for $i = 1, i - 1$ is understood to be 3. Trivially, one has

$$\frac{d}{dt}EU^4(\mu_t)(0) = 0.$$

To show that the transaction μ_t {1,2,3}-dominates μ, by (34) and (35) it is enough to show

$$(\chi_{\{1,2\}}\beta - 1)\epsilon - \eta_i + \beta\rho\eta_{i-1} \geq 0 \tag{36}$$

for an appropriate choice of η_i for each $i = 1, 2, 3$.[14]

Start at $i = 2$ and set $\eta_2 = \epsilon$. Set η_3 so that

$$\begin{aligned} -\epsilon - \eta_3 + \beta\rho\eta_2 = 0 \quad &\text{or} \\ \eta_3 = (\beta\rho - 1)\epsilon. \end{aligned} \tag{37}$$

By setting β to be $\min\{\beta, 2\}$ if necessary, one can assume w.l.o.g. that $\beta \leq 2$. [15] For $i = 1$, then, set η_1 so that

$$(\beta - 1)\epsilon - \epsilon + \beta\rho\eta_1 = 0. \tag{38}$$

Thus, one obtains

$$\eta_1 = \frac{2-\beta}{\beta\rho}\epsilon. \tag{39}$$

By the way η_i's, i =1,2,3, are set, we have

$$\frac{d}{dt}EU^i(\mu_t)(0) > 0 \tag{40}$$

for $i = 2, 3$. We show

$$\frac{d}{dt}EU^1(\mu_t)(0) > 0 \tag{41}$$

by showing (36) for $i = 2$. By substituting the values of η_i's in (37) and (39) into (36) for $i = 2$, one obtains

$$\begin{aligned} &(\beta - 1)\epsilon - \frac{2-\beta}{\beta\rho}\epsilon + \beta\rho(\beta\rho - 1)\epsilon \\ &\quad = \frac{1}{\beta\rho}[\beta\rho(\beta - 1) + \beta^2\rho^2(\beta\rho - 1) + \beta - 2]\epsilon \\ &\quad = \frac{1}{\beta\rho}[(\beta - 1)(\beta\rho + 1) + \beta^2\rho^2(\beta\rho - 1) - 1]\epsilon \\ &\quad > \frac{1}{\beta\rho}(\beta^3\rho^3 - 2)\epsilon \geq 0 \end{aligned} \tag{42}$$

[14] Note that we are ignoring the first term of the R.H.S. of the equations (34) and (35). We avoided this to simplify calculations. The cost paid is a tighter requirement of $\beta\rho > \sqrt[3]{2}$.

[15] The requirement that $\beta\rho > \sqrt[3]{2}$ must be met with this new β.

as $\beta\rho \geq \sqrt[3]{2}$. This proves that μ_t {1,2,3}-dominates μ. [16] ∎

Proof of the lemma 2

Assume $\mu_i^S \mu_i^F > 0$ for some $i \in \{1, 2, 4, 5\}$. Let $\epsilon = \min\{\mu_i^S, \mu_i^F\}$. Then, define μ_ϵ by letting $\mu_{\epsilon j} = \mu_j$ for all $j \neq i$ and $\mu_{\epsilon i}^1 = \mu_i^1 + \epsilon, \mu_{\epsilon i}^S = \mu_i^S - \epsilon, \mu_{\epsilon i}^F = \mu_i^F - \epsilon$. μ_ϵ is feasible and $EU^i(\mu_\epsilon) = EU^i(\mu), EU^{i+1}(\mu_\epsilon) > EU^{i+1}(\mu)$ so that μ_ϵ $\{i, i+1\}$-dominates μ. ∎

Proof of the lemma 3

Let μ be a feasible transaction. If trader 3 is sending all of its endowments to other parties, it is easier to show that trader 2 must be sending some of its endowments to 3. Thus, it is sufficient to consider the case where trader 1 is sending all of its endowments. W.l.o.g. assume $\mu_1^1 = 1$. To simplify arguments, assume $\mu_3^S = \mu_3^F = 0$. To maintain the individual rationality of trader 1, one must have $\mu_3^1 \geq 1/\beta$. Assume trader 2 sends nothing to trader 3. Then, again by the consideration of individual rationality for trader 3 and 4, inequalities below need be satisfied.

$$\begin{aligned}
\varphi\mu_5^1 &\geq \mu_4^1 \\
\gamma\mu_4^1 &\geq \mu_3^1 + \mu_5^1.
\end{aligned} \tag{43}$$

It follows that one must have

$$\beta(\varphi\gamma - 1)\mu_5^1 \geq 1, \tag{44}$$

which is impossible since $\beta > 0, \mu_5^1 \geq 0$, and $0 < \varphi\gamma < 1$. ∎

Proof of the lemma 4

For any ϵ in a neighborhood of zero (could be either positive or negative), we set either η_i^S or η_i^F or both so as to make

$$\frac{d}{dt} EU^i(\mu_t^i)(0) = 0. \tag{45}$$

But we have

$$\frac{d}{dt} EU^i(\mu_t^i)(0) = \frac{\sigma}{C_i^S}(-\epsilon - \eta_i^S) + \frac{1-\sigma}{C_i^F}(-\epsilon - \eta_i^F) = 0. \tag{46}$$

where C_i^S and C_i^F are defined as in (9) and (10). It follows that η_i^S and η_i^F are set so that one has

$$\frac{\sigma}{C_i^S}\eta_i^S + \frac{1-\sigma}{C_i^F}\eta_i^F = -\left(\frac{\sigma}{C_i^S} + \frac{1-\sigma}{C_i^F}\right)\epsilon. \tag{47}$$

We then look at a change in expected utility of trader $i+1$ resulting from a shift in transfer induced by ϵ, η_i^S, and η_i^F, where $i+1$ is 1 for $i = 3$, and 4 for $i = 5$. We have

$$\begin{aligned}
\frac{d}{dt} EU^{i+1}(\mu_t^i)(0) &= \frac{\sigma}{C_{i+1}^S}(\beta\epsilon + \beta\rho\eta_i^S) + \frac{1-\sigma}{C_{i+1}^F}(\chi_{\{1,3,4,5\}}\beta\epsilon + \beta\rho\eta_i^F) \\
&= \beta\left[\left(\frac{\sigma}{C_{i+1}^S} + \chi_{\{1,3,4,5\}}\frac{1-\sigma}{C_{i+1}^F}\right)\epsilon + \rho\left(\frac{\sigma}{C_{i+1}^S}\eta_i^S + \frac{1-\sigma}{C_{i+1}^F}\eta_i^F\right)\right]
\end{aligned} \tag{48}$$

where $\chi_{\{1,3,4,5\}}$ =0 for i=2 and =1 otherwise.

[16] If one is willing to assume $\beta\rho(\beta - 1) > 1$, then in the above proof one can show that μ_t dominates μ with $\eta_1 = \eta_3 = 0$ and $\eta_2 = (\beta - 1)\epsilon$. In other words, by increasing μ_i for $i = 1, 2, 3$ and μ_2^F, μ is dominated.

Let us start from $i = 2$ and consider four cases.

Case 1: $\eta_2^F = \eta_2^S = \eta$ (A simultaneous change in consumption in both events.)
In this case η is set equal to $-\epsilon$ from (47). Substituting this value of $\eta = \eta_2^F = \eta_2^S$ into (48), one obtains

$$\frac{d}{dt}EU^3(\mu_t^2)(0) = \frac{\beta\sigma}{C_3^S}\left[(1-\rho) - \rho\left(\frac{1-\sigma}{\sigma}\right)\frac{C_3^S}{C_3^F}\right]\epsilon \ . \tag{49}$$

Since the inside of the bracket is positive if and only if

$$\frac{\sigma C_3^F}{(1-\sigma)C_3^S} > \frac{\rho}{1-\rho} \ ,$$

it follows that

$$sgn\left[\frac{d}{dt}EU^3(\mu_t^2)(0)\right] = sgn[\epsilon] sgn\left[\frac{C_3^F}{C_3^S} - r^{\sigma\rho}\right] \ . \tag{50}$$

Case 2: $\eta_2^F = 0$ (μ_2^F held constant.)
It follows from (47) and (48) that

$$\begin{aligned}\frac{d}{dt}EU^3(\mu_t^2)(0) &= \beta\left[\frac{\sigma}{C_3^S} - \rho\frac{\sigma}{C_3^S}\left(\frac{C_2^S}{\sigma}\right)\left(\frac{\sigma}{C_2^S} + \frac{1-\sigma}{C_2^F}\right)\right]\epsilon \\ &= \frac{\beta\sigma}{C_3^S}\left[(1-\rho) - \rho\left(\frac{1-\sigma}{\sigma}\right)\frac{C_2^S}{C_2^F}\right]\epsilon \ .\end{aligned} \tag{51}$$

Thus, one obtains

$$sgn\left[\frac{d}{dt}EU^3(\mu_t^2)(0)\right] = sgn[\epsilon] sgn\left[\frac{C_2^F}{C_2^S} - r^{\sigma\rho}\right] . \tag{52}$$

Case 3: $\eta_2^S = 0$ (μ_2^S held constant.)
We obtain from (47) and (48) that

$$\begin{aligned}\frac{d}{dt}EU^3(\mu_t^2)(0) &= \beta\left[\frac{\sigma}{C_3^S} - \rho\frac{1-\sigma}{C_3^F}\left(\frac{C_2^F}{1-\sigma}\right)\left(\frac{\sigma}{C_2^S} + \frac{1-\sigma}{C_2^F}\right)\right]\epsilon \\ &= \frac{\beta\sigma}{C_3^S}\left[1 - \rho\left(\frac{C_3^S}{C_3^F}\right)\frac{C_2^F}{C_2^S} - \rho\left(\frac{1-\sigma}{\sigma}\right)\frac{C_3^S}{C_3^F}\right]\epsilon \\ &= \frac{\beta\sigma}{C_3^S}\left[(1-\rho R_2) - \rho\left(\frac{1-\sigma}{\sigma}\right)\frac{C_3^S}{C_3^F}\right]\epsilon \ .\end{aligned} \tag{53}$$

It then follows that

$$\begin{aligned}sgn\left[\frac{d}{dt}EU^3(\mu_t^2)(0)\right] &= sgn[\epsilon] sgn\left[\frac{C_3^F}{C_3^S} - \rho\left(\frac{1-\sigma}{\sigma}\right) - \rho\frac{C_2^F}{C_2^S}\right] \\ &= sgn[\epsilon] sgn\left[\frac{C_3^F}{C_3^S} - r_2^{\sigma\rho}\right] .\end{aligned} \tag{54}$$

Case 4: $\epsilon = 0$ (μ_2^1 held constant.)
In this case from (47) we have

$$\eta_2^F = -\left(\frac{\sigma}{1-\sigma}\right)\frac{C_2^F}{C_2^S}\eta_2^S \ .$$

Substituting this value into (48) one gets

$$\begin{aligned}\frac{d}{dt}EU^3(\mu_t^2)(0) &= \beta\rho\left[\frac{\sigma}{C_3^S} - \frac{1-\sigma}{C_3^F}\left(\frac{\sigma}{1-\sigma}\right)\frac{C_2^F}{C_2^S}\right]\eta_2^S \\ &= \frac{\beta\rho\sigma}{C_3^S}(1-R_2)\eta_2^S. \qquad (55)\end{aligned}$$

It then follows that

$$\begin{aligned}sgn\left[\frac{d}{dt}EU^3(\mu_t^2)(0)\right] &= sgn[\eta_2^S]sgn[1-R_2] \\ &= sgn[\eta_2^F]sgn[R_2-1] \ . \qquad (56)\end{aligned}$$

This completes the proof of the lemma. ∎

Proof of the lemma 5.

All the listed properties of μ are consequences of a consideration of the coalition of traders 2 and 3 not to dominate the given transaction μ by soley changing the transfer from 2 to 3. In particular, we use (47) and (48) to see whether the coalition {2,3} can dominate μ. We indicate below which equations are used for this purpose.

(21) follows from (49) and (20).

(22) follows from (49), (17), and (20). To show (23), let

$$a = \frac{C_2^F}{C_2^S}, \quad b = \frac{C_3^F}{C_3^S}.$$

Then, since $b = r_2^{\sigma\rho}$, one has

$$b = \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\rho}{1-(a\rho/b)}\right).$$

It follows that

$$b = \left(\frac{\rho}{1-\rho}\right)a = \left(\frac{1}{1-\rho}\right)b - r^{\sigma\rho}.$$

Thus, one obtains

$$\left(\frac{\rho}{1-\rho}\right)(a-b) = \left(\frac{1}{1-\rho}\right)b - r^{\sigma\rho} - \left(\frac{\rho}{1-\rho}\right)b = b - r^{\sigma\rho}$$

which gives (23).

(24) follows from (49) and (16).

The last properties of the lemma follow from (49), (16), and (20). ∎

Proof of the lemma 6

Given a feasible transaction μ, define another feasible transaction μ_t^i as in (12). Then, for $i = 1, 3$ we have by (48)

$$\frac{d}{dt}EU^{i+1}(\mu_t^i)(0) = \beta\left[\left(\frac{\sigma}{C_{i+1}^S} + \frac{1-\sigma}{C_{i+1}^F}\right)\epsilon - \rho\left(\frac{\sigma}{C_{i+1}^S}\eta_i^S + \frac{1-\sigma}{C_{i+1}^F}\eta_i^F\right)\right]. \tag{57}$$

Let us consider three cases.

Case 1: $\eta_i^F = 0$ (μ_i^F held constant.)
Setting $\eta_i^F = 0$ in (47), and substituting the value of η_i^S into (57), one obtains

$$\begin{aligned}
\frac{d}{dt}EU^{i+1}(\mu_t^i)(0) &= \beta\left[\left(\frac{\sigma}{C_{i+1}^S} + \frac{1-\sigma}{C_{i+1}^F}\right) - \rho\left(\frac{\sigma}{C_{i+1}^S}\right)\left(\frac{C_i^S}{\sigma}\right)\left(\frac{\sigma}{C_i^S} + \frac{1-\sigma}{C_i^F}\right)\right]\epsilon \\
&= \frac{\beta\sigma}{C_{i+1}^S}\left[(1-\rho) + \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{C_{i+1}^S}{C_{i+1}^F} - \rho\frac{C_i^S}{C_i^F}\right)\right]\epsilon \\
&= \frac{\beta(1-\sigma)\rho}{C_{i+1}^S}\left[\left(\frac{\sigma}{1-\sigma}\right)\left(\frac{1-\rho}{\rho}\right) - \left(\frac{C_i^S}{C_i^F} - \left(\frac{1}{\rho}\right)\frac{C_{i+1}^S}{C_{i+1}^F}\right)\right]\epsilon \qquad (58)\\
&= \frac{\beta\rho(1-\sigma)}{r^{\sigma\rho}C_{i+1}^F}\left[\frac{C_{i+1}^F}{C_{i+1}^S} - r^{\sigma\rho}\left(\frac{1}{R_i} - \frac{1}{\rho}\right)\right]\epsilon\ .
\end{aligned}$$

Hence, it follows

$$\begin{aligned}
sgn\left[\frac{d}{dt}EU^{i+1}(\mu_t^i)(0)\right] &= sgn[\epsilon]sgn\left[\left(\frac{\sigma}{1-\sigma}\right)\left(\frac{1-\rho}{\rho}\right) - \left(\frac{C_i^S}{C_i^F} - \left(\frac{1}{\rho}\right)\frac{C_{i+1}^S}{C_{i+1}^F}\right)\right] \\
&= sgn[\epsilon]sgn\left[\frac{C_{i+1}^F}{C_{i+1}^S} - r^{\sigma\rho}\left(\frac{1}{R_i} - \frac{1}{\rho}\right)\right]. \qquad (59)
\end{aligned}$$

Case 2: $\eta_i^S = 0$ (μ_i^S held constant.)
Again letting $\eta_i^S = 0$ in (47) we obtain from (57)

$$\begin{aligned}
\frac{d}{dt}EU^{i+1}(\mu_t^i)(0) &= \beta\left[\left(\frac{\sigma}{C_{i+1}^S} + \frac{1-\sigma}{C_{i+1}^F}\right) - \rho\left(\frac{1-\sigma}{C_{i+1}^F}\right)\left(\frac{C_i^F}{1-\sigma}\right)\left(\frac{\sigma}{C_i^S} + \frac{1-\sigma}{C_i^F}\right)\right]\epsilon \\
&= \frac{\beta(1-\sigma)}{C_{i+1}^F}\left[(1-\rho) + \left(\frac{\sigma}{1-\sigma}\right)\left(\frac{C_{i+1}^F}{C_{i+1}^S} - \rho\frac{C_i^F}{C_i^S}\right)\right]\epsilon \\
&= \frac{\beta\sigma}{C_{i+1}^F}\left[\left(\frac{1-\sigma}{\sigma}\right)\left(\frac{1-\rho}{\rho}\right) - \left(\frac{C_i^F}{C_i^S} - \left(\frac{1}{\rho}\right)\frac{C_{i+1}^F}{C_{i+1}^S}\right)\right]\epsilon \qquad (60)\\
&= \frac{\beta\sigma}{C_{i+1}^F}\left[\left(\frac{1-\sigma}{\sigma}\right)(1-\rho) - \frac{C_{i+1}^F}{C_{i+1}^S}(\rho R_i - 1)\right]\epsilon\ .
\end{aligned}$$

It then follows

$$sgn\left[\frac{d}{dt}EU^{i+1}(\mu_t^i)(0)\right] = sgn[\epsilon]sgn\left[\left(\frac{1-\sigma}{\sigma}\right)\left(\frac{1-\rho}{\rho}\right)-\left(\frac{C_i^F}{C_i^S}-\left(\frac{1}{\rho}\right)\frac{C_{i+1}^F}{C_{i+1}^S}\right)\right]$$
$$= sgn[\epsilon]sgn\left[\left(\frac{1-\sigma}{\sigma}\right)(1-\rho)-\frac{C_{i+1}^F}{C_{i+1}^S}(\rho R_i-1)\right] \ . \tag{61}$$

Case 3: $\epsilon = 0$ (μ_i^1 held constant.)

This case for $i \neq 2$ is exactly the same as that for $i = 2$. Hence, from (55) and (20) we have

$$sgn\left[\frac{d}{dt}EU^{i+1}(\mu_t^2)(0)\right] = sgn[\eta_i^F]sgn[R_i-1]$$
$$= sgn[\eta_i^S]sgn[1-R_i] \ . \tag{62}$$

This completes the proof of the lemma. ∎

Proof of the lemma 7

Case 1: $\eta_i^F = 0, \epsilon_4 \neq 0, \epsilon_5 \neq 0.$

Set ϵ_4, ϵ_5 so that we have

$$\frac{d}{dt}EU^4(\mu_t)(0) = -\left(\frac{\sigma}{C_4^F}+\frac{1-\sigma}{C_4^S}\right)\epsilon_4+\varphi\left(\frac{\sigma}{C_4^F}+\frac{1-\sigma}{C_4^S}\right)\epsilon_5 = 0 \ , \tag{63}$$

or $\epsilon_4 = \varphi\epsilon_5$. Then, using this value of ϵ_4, we obtain

$$\frac{d}{dt}EU^3(\mu_t)(0) = \left(\frac{\sigma}{C_4^F}+\frac{1-\sigma}{C_4^S}\right)(\varphi\gamma-1)\,\epsilon_5 \ .$$

Case 2: $\eta_4^F \neq 0, \epsilon_5 \neq 0, \epsilon_4 = \eta_5^F = 0.$

Set η_4^F and ϵ_5 so as to make

$$\frac{d}{dt}EU^4(\mu_t)(0) = \varphi\left(\frac{\sigma}{C_4^S}+\frac{1-\sigma}{C_4^F}\right)\epsilon_5-\frac{1-\sigma}{C_4^F}\eta_4^F = 0 \ .$$

It follows that

$$\eta_4^F = \frac{\varphi C_4^F}{1-\sigma}\left(\frac{\sigma}{C_4^S}+\frac{1-\sigma}{C_4^F}\right)\epsilon_5 \ .$$

Using this value of η_4^F, one sees that

$$\frac{d}{dt}EU^3(\mu_t)(0) = -\left(\frac{\sigma}{C_3^S}+\frac{1-\sigma}{C_3^F}\right)\epsilon_5+\frac{\gamma\rho(1-\sigma)}{C_3^F}\eta_4^F$$
$$= \left[\frac{\gamma\rho(1-\sigma)}{C_3^F}\left(\frac{\varphi C_4^F}{1-\sigma}\right)\left(\frac{\sigma}{C_4^S}+\frac{1-\sigma}{C_4^F}\right)-\left(\frac{\sigma}{C_3^S}+\frac{1-\sigma}{C_3^F}\right)\right]\epsilon_5$$
$$= \frac{C_4^F}{C_3^F}\left[\varphi\gamma\rho\left(\frac{\sigma}{C_4^S}+\frac{1-\sigma}{C_4^F}\right)-\frac{1-\sigma}{C_4^F}\left(1+\left(\frac{\sigma}{1-\sigma}\right)\frac{C_3^F}{C_3^S}\right)\right]\epsilon_5 \ .$$

We thus obtain

$$sgn\left[\frac{d}{dt}EU^3(\mu_t)(0)\right] = sgn[\epsilon_5]sgn\left[\varphi\gamma\rho\left(\frac{\sigma}{C_4^S}+\frac{1-\sigma}{C_4^F}\right)-\frac{1-\sigma}{C_4^F}\left(1+\left(\frac{\sigma}{1-\sigma}\right)\frac{C_3^F}{C_3^S}\right)\right] . \tag{64}$$

This completes the proof. ∎

Proof of the proposition 1

Let μ be a transaction satisfying the three conditions stipulated in the statement of the proposition. Then, μ is feasible, individually rational and satisfies $(\forall i = 1,2,3) 0 < \mu_i^1 < 1, \mu_i^F = \mu_i^S = 0$, and $\mu_j^1 = \mu_j^F = \mu_j^S = 0$ for $j = 4,5$. It follows that we have

$$\begin{aligned} C_i^F &= C_i^S \qquad \text{for } i = 1,2,4 \\ C_3^F &= 1-\mu_3^1 \qquad C_3^S = 1-\mu_3^1+\beta\mu_2^1 \end{aligned}$$

where one has

$$\mu_2^1 \geq \frac{1}{\beta\sigma}\mu_3^1$$

by the individual rationality of trader 3. Thus,

$$\begin{aligned} &\frac{C_i^F}{C_i^S} = 1 \qquad \text{for } i = 1,2,4, \text{ and} \\ &0 < \frac{C_3^F}{C_3^S} < 1 . \end{aligned}$$

If trader 3 sends almost all its endowments so that μ_3^1 is very close to 1, because $\mu_2^1 \geq (1/\beta\sigma)\mu_3^1$, the ratio C_3^F/C_3^S becomes arbitrarily close to 0. We check trader by trader that μ can be dominated by changing μ to be state contingent.

- **i=1:** Since we have

$$\frac{C_1^S}{C_1^F} - \left(\frac{1}{\rho}\right)\left(\frac{C_2^S}{C_2^F}\right) = \frac{C_1^F}{C_1^S} - \left(\frac{1}{\rho}\right)\left(\frac{C_2^F}{C_2^S}\right) = 1-\frac{1}{\rho} < 0 ,$$

  it follows from (25) and (26) that μ_1^F and μ_1^S must stay zero, otherwise it will be {1,2}-dominated.

- **i=2:** Since $C_2^F/C_2^S = 1 > r^{\sigma\rho}$ for $\sigma > \rho$, μ_2^S must stay zero by (16). On the other hand, one has

$$\rho\left(\frac{1-\sigma}{\sigma}\right)+\rho\left(\frac{C_2^F}{C_2^S}\right) = \rho\left(\frac{1-\sigma}{\sigma}+1\right) = \frac{\rho}{\sigma} .$$

  Therefore, by (17), if

$$\frac{C_3^F}{C_3^S} < \frac{\rho}{\sigma} ,$$

  then μ will be {2,3}-dominated by increasing μ_2^F above zero and decreasing μ_2^1. But since we have

$$\frac{C_3^F}{C_3^S} = \frac{1-\mu_3^1}{1-\mu_3^1+\beta\mu_2^1} ,$$

the above inequality amounts to having

$$1-\mu_3^1 < \frac{\rho}{\sigma}(1-\mu_3^1+\beta\mu_2^1) \ ,$$

or

$$\mu_3^1 + \left(\frac{\beta\rho}{\sigma-\rho}\right)\mu_2^1 > 1 \ . \tag{65}$$

For example, for parameter values $\sigma = 0.9, \rho = 0.8$, and $\beta = 1.6$, the coefficient of μ_2^1 in the inequality is 12.8 so that if the transfer from trader 2 exceeds $\frac{5}{64}$, regardless of the amount of transfer from trader 3 the inequality is satisfied.

Thus, all those transactions satisfying (31) will be {2,3}-dominated by increasing the transfer from trader 2 in event F and decreasing the first round transfer from 2.

- **i=3:** Since inequality

$$\frac{C_3^F}{C_3^S} < \left(\frac{1}{\rho}\right)\frac{C_1^F}{C_1^S} = \frac{1}{\rho}$$

always hold for $0<\rho<1$, by (26) an increase in μ_3^F will not dominate the given transaction μ so that μ_3^F must stay zero.

On the other hand, we have

$$\begin{aligned}
\frac{C_3^S}{C_3^F} - \left(\frac{1}{\rho}\right)\frac{C_1^S}{C_1^F} = \frac{C_3^S}{C_3^F} - \frac{1}{\rho} \quad &> \quad \left(\frac{\sigma}{1-\sigma}\right)\left(\frac{1-\rho}{\rho}\right) \\
\Leftrightarrow \quad -(1-\rho)(1-\mu_3^1)+\beta\rho\mu_2^1 \quad &> \quad \left(\frac{\sigma}{1-\sigma}\right)\left(\frac{1-\rho}{\rho}\right)(1-\mu_3^1) \\
\Leftrightarrow \mu_3^1 + \beta(1-\sigma)\left(\frac{1-\rho}{\rho}\right)\mu_2^1 \quad &> \quad 1 \ .
\end{aligned} \tag{66}$$

It thus follows from (26) that for transfers satisfying (66), μ will be {1,3}-dominated by increasing μ_3^S and decreasing μ_3^1. For the parameter values we specified in an earlier example, the condition above is met if $\mu_2^1 \geq 0.02$, for example, regardless of the value of μ_3^1.

- **i=4,5:** We have $C_4^F = C_4^S = 1$. Let us compute the inside the bracket of the second term on L.H.S. of (64). It becomes

$$\begin{aligned}
& \varphi\gamma\rho - (1-\sigma) - \sigma\left(\frac{C_3^F}{C_3^S}\right) \\
= \quad & \varphi\gamma\rho - (1-\sigma) - \sigma\left(\frac{1-\mu_3^1}{1-\mu_3^1+\beta\mu_2^1}\right) \\
= \quad & \frac{\rho}{1-\mu_3^1+\beta\mu_2^1}\left[\beta\left(\varphi\gamma - \frac{1-\sigma}{\rho}\right)\mu_2^1 - \left(\frac{1}{\rho}-\varphi\gamma\right)(1-\mu_3^1)\right] .
\end{aligned} \tag{67}$$

It thus follows that μ is {3,4}-dominated by increasing both μ_4^F and μ_5^1 by appropriate amounts when

$$\beta\left(\varphi\gamma - \frac{1-\sigma}{\rho}\right)\mu_2^1 > \left(\frac{1}{\rho}-\varphi\gamma\right)(1-\mu_3^1) \ , \tag{68}$$

where parameter values satisfy

$$\varphi\gamma > \frac{1-\sigma}{\rho} \ . \tag{69}$$

By summarizing the arguments, we obtain the statement of the proposition 1. ∎

Proof of the proposition 2 and the proposition 3

The first part of the proposition 3 is a direct consequence of lemma 1. To show the proposition 2 and the second part of the proposition 3, we will look for characteristics of typical core transactions by modifying state non-contingent transactions.

Step 1: We modify a feasible and individually rational transaction μ, having only state non-contingent transfers, so as to avoid domination by coalitions stated in proposition 1.

Thus, assume that μ is a feasible and undominated transaction satisfying

$$\begin{gathered} \mu_i^1 > 0 \text{ for } i = 1, 2, 3, 5, \\ \mu_2^F > 0, \mu_3^S > 0, \mu_4^F > 0, \\ \text{all other } \mu_i^A\text{'s are 0 for } A = 1, F, S \ . \end{gathered}$$

We will look at traders 1 and 2 to check whether further state contingent transfers are called for.

Since we have

$$\begin{aligned} \frac{\rho C_1^F}{C_1^S} - \frac{C_2^F}{C_2^S} &= \frac{\rho(1-\mu_1^1+\beta\mu_3^1)}{1-\mu_1^1+\beta\mu_3^1+\beta\rho\mu_3^S} - \frac{1-\mu_2^1+\beta\mu_1^1-\mu_2^F}{1-\mu_2^1+\beta\mu_1^1} \\ &> \left(\frac{1-\sigma}{\sigma}\right)(1-\rho) \\ \Leftrightarrow \quad & \frac{\mu_2^F}{1-\mu_2^1+\beta\mu_1^1} - \frac{\rho}{1+((1-\mu_1^1+\beta\mu_3^1)/(\beta\rho\mu_3^S))} > \frac{1-\rho}{\sigma} \ , \end{aligned} \tag{70}$$

by (26), when the inequality (70) holds, μ is {1,2}-dominated by increasing μ_1^F and decreasing μ_1^1 so that μ must specify $\mu_1^F > 0$ in order to be undominated.

The term $(1-\rho)/\sigma$ may become arbitrarily close to 0 for σ sufficiently close to 1 and ρ sufficiently close to σ. On the other hand, the second term in the last inequality in (70) may be reduced very close to 0 for sufficiently small μ_3^S. Thus, in general, we expect that when the transfer μ_2^F of trader 2 in event F is not too small relative to the benefit $(1-\mu_2^1+\beta\mu_1^1)$ of his consumption in event F, undominated transfer μ must specify trader 1 to effect positive transfer μ_1^F to trader 2 in event F.

By (14) of the lemma 4, when we have

$$\frac{C_3^F}{C_3^S} = \frac{1-\mu_3^1-\mu_5^1+\rho(\beta\mu_2^F+\gamma\mu_4^F)}{1-\mu_3^1-\mu_5^1-\mu_3^S+\beta\rho\mu_2^1} < r^{\sigma\rho} \ ,$$

μ will be {2,3}-dominated by increasing both μ_2^F and μ_2^S by the amount of a decrease in μ_2^1. It means that a core transaction will guarantee in event F approximately 100 $r^{\sigma\rho}$ %

of the utility level trader 3 gets in event S. Recall that

$$r^{\sigma\rho} = \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\rho}{1-\rho}\right)$$

and as a function of ρ it increases from 0 to 1 as ρ increases from 0 to σ. In case of a failure in transfer from trader 2 to trader 3, μ_2^1 is lost in event F. If the compensation of this loss is not done sufficiently, then an increase in second round transfer from trader 2 accompanied by a decrease in first round transfer will make both traders better off.

Step 2: Following the arguments of the previous step, let μ be a feasible and undominated transaction as specified in Step 1 except for μ_2^S which we now consider to be positive. Then, since all three of μ_2^1, μ_2^F and μ_2^S become positive, by (21) of the lemma 5, we obtain

$$\frac{C_2^F}{C_2^S} = \frac{C_3^F}{C_3^S} = r^{\sigma\rho} \ . \tag{71}$$

By (71), when we have

$$\begin{aligned} \frac{C_3^F}{C_3^S} - \left(\frac{1}{\rho}\right)\frac{C_1^F}{C_1^S} &> \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{1-\rho}{\rho}\right) \\ \Leftrightarrow \left(\frac{1-\sigma}{\sigma}\right)\frac{2\rho-1}{1-\rho} &> \frac{C_1^F}{C_1^S} \ . \end{aligned} \tag{72}$$

Thus, by (26), when this inequality holds μ_3^F must be positive to avoid domination by an increase in μ_3^F and an accompanying decrease in μ_3^1. But since both μ_3^F and μ_3^S cannot be positive by the lemma 2, we need to look at (62) to check which of μ_3^F and μ_3^S need be positive. By (62) and (71), μ_3^F must be positive if

$$\frac{C_1^F}{C_1^S} < r^{\sigma\rho} \ . \tag{73}$$

Since $0 < \rho < 1$ implies $r^{\sigma\rho} > (2\rho-1)/(1-\rho)$, (73) but not (72) gives the criterion of when μ_3^F must be positive.

Step 3: Following the arguments of Step 3, we now take μ_3^F to be positive instead of μ_3^S. We go back to Step 1 to consider transfers of $i = 1$. We have

$$\begin{aligned} \frac{\rho C_1^F}{C_1^S} - \frac{C_2^F}{C_2^S} &= \frac{\rho(1-\mu_1^1-\mu_1^F+\beta\mu_3^1+\beta\rho\mu_3^F)}{1-\mu_1^1+\beta\mu_3^1} \\ &\quad -\frac{1-\mu_2^1-\mu_2^F+\beta\mu_1^1+\beta\rho\mu_1^F}{1-\mu_2^1-\mu_2^S+\beta\mu_1^1} \\ &= \frac{\beta\rho^2\mu_3^F-\rho\mu_1^F}{1-\mu_1^1+\beta\mu_3^1} - \frac{\beta\rho\mu_1^F-\mu_2^F+\mu_2^S}{1-\mu_2^1-\mu_2^S+\beta\mu_1^1} \\ &\quad -(1-\rho) \geq \frac{1-\sigma}{\sigma}(1-\rho) \\ \Leftrightarrow \frac{\mu_2^F-\mu_2^S-\beta\rho\mu_1^F}{1-\mu_2^1-\mu_2^S+\beta\mu_1^1} &- \frac{\rho\mu_1^F-\beta\rho^2\mu_3^F}{1-\mu_1^1+\beta\mu_3^1} \\ &\geq \frac{1-\rho}{\sigma} \ . \end{aligned} \tag{74}$$

Thus, by (61), if μ is undominated, it must satisfy the above weak inequality where the equality holds when $\mu_1^1 + \mu_1^F < 1$.

Step 4: By step-by-step arguments we have been led to consider feasible and undominated transactions μ satisfying

$$\begin{array}{lcll} \mu_i^1 > 0 & \text{for} & i = 1, 2, 3, 5, & \\ \mu_i^F > 0 & \text{for} & i = 1, 2, 3, 4 & \\ \mu_2^S > 0, & \text{and all other} & \mu_i^S \text{'s and } \mu_i^4 & \text{are } 0. \end{array}$$

We finally come back to $i = 4$ and check transfers between trader 4 and trader 3. We have

$$\begin{aligned} \frac{C_3^F}{C_3^S} &= r^{\sigma\rho} \\ C_4^F &= 1 - \mu_4^F + \beta\mu_5^1 \\ C_4^S &= 1 + \beta\mu_5^1 \,. \end{aligned} \tag{75}$$

Hence, the terms inside the second bracket in (64) are given by

$$\begin{aligned} &\varphi\gamma\rho\left(\frac{\sigma}{C_4^S} + \frac{1-\sigma}{C_4^F}\right) - \frac{1-\sigma}{C_4^F}\left(1 + \left(\frac{\sigma}{1-\sigma}\right)r^{\sigma\rho}\right) \\ &\quad = \frac{\varphi\gamma\sigma\rho}{C_4^F}\left[\frac{C_4^F}{C_4^S} - \frac{1-\sigma}{\sigma}\left(\frac{1}{\varphi\gamma\rho(1-\rho)} - 1\right)\right]. \end{aligned} \tag{76}$$

Here one may note that $0 < \varphi\gamma\rho(1-\rho) < 1/4$ since $0 < \varphi\gamma < 1$ and the maximum value for $\rho(1-\rho)$ is $1/4$ for $0 < \rho \leq \sigma$ and $\sigma \geq 1/2$. Therefore, for an undominated transfer μ having $\mu_4^F > 0$ and $\mu_5^1 > 0$, we have

$$\frac{C_4^F}{C_4^S} \geq \frac{1-\sigma}{\sigma}\left(\frac{1}{\varphi\gamma\rho(1-\rho)} - 1\right) \tag{77}$$

with strict equality holding for an interior solution.

It follows from (75) and (77) that we have

$$\mu_4^F \leq \left(\frac{\varphi\gamma\rho(1-\rho) - (1-\sigma)}{\sigma\varphi\gamma\rho(1-\rho)}\right)(1 + \beta\mu_5^1) \tag{78}$$

where strictly equality corresponds to that in (77). The R.H.S. of this weak inequality is less than or equal to 1 if and only if

$$\mu_5^1 \leq \frac{(1-\sigma)\,(1 - \varphi\gamma\rho(1-\rho))}{\beta\left[\varphi\gamma\rho(1-\rho) - (1-\sigma)\right]} = v(\varphi\gamma, \sigma, \rho, \beta)\,. \tag{79}$$

We like to impose a parameter restriction so that the nominator of the first term in the R.H.S. of (78) is positive, i.e., $\varphi\gamma\rho(1-\rho) - (1-\sigma) > 0$. Thus, we assume

$$\varphi\gamma > \left(\frac{1-\sigma}{\rho}\right)\frac{1}{1-\rho}\,. \tag{80}$$

For example, this condition is satisfied if $\varphi\gamma > 0.63$ when $\sigma = 0.9$ and $\rho = 0.8$. It is also satisfied whenever we have $\varphi\gamma > r^{\sigma\rho}$.

(79) means that as long as the value of μ_5^1 is set to satisfy this weak inequality, the value of μ_4^F can be set to satisfy (78) with equality because it becomes feasible to do so. The value of $v(\varphi\gamma, \sigma, \rho, \beta)$, for example, is 0.456 for $\varphi\gamma = 0.9, \sigma = 0.9, \rho = 0.7, \beta = 2$, and is 3.7 for $\varphi\gamma = 0.7, \sigma = 0.9, \rho = 0.8, \beta = 2$. Thus, whenever the value of μ_5^1 can be feasibly increased within the bound of $v(\varphi\gamma, \sigma, \rho, \beta)$, then the value of μ_5^1 is given by $v(\varphi\gamma, \sigma, \rho, \beta)$ and if μ_5^1 cannot be increased feasibly up to the value of $v(\varphi\gamma, \sigma, \rho, \beta)$, then strict inequality holds in (79). But in either case the value of μ_4^F is given by (78) with equality. It therefore follows that in (77) the equality always hold. We are now done with step-by-step arguments.

The second part of the proposition 3 follows from the arguments leading to (78) and (79) in Step 4 by noting that (64) is derived on the basis of transferring to trader 4 just enough to make him indifferent in expected utility to transferring μ_4^F to trader 3 in event F.

The statement of the proposition 2 summarizes the results of the above arguments. ∎