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Stochastically Stable Equilibria in Coordination Games with Multiple Populations

by

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ABSTRACT: We investigate the equilibrium selection problem in n-person binary coordination games by means of adaptive play with mistakes (Young 1993). The size and the depth of a particular type of basins of attraction are found to be the main factors in determining the selection outcome. The main result shows that if a strategy has the larger basin of attraction, and if it is deep enough, then the strategy constitutes a stochastically stable equilibrium. The existence of games with multiple stochastically stable equilibria is an immediate consequence of the result. We explicitly address the qualitative difference between selection results in multi-dimensional stochastic evolution models and those in single dimensional models, and shed some light on the source of the difference.

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1 Introduction

In a coordination game with many players, the payoff that a particular strategy generates is a nondecreasing function of the number of players who adopt that strategy. This class of games can be used to model a number of economic environments with strategic complementarities (e.g., Cooper 1999, Schelling 1978). For example, consider a situation in which there are several distinct technical standards of an emerging network product. If there are n producers in the market, each facing a problem in determining which standard it adopts for its product, then their environment defines an n-person coordination game. Another example is a decision problem over multiple alternatives in a project team consisting of n members. The characteristic feature of these environments is, of course, the presence of multiple strict equilibria. Which equilibrium, if any, is the robust prediction of the model? To deal with the equilibrium selection problem, a systematic analysis is called for.

This paper investigates the equilibrium selection problem in n-person asymmetric binary coordination games by means of the stochastic evolution analysis introduced by Foster and Young (1991), Kandori, Mailath, and Rob (1993) and Young (1993). The main contributions are as follows. First, we develop a systematic analysis into stochastic evolution of many-person stage games, and compare our results with existing results. Second, we explicitly address the qualitative difference between selection results obtained in stochastic evolution models with multi-dimensional state spaces and those in models with single dimensional state spaces, and shed some light on the source of the difference. For evolutionary analyses that built around the versions of replicator dynamics, it is well known that there are qualitative difference between multi-population and single population models (e.g., Weibull 1995). However, it does not appear to be well recognized that an analogous difference is present for stochastic evolution models. Our analysis is based on the formulations of and the investigations into appropriate linear and non-linear minimization programs, which formalize the familiar mistake counting argument. Authors have proposed varieties of dynamics that employ the mistake counting argument. Each dynamic requires a particular minimization program that corresponds to the mistake counting in that particular dynamic. In this paper, we focus on the following dynamics: adaptive play (Young 1993), multi-population random matching (Young 1998), and single population random matching (Kandori et al 1993, Kim 1996).

Although the stochastic evolution has grown into an established literature, it should be stressed that the focus of the paper is on the class of games that appears to have been almost left untouched: asymmetric, many-person games. To the best of our knowledge, the explicit equilibrium selection results obtained thus far restrict the stage game to be either symmetric, two-person, or both.[1] In contrast, the present paper develops a systematic analysis of n-person games. To keep the analysis tractable we restrict our attention to a binary coordination game, in which each player has only two strategies and the two unanimous strategy profiles constitute strict equilibria. As far as formal equilibrium selection results concerned, we focus on the adaptive play model, mainly because it is the least analyzed among the three representative models. The main result offers a sufficient condition for stochastically stable equilibrium in an n-person asymmetric binary coordination game, which leads to a generalization of the existing result for two-person games. More importantly, the main result is used to derive a multiplicity result, which states that if the binary coordination game has strategy profiles at which two strategies generate the same payoff, then both equilibria may well be stochastically stable, even if each strategy pays off quite differently elsewhere.

The existence of multiple stochastically stable equilibrium is first discovered by Young (1998) for n-person binary unanimity games played in multi-population random matching. In several ways our result generalizes that of Young (1998). First, it applies to a general binary coordination game, which is not necessarily a unanimity game. Second, our result and that of Young (1998) together show that multiplicity is not specific to a particular selection dynamic. On the other hand, in a rare study of equilibrium selection in a many-person game, Kim (1996) obtains, among other things, a uniqueness of stochastically stable equilibrium for a symmetric n-person binary coordination game played in single population random matching.

Thus, multiplicity may occur in both adaptive play and multi-population random matching, whereas it does not in single population random matching. One may suspect that the difference is related somehow to the difference in the dimensions of the respective state spaces. It is not immediately clear, however, how exactly the difference in dimension leads to the presence or absence of multiplicity. We try to answer this question by analyzing appropriate mistake minimization programs.

The remainder of the paper is organized as follows. In the next section we offer an intuitive explanation of the issues and our results. In section 3 we derive preliminary results for adaptive play of a many-person binary coordination game. The equilibrium selection results are proved in Section 4. In the final section we discuss the source of the aforementioned difference in terms of relevant mistake minimization programs.

k	0	1	2	3
A	a_1	a_2	a_3	a_4
B	b_4	b_3	b_2	b_1

G_0

k	0	1	2	3
A	0	a_2	a_3	a_4
B	b_4	0	0	0

G_1

k	0	1	2	3
A	0	ε	ε	a
B	$a+\delta$	0	0	0

G_2

k	0	1	2	3
A	0	0	0	$a+\delta$
B	a	ε	ε	0

G_3

k	0	1	2	3
A	0	0	0	a_4
B	b_4	0	0	0

G_4

k	0	1	2	3
A	0	0	0	a
B	$a+\delta$	ε	ε	0

G_5

Figure 1: Payoff tables for binary coordination games, where k is the number of other players who choose A.

2 Intuition behind the results

The game G_0 in Figure 1 describes a symmetric four-person game. Strategies are named A and B, and the table represents the payoff function of a player. The top row shows the number of other A players. If a player chooses A and there are two others who do the same, then the payoff for the player is a_3, and so on. It is a binary coordination game if a_k and b_k are nondecreasing in k and if $a_4 > b_1$ and $b_4 > a_1$. It follows that $(A, \ldots, A)$ and $(B, \ldots, B)$ are strict equilibria.

Consider a situation in which players play a game over time. The dynamic is a myopic best response with mistakes, which we call the *prototype*. On each day, each player chooses a best response against the profile realized yesterday, but sometimes one may choose otherwise by mistake. Let the game they play be G_1 and let $(A, \ldots, A)$ be the realized profile yesterday. Although everyone is supposed to play A today, assume that exactly one of them chooses B. What will happen tomorrow? If there are no further mistakes, they will return to the original equilibrium. One mistake is not enough to drive the players to switch their actions. Alternatively, assume that $(B, \ldots, B)$ was the profile realized yesterday and that exactly one player makes a mistake today. Then, without further mistakes, players will find themselves at the other equilibrium the day after tomorrow. One mistake is enough to drive them to switch their actions. Using this type of "mistake counting argument", let us measure the size of the "basins of attraction" of each strategy. It is crucial to observe that the size of a basin

[1]In stochastic evolution analysis, a selection result is explicit if conditions for a stochastically stable equilibrium are stated only in terms of payoffs and other primitives in the stage game, without mentioning any intermediate concepts such as costs of transition or potential minimizing trees.

in the prototype depends solely on the best response structure of the given game and is not related whatsoever to any "enriched" structure, such as random matching in a population or a truncated fictitious play, into which the given game would be embedded in a full-fledged stochastic evolution analysis. It is also important to recognize that the mistake counting argument in the prototype would not discriminate equilibria at all if the given game were a two-person game, so the issues addressed here are specific to many-person games.

Thus, we say that A has the larger basin of attraction in game G_1. In such a case, is it natural to expect that a reasonable equilibrium selection criterion would single out equilibrium $(A, \ldots, A)$? Not necessarily, since payoffs generated by strategy A may not be large enough compared to that of B. For example, a_2 and a_3 may be vanishingly small and a_4 may be smaller than b_4. Intuitively, even when the basin of attraction of A is large, it need not be "deep" enough. Thus an equilibrium selection model may determine the outcome as a balance of the two factors: comparison of the size and depth of the basins of attraction.[2] In game G_2, the balance of the two factors boils down to that of $\varepsilon > 0$ and $\delta > 0$. The way these two parameters work in determining the selection outcome depends on the "enriched" structure, or the particular dynamic under study.

Working in the adaptive play with mistakes, our main result states that if a strategy has the larger basin of attraction in the prototype, and if the basin is also deep enough, then the strategy constitutes a stochastically stable equilibrium. This sounds simple and intuitive, but it has a somewhat unexpected consequence. For $(A, \ldots, A)$ to be stochastically stable in game G_2, the depth consideration requires that δ should not be as large. Consequently, the main result implies that there is $\overline{\delta} > 0$ such that for every $0 < \delta \leq \overline{\delta}$ and every $\varepsilon > 0$, $(A, \ldots, A)$ is stochastically stable. Likewise, $(B, \ldots, B)$ is stochastically stable in Game G_3. An immediate consequence is the existence of multiple stochastically stable equilibria: both equilibria are stable in G_4 provided a_4 and b_4 are relatively close. If the size of the basins is the same, both equilibria are stable unless the difference in depth is overwhelming.

Now we return to the prototype. To determine the basin of attraction, we search for sequences of strategy profiles from one equilibrium to the other, and then identify which of these contains the fewest mistakes. Such sequences contain asymmetric strategy profiles in general. Given a full-fledged stochastic evolution dynamic, consider its state space. If it contains states that correspond to asymmetric stage game strategy profiles, then the mistake counting in the prototype can be replicated as a mistake counting in the full-fledged model.

[2] In stochastic evolution models in which the mistake rate or the speed of adjustment is state dependent, the "depth" of a basin of attraction has been found to be a factor in determining the selection outcome (e.g., Binmore and Samuelson 1997, Kandori 1997). When the stage game is a many-person game, our analysis shows that the depth matters even in state independent models.

Moreover, such a replication may well be optimal in the mistake minimizing problem in the full-fledged model. In this way, mistake counting in the prototype may become relevant or even decisive. The adaptive play is a fictitious play with a finite memory size, so the player role in the stage game remains intact. Thus, there is no problem in translating a sequence of stage game strategy profiles into a path of states in the adaptive play. The same is true for the multi-population random matching model. In contrast, the single population random matching model is incapable of expressing an asymmetric stage game strategy profile as its own state, since the player role is absent in the model. Therefore, in the single population random matching model games G_2 and G_5 are treated as almost identical provided ε is sufficiently small. Thus the mistake counting argument in the prototype can never be a factor.

3 Preliminaries

3.1 The game

There are n players, denoted by $i \in I = \{1, \dots, n\}$, $n \geq 2$. Each player chooses her strategy $\sigma^i \in \{A, B\}$. A generic strategy profile is denoted by $\sigma \in \Sigma = \{A, B\}^n$. Let $|\sigma|_X$ be the number of players employing $X \in \{A, B\}$ in σ. The payoff for player i is:

$$u^i(\sigma) = \begin{cases} a^i_{|\sigma|_A}, & \text{if } \sigma^i = A, \\ b^i_{|\sigma|_B}, & \text{if } \sigma^i = B, \end{cases}$$

where a^i_k and b^i_k are functions defined on $\{1, \dots, n\}$ such that

(**G1**) a^i_k and b^i_k are nondecreasing in k,

(**G2**) $a^i_n > b^i_1$ and $b^i_n > a^i_1$.

The game thus defined is called a *binary coordination game*. The condition (**G1**) implies that the payoff associated with a particular strategy depends only on the number of players who adopt that strategy. By (**G2**), both $(A, \dots, A)$ and $(B, \dots, B)$ are strict equilibria. Following Harsanyi and Selten (1988), let us call $\alpha^i = a^i_n - b^i_1$ the *deviation loss* of $i \in I$ at equilibrium $(A, \dots, A)$. The deviation loss at $(B, \dots, B)$ is $\beta^i = b^i_n - a^i_1$. The game is *symmetric* if players have identical payoff parameters. If the game is symmetric or $n \leq 3$, it has exactly two strict equilibria. The game may well have more than two strict equilibria in general.

3.2 Convergence in the adaptive play

We employ the adaptive play model of Young (1993) for equilibrium selection. The *adaptive play without mistakes* is a dynamic adjustment model in discrete time in which the stage game is played once in each period. The state of the dynamic in a given period is the most recent history (*i.e.*, the sequence of strategy profiles most recently realized) of length T. Each player chooses a best response against her sample, which is a randomly chosen s-length subsequence of the current state, where $s \leq T$. Owing to random sampling, the adaptive play without mistakes is a finite-state Markov chain. A notable property of the chain is that a state is absorbing if and only if it is a T-fold concatenation of a strict equilibrium in the stage game.

The following definitions and facts from Markov chain theory are relevant to the current discussion.[3] A *recurrent class* is a set of states that is minimal with respect to set inclusion among the sets with the property that once the chain enter into the set, it will remain within thereafter. A finite-state Markov chain has nonempty recurrent classes. A state is *absorbing* if by itself it forms a singleton recurrent class. A chain is *absorbing* if, starting from any state, the chain will reach an absorbing state in a finite number of steps with probability one.

We now introduce some noise as follows. In each period a player may fail to choose a best response and end up with a random strategy choice with probability $\epsilon > 0$. If the randomly chosen strategy is not a best response to any sample that might be drawn, then the strategy is called a *mistake*. The resulting process is called the *adaptive play with mistakes*. The crucial property of the play with mistakes is that it has a unique stationary distribution μ_ϵ, to which the distribution of play converges in the long run. Young (1993) shows that the limit $\mu^* = \lim_{\epsilon \to 0} \mu_\epsilon$ is a stationary distribution of the adaptive play without mistakes. A state is *stochastically stable* if the limiting distribution μ^* puts a positive weight on it.

In principle, the stochastically stable states can be identified by invoking the mistake counting argument. It involves evaluating the *resistance* from a recurrent class to another, which is the minimum number of mistakes for the adaptive play to travel from the origin to the destination. Due to the potential presence of intermediate recurrent classes, the evaluation of resistance can be quite complex in general. If there are exactly two recurrent classes, however, the argument is straightforward, as we only need to consider the direct paths from one recurrent class to the other.

The stochastic stability analysis for the adaptive play generates equilibrium selection results when the play without mistakes is absorbing. In this case, the stochastically stable distribution can be viewed as a probability measure on the set of strict equilibria in the stage game.

A strategic game is *weakly acyclic* if, starting from any strategy profile, a strict equilibrium can be reached via a sequence of strategy profiles such that there is exactly one player who plays differently between each profile and its immediate predecessor, and the new strategy is a best response against the predecessor. Young (1993, Theorem 1) shows that if the stage game is weakly acyclic, then the adaptive play without mistakes is absorbing with appropriate

[3]Basic results for finite Markov chains can be found in, for example, Kemeny and Snell (1976).

k	0	1	2	3	4
A	0	ε_2	ε_3	ε_4	α
B	β	0	0	0	0

u

k	0	1	2	3	4
A	0	0	0	0	α
B	β	ε_4	ε_3	ε_2	0

v

k	0	1	2	3	4
A	0	0	0	0	α
B	β	0	0	0	0

w

Figure 2: Payoffs in a five-person game, in which $0 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < \min\{\alpha, \beta\}$.

choices of T and s. One can verify that a binary coordination game is weakly acyclic if it is symmetric or $n \leq 4$. In general, however, it may not be weakly acyclic. In fact, the adaptive play need not be absorbing.

Example 1. There are five players. Players $i = 1, 2$ have payoff function u, player $i = 3$ has payoff function w, and players $i = 4, 5$ have payoff function v in Figure 2. The five-person game has exactly two strict equilibria, because player 3 has a unique best response only if all the others make a unanimous choice. Now consider the strategy profiles (A, A, A, B, B) and (A, A, B, B, B). Each is a non-strict equilibrium, in which all players but 3 play their unique best responses. Since all the states consisting solely of these equilibria form a non-singleton recurrent class, the adaptive play is not absorbing.

In view of this example, it is important to identify a condition that makes the adaptive play without mistakes absorbing. In addition, we would like to find a condition under which there are exactly two strict equilibria. In this paper, we offer a single condition that ensures these two desirable properties. To state the condition, we need to introduce some notation. For every $i \in I$, define $k^i = \max\{\, k \mid b^i_{n-k} > a^i_{k+1} \,\}$. In Figure 3, which depicts the payoff parameters of $i \in I$, the number k^i is $m - 1$. The threshold $k^i = k$ means that at least $k + 1$ others must be present in order for i to play A optimally. Let $k = |\sigma^{-i}|_A$ be the number of others adopting A and let $BR^i(\cdot)$ be the pure best response correspondence. It follows that $BR^i(\sigma) = \{B\}$ if $k \leq k^i$ and $A \in BR^i(\sigma)$ otherwise. Note that $0 \leq k^i \leq n - 2$ for every $\in I$. For every $k = 0, 1, \ldots, n - 2$, let

$$\bar{I}(k) = \{\, i \in I \mid k^i \leq k \,\}, \quad \underline{I}(k) = \{\, i \in I \mid k^i \geq k \,\}, \quad I(k) = \{\, i \in I \mid k^i = k \,\}.$$

The first set is the set of players for which A is a best response if there are $k+1$ others who play A. The second is the set of players for which B is a unique best response if there are k others who play A. $I(k)$ is the intersection of the two. The key condition concerns the distribution of thresholds k^i.

(**G3**) If there is k, $2 \leq k \leq n - 2$, such that $|\bar{I}(k - 2)| = k$, then $I(k - 1) \neq \varnothing$.

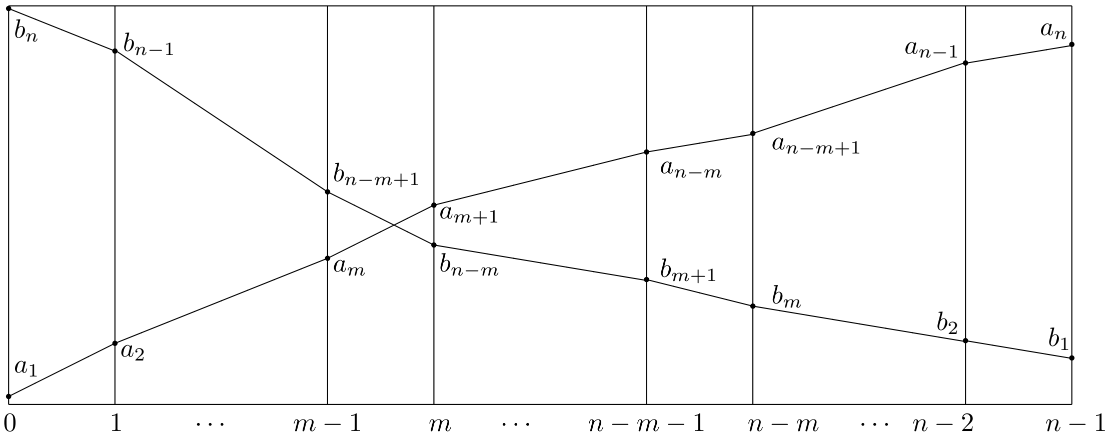

Figure 3: Payoff parameters in an n-person simple binary coordination game.

In words, (**G3**) can be explained as follows. Assume that $|\bar{I}(k-2)| = k$ and consider any strategy profile in which every $i \in \bar{I}(k-2)$ chooses A. By definition of k^i, A by i is a best response, regardless of what the $n-k$ others do. What about the choices by the $n-k$ others? Condition (**G3**) stipulates that there is some $j \notin \bar{I}(k-2)$ such that playing A is a best response as long as every $i \in \bar{I}(k-2)$ chooses A.

Lemma 1. *If a binary coordination game satisfies* (**G3**)*, then it has exactly two strict equilibria. For a binary coordination game that does not involve alternative best responses, the converse is also true.*

The proof of Lemma 1 is left to the reader. Recall that the size of a state and of a sample in the adaptive play are denoted by T and s, respectively.

Lemma 2. *For a binary coordination game satisfying* (**G3**)*, the adaptive play without mistakes is absorbing whenever $s \leq T/2$.*

The proof is given in the Appendix. Under (**G3**), therefore, the equilibrium selection problem takes the simplest form. Let $\mathbf{A}$ and $\mathbf{B}$ denote the T-fold concatenations of $(A, \ldots, A)$ and $(B, \ldots, B)$. The resistance from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $r(A, B)$, and the resistance for the other direction is $r(B, A)$. $(A, \ldots, A)$ is uniquely stochastically stable if and only if $r(A, B) > r(B, A)$. In what follows, we always assume that a binary coordination game satisfies (**G3**) and that $s \leq T/2$.[4]

[4]Even if (**G3**) fails, the results in the next section apply as long as the binary coordination game has exactly two strict equilibria to which the adaptive play without mistakes converges.

4 Equilibrium Selection

4.1 Linear program for evaluating resistance

Consider the adaptive play with mistakes for a binary coordination game. The current state is $\mathbf{A}$. In any path from $\mathbf{A}$ to $\mathbf{B}$, there is a player who optimally chooses strategy B for the first time. Let us call that player a *first deviator*. The first deviator $i \in I$ must have a sample against which playing B is optimal. Such a sample must contain considerable number of B strategies played by others. Since player i is a first deviator, all such B strategies are mistakes. We set up a linear program that gives us the minimum number of B strategies that i must face. Its optimal solution not only gives us the number, but also reveals the way in which mistakes occur. In many-person games not only the number but also the *distribution* of mistakes matters. The linear program introduced below takes care of the case in point.

Fix a player $i \in I$. Set

$$z^i_k = a^i_n - a^i_{n-k} + b^i_{k+1} - b^i_1 = \alpha^i + b^i_{k+1} - a^i_{n-k}$$

for $k = 1, \dots, n-1$. Recall that $\alpha^i = a^i_n - b^i_1$ is the deviation loss at equilibrium $(A, \dots, A)$. Note that z^i_k is nonnegative and nondecreasing in k. The linear program is as follows:

$$(\mathbf{P^i_A}) \qquad \min x_1 + 2x_2 + \cdots + (n-1)x_{n-1}$$

$$s.t. \quad x_1 + \cdots + x_{n-1} \le s, \quad \textstyle\sum_{k=1}^{n-1} z^i_k x_k \ge s\alpha^i, \quad x_k \ge 0.$$

In this program, x_k is the number of profiles that contain exactly k mistakes. $\sum_k x_k$ is the number of profiles that contain at least one mistake. The first constraint comes from the fact that this number cannot exceed the sample size. The second constraint expands to

$$b^i_n x_{n-1} + \cdots + b^i_2 x_1 + (s - \textstyle\sum_{k=1}^{n-1} x_k) b^i_1 \ge a^i_1 x_{n-1} + \cdots + a^i_{n-1} x_1 + (s - \sum_{k=1}^{n-1} x_k) a^i_n.$$

Thus, it ensures that strategy B is a best response against the sample. The objective function gives the total number of mistakes in the sample. It is clear that $(\mathbf{P^i_A})$ has an optimal solution.

The stochastic stability analysis hinges on the number of mistakes. Do we need additional integer constraints? For most of our purpose,[5] we do not need them, as we only need the following implications. By the definition of the first deviator, if the optimal value of $(\mathbf{P^i_A})$ is at least v for every $i \in I$, then v is a lower bound of the resistance $r(A, B)$. If the optimal value of $(\mathbf{P^i_A})$ is strictly greater than v for every $i \in I$, then the resistance $r(A, B)$ is strictly greater than v.[6]

[5] The only exception is part (2) of Proposition 2.

[6] In general, the resistance $r(A, B)$ need not be the optimal value on integer solutions of some $(\mathbf{P^i_A})$.

4.2 Main result

Define

$$m_A = \min \left\{ k \mid a^i_{k+1} \geq b^i_{n-k} \text{ for every } i \in I \right\}.$$

In terms of k^i, which was defined in Section 2.2, $m_A = 1 + \max_{i \in I} k^i$. We are ready to show the main result of the paper.

Theorem. *Consider an n-person binary coordination game. If* (**A1**) *and* (**A2**) *are satisfied, then* $(A, \ldots, A)$ *is stochastically stable:*

(**A1**) $m_A \leq \dfrac{n-1}{2}$,

(**A2**) $m_A z^i_k \leq k\alpha^i$ *for every* $k = 1, \ldots, n-1$ *and every* $i \in I$.

If all inequalities in (**A2**) *are strict, then it is a unique stochastically stable equilibrium.*

Proof. Write $m_A = m$. The result is a consequence of the following facts.

(1) Under (**A1**), the resistance from $(B, \ldots B)$ to $(A, \ldots A)$ is at most sm.

(2) Under (**A2**), the optimal value of $(\mathbf{P^i_A})$ is at least sm for every $i \in I$. If the inequalities are strict, then the optimal value is greater than sm.

To prove (1), it suffices to construct a path from **B** to **A** in which there are exactly sm mistakes. In Phase 2 in Figure 4, let every player sample Phase 1. All the A^* strategies in the figure are mistakes and there are exactly sm of them. Let $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, n\}$. By definition of m, $a^l_{m+1} \geq b^l_{n-m}$ for $l = i, j$. In Phase 3, let j sample Phase 2. Then, since $a^j_{m+1} \geq b^j_{n-m}$, A is a best response for j. In Phase 3, strategies $X \in \{A, B\}$ by i means that they are immaterial to the argument. In Phase 4, let i sample Phase 3. Then the respective strategies yield a^i_{n-m+1} and b^i_m. Now (**A1**) implies that $n - m \geq m$. Since $a^i_{m+1} \geq b^i_{n-m}$, (**G1**) implies that $a^i_{n-m+1} \geq a^i_{m+1} \geq b^i_{n-m} \geq b^i_m$, which allows i to choose A. Letting j sample the final available segment of Phase 2 and the initial segment of Phase 4, we make her choose A in Phase 4 as well. Finally, note that these sample assignments are possible as long as $s \leq T/2$.

To prove (2), pick $i \in I$ and consider the program $(\mathbf{P^i_A})$. Take a nonnegative, nonzero vector $(x_1, \ldots, x_{n-1})$. By (**A2**),

$$m z^i_k x_k \leq k\alpha^i x_k$$

for every $k = 1, \ldots, n-1$. Assume that the objective value of the vector is less than sm. Then

$$m \textstyle\sum_k z^i_k x_k \leq \alpha^i \sum_k k x_k < \alpha^i sm. \qquad (\star)$$

Therefore $\sum_k z^i_k x_k < s\alpha^i$, which means that the vector is infeasible. If the inequalities in (**A2**) are strict, then the left inequality in $(\star)$ becomes strict, which in turn implies that the vector

	Phase 1 $\overbrace{\qquad}^{T}$			Phase 2 $\overbrace{\qquad}^{s}$			Phase 3 $\overbrace{\qquad}^{s}$			Phase 4 $\overbrace{\qquad}^{s}$		
σ_1	B	$\cdots$	B	A^*	$\cdots$	A^*	X	$\cdots$	X	A	$\cdots$	A
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_m	B	$\cdots$	B	A^*	$\cdots$	A^*	X	$\cdots$	X	A	$\cdots$	A
σ_{m+1}	B	$\cdots$	B	B	$\cdots$	B	A	$\cdots$	A	A	$\cdots$	A
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_n	B	$\cdots$	B	B	$\cdots$	B	A	$\cdots$	A	A	$\cdots$	A

Figure 4: Path from **B** to **A**.

$(x_1, \ldots, x_{n-1})$ is infeasible, even if the right inequality in $(\star)$ is a weak inequality. Thus, in this case the optimal value is greater than sm. $\square$

Part (1) of the proof establishes a link between the discussion on the prototype in Section 2 and the formalism developed here for the adaptive play. The path from **B** to **A** constructed in Figure 4 replicates the sequence from $(B, \ldots, B)$ to $(A, \ldots, A)$ in the prototype in which exactly m players make mistakes in the first period.

What (**A1**) requires is clear from the definition of m_A; in Figure 3, the two payoff curves intersect in the left half of the domain. The condition dictates that strategy A has the larger basin of attraction, in the sense that we used the term in the discussion of the prototype.

Condition (**A2**), meanwhile, concerns the "depth" of the basins. The notion of depth in turn concerns the deviation loss, which is defined at each strategy profile to be the difference between the best response payoff and the suboptimal payoff. Rewrite (**A2**) as

$$(m-k)\alpha \leq m(a_{n-k} - b_{k+1}).$$

For $k = m, \ldots, n-m-1$, as Figure 3 shows, the left hand side is nonpositive and the right hand side is nonnegative under (**A1**). This means that the inequality is satisfied by (**A1**) in the middle part of Figure 3. The genuine requirement of (**A2**) thus only concerns the left and the right portions of the figure. For $k = 1, \ldots, m-1$, the inequality requires the deviation loss to be sufficiently large relative to the equilibrium deviation loss, α. For $k = n-m, \ldots, n-1$, the inequality can be rewritten as

$$(k-m)\alpha \geq m(b_{k+1} - a_{n-k}),$$

which states that deviation loss should not be too large relative to α. In particular, it dictates that

$$\beta \leq \frac{(n - m_A - 1)\alpha}{m_A},$$

which means that the deviation loss at $(B, \ldots, B)$ is not overwhelming compared to the deviation loss at $(A, \ldots, A)$.

In intuitive terms, (**A2**) requires that the basin of A is relatively deeper than that of B. We can describe the result as follows. If strategy A has the larger basin of attraction and if the basin is relatively deep enough, then $(A, \ldots, A)$ is stochastically stable.

Example 2. Consider the n-person linear symmetric binary coordination game in which

$$a_k = a(k-1), \quad b_k = b(k-1), \quad \text{for } k = 1, \ldots, n,$$

where $a > 0$ and $b > 0$. Then $m_A = \lceil b\,(n-1)\,/\,(a+b) \rceil$, where $\lceil x \rceil$ is the least integer that is greater than or equal to x. Without loss of generality, we may assume that $a > b$. Then, ignoring the rounding problem, (**A1**) holds true.[7] It follows from $z_k = (a+b)k$ that

$$m_A z_k = m_A(a+b)k \leq \frac{(a+b)(n-1)k}{2} < (n-1)ak = k\alpha,$$

which shows that (**A2**) is also satisfied in strict inequalities. Hence, the unique stochastically stable equilibrium is $(A, \ldots, A)$. This example indicates that the main theorem generalizes the entirely natural observation that $(A, \ldots, A)$ is uniquely stochastically stable in the linear coordination game in which $a > b$.

4.3 Multiplicity of stochastically stable Equilibria

Interchanging a_k^i and b_k^i and replacing z_k^i with $w_k^i = b_n^i - b_{n-k}^i + a_{k+1}^i - a_1^i$, the main theorem generates a pair of conditions that ensures the stochastic stability of $(B, \ldots, B)$:

(**B1**) $m_B \leq \dfrac{n-1}{2}$,

(**B2**) $m_B w_k^i \leq k\beta^i \quad$ for every $k = 1, \ldots, n-1$ and every $i \in I$,

where $m_B = \min\left\{ k \mid b_{k+1}^i \geq a_{n-k}^i \text{ for every } i \in I \right\}$.

If all the conditions for the respective equilibria are satisfied within a single game, then multiple stochastically stable equilibria arise. The four conditions are jointly equivalent to the next two.[8]

Proposition 1. *Consider an n-person binary coordination game. If there is a positive integer m such that*

(**M1**) $m = m_A = m_B \leq \dfrac{n-1}{2}$,

[7] If n is either odd or sufficiently large, (**A1**) is in fact true.

[8] Note that (**A2**) implies that $m_A \leq m_B$. Hence, (**A2**) and (**B2**) imply that $m_A = m_B$.

$$(\mathbf{M2}) \quad \left(\frac{m-k}{m}\right)\alpha^i \leq a^i_{n-k} - b^i_{k+1} \leq \left(\frac{n-m-k-1}{m}\right)\beta^i$$

for every $k = 0, 1, \ldots, n-1$ and every $i \in I$, then both $(A, \ldots, A)$ and $(B, \ldots, B)$ are stochastically stable.

An important implication of (**M1**) is that

$$b^i_{m+1} = \cdots = b^i_{n-m} = a^i_{m+1} = \cdots = a^i_{n-m}.$$

Following the interpretation we gave for (**A1**), we take (**M1**) as stating that the basins of attraction of the respective equilibria are equal in size.

Roughly speaking, therefore, if the size of the basins of attraction is the same, then both equilibria are stochastically stable unless one of the two basins is considerably deeper than the other. In some games, seemingly considerable difference in depth may not be reflected in the selection outcome. The class of unanimity games offers itself as such an example.

A binary *unanimity game* is a binary coordination game in which $m = m_A = m_B = 1$. For the game G_0 in Figure 1, for example, the definition implies that $a^i_2 = a^i_3 = b^i_2 = b^i_3$. Many authors appear to define a unanimity game with additional conditions that $a^i_1 = a^i_2$ and $b^i_1 = b^i_2$. Being more general, our definition qualifies any two-person binary coordination game as a unanimity game. For unanimity games, a complete characterization of the selection outcome is available.

Proposition 2. *Consider an n-person binary unanimity game, $n \geq 2$. If*

(1) *$\beta^i \leq (n-2)\alpha^i$ and $\alpha^i \leq (n-2)\beta^i$ for every $i \in I$, then both equilibria are stochastically stable.*

(2) *there is $i \in I$ such that $\alpha^i > (n-2)\beta^i$ or $\beta^i > (n-2)\alpha^i$, then, assuming that the sample size s is sufficiently large,[9] $(A, \ldots, A)$ is stochastically stable if and only if*

$$\min_{i \in I} \frac{\beta^i}{\alpha^i + \beta^i} \leq \min_{i \in I} \frac{\alpha^i}{\alpha^i + \beta^i}.$$

The first case follows from Proposition 1. The second case is proved in the Appendix. According to (1), multiplicity arises if $\beta^i \in [\alpha^i/(n-2), (n-2)\alpha^i]$. As n increases, the condition becomes increasingly generous. If $n = 2$, Proposition 1 is not applicable since (**M1**) fails. Nonetheless, Proposition 2 remains valid. A well-known result of Young (1993, Theorem 3) states that risk dominant equilibrium in a two-person binary coordination game is stochastically stable. For two-person games, (1) fails and the risk dominance is equivalent to the inequality condition in (2). Viewed thus, Proposition 2 generalizes Young (1993, Theorem 3).

[9] Recall that when s is small, the mistake counting in the adaptive play may not be able to discriminate equilibria at all. In the extreme case of $s = 1$, both equilibria are stable even in two-by-two coordination games.

5 Discussion

We have shown equilibrium selection results from which the multiplicity of stochastically stable equilibria follows in a class of games. Given a unanimity game with multiple stochastically stable equilibria, the main theorem implies that each neighborhood of the game contains a game in which $(A, \ldots, A)$ is uniquely stochastically stable and another in which $(B, \ldots, B)$ is uniquely stochastically stable, as games G_1, G_2 and G_3 in Figure 1 illustrate. As a correspondence, the stochastically stable equilibrium is not lower hemicontinuous.[10]

Now let us return to the prototype in Section 2. Assuming that everyone chose action B yesterday, how many mistakes today shall allow a player to optimally choose A tomorrow? Let us say that the answer is m. With m mistakes today, at least one player will choose A optimally. Do subsequent optimal switches to A accumulate enough to reach the other equilibrium? It may well be the case that they do. It is such a sequence of profiles in the prototype that the path in Figure 4 reproduces in the adaptive play, and the number m of mistakes required today translates into a feasible solution in the linear program such as $(\mathbf{P^i_A})$, which was formulated in Section 4, of the form

$$(x_1, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_{n-1}) = (0, \ldots, 0, s, 0, \ldots, 0).$$

In general, mistakes in the prototype that trigger a transition from an equilibrium to the other can be embedded into the adaptive play by this type of solutions, which we call *corner solutions*. In this way, mistake counting in the prototype carries over to that in the adaptive play. To understand our selection results, it is crucial to note that a corner solution can be optimal in the relevant linear program. Roughly speaking, multiple stochastically stable equilibria arise if corner solutions are optimal in both directions, i.e., from one equilibrium state to the other and vice versa. Notice that if the stage game were a two-person game, all these considerations would be irrelevant as the prototype would not discriminate any equilibrium. Thus the relationship between the prototype and the adaptive play emerges specifically in analysis of many-person games.

An entirely analogous relationship can be established between the prototype and the multi-population random matching model. For simplicity, consider a symmetric n-person binary coordination game. For each player role j, $j = 1, \ldots, n$, there is a population C_j consisting of N agents. The state space of the model is $\{0, 1, \ldots, N\}^n$, which keeps track of the number of B-players in each population. In each period, an agent in C_j is informed of the state in the previous period, and is randomly matched with $n - 1$ agents, each of whom are chosen from a different population, and they play the stage game. The agent intends to play the best response

[10] This feature reminds us of the equilibrium refinement literature; see, for example, Okada (1981).

against the previous state, but may occasionally make a mistake. Now set the previous state be $(0, \ldots, 0)$, which corresponds to $(A, \ldots, A)$ equilibrium in the stage game. Focus on an agent in population C_n and consider what kind of current period mistakes by others allow her to play B in the next period. Specifically, assume that exactly x_j agents in C_j make mistakes, $j \neq n$. Since the previous state is $(0, \ldots, 0)$, the current state for the particular agent in C_n is $\mathbf{x}_{-n} = (x_1, \ldots, x_{n-1})$. In the next period, the probability of the agent facing exactly k others who are playing B in the n-match is given by $p(k, \mathbf{x})/N^{n-1}$, where

$$p(k, \mathbf{x}_{-n}) = \sum_{\substack{j_1,\ldots,j_k \in J \\ \text{all distinct}}} \prod_{h=1}^{k} x_{j_h} \prod_{j \in J \setminus \{j_1,\ldots,j_k\}} (N - x_j).$$

and $J = \{1, \ldots, n-1\}$. Now we can write down the nonlinear program that works for evaluating the resistance from $(A, \ldots, A)$ to $(B, \ldots, B)$ as follows:

$$\begin{aligned} & \min_{\mathbf{x}_{-n}=(x_1,\ldots,x_{n-1})} \sum_{j=1}^{n-1} x_j \\ \text{subject to} \quad & 0 \leq x_j \leq N, \quad j = 1, \ldots, n-1, \\ & \sum_{k=0}^{n-1} p(k, \mathbf{x}_{-n}) b_{k+1} \geq \sum_{k=0}^{n-1} p(k, \mathbf{x}_{-n}) a_{n-k}. \end{aligned}$$

A feasible solution of the program is called an *interior solution* if $0 < x_j < N$ for every $j \neq n$. Otherwise, it is called a *corner solution.* Taking this program as the point of departure, we could derive selection results for the multi-population random matching model by following the line of reasoning in Section 4. For example, the program takes the simplest form if the stage game is a unanimity game. In such a case, one can prove that corner solutions are optimal in both directions, from one equilibrium state to the other and vice versa, if and only if

$$\max\{a_n/b_n, b_n/a_n\} \leq (n-2)^{n-1},$$

which is precisely the condition found in Young (1998) under which both equilibria are stochastically stable.

To relate the program to the single population random matching model, we introduce an extra constraint that

$$x_1 = x_2 = \cdots = x_{n-1} = x,$$

which dictates that populations are indistinguishable from each other. For such states, the map $p(k, \mathbf{x}_{-n})$ simplifies to

$$p(k, \mathbf{x}) = \binom{n-1}{k} x^k (N - x)^{n-k-1},$$

which is precisely the function that appeared in Kim (1996). With the extra condition, the program reduces to

$$\min x$$
$$\text{subject to} \quad 0 \le x \le N \quad \text{and} \quad \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (N-x)^{n-k-1}(b_{k+1} - a_{n-k}) \ge 0.$$

It is obvious that the reduced program always has an interior optimal solution. With just a single variable, the mistake counting argument in single population model is not capable of accommodating that in the prototype. In particular, as Kim (1996) shows, multiplicity does not arise.

These formulations clarify why the prediction of equilibrium selection by mistake counting analysis differs for different dynamics. In multi-dimensional state models, the mistake counting argument becomes richer than that in single population models in that an optimal solution of the associated minimization program may be a corner solution, which reflects the mistake counting argument in the prototype. Consequently, if two equilibria possess basins of attraction of the same size in the prototype, then both may well be stochastically stable in multi-dimensional state models, but they never be in single population random matching model.

Appendix

A.1 Proof of Lemma 2

Recall that T and s are the history size and the sample size of the adaptive play, respectively. Lemma 2 states that the adaptive play without mistakes is absorbing in a binary coordination game whenever $s \le T/2$. This is a consequence of Lemma 3 below. For definitions of symbols and mathematical conditions that appear in the proof of Lemma 3, see section 2.

Lemma 3. *In a binary coordination game, the adaptive play without mistakes is absorbing if* $T = 2$ *and* $s = 1$.

Proof. Consider the adaptive play with $(T, s) = (2, 1)$. Let $\sigma \in \Sigma$ be the sample given to $i \in I$ on a particular day. To avoid ambiguities caused by multiple best responses, let the player choose B if and only if B is a unique best response against σ. Formally,

$$br^i(\sigma) = \begin{cases} A, & \text{if } i \in \bar{I}(k-1), \\ B, & \text{if } i \in \underline{I}(k), \end{cases} \tag{$\star$}$$

where $k = |\sigma^{-i}|_A$, the number of others that adopt A. Clearly, $br^i(\sigma) \in BR^i(\sigma)$.

Pick $\sigma_1 \in \Sigma$ and assume that each player chooses σ_1^i on day 1. Starting from σ_1, we construct a path that leads to either $(A, \ldots, A)$ or $(B, \ldots, B)$. Set $|\sigma_1|_A = k_1$ so that we can depict it, with an appropriate permutation, as follows:

$$\sigma_1 = (\overbrace{A, \ldots, A}^{k_1}, B, \ldots, B).$$

We can assume that $1 \leq k_1 \leq n-1$. On day 2, let everyone sample σ_1. Following $(\star)$, they play $\sigma_2^i = br^i(\sigma_1)$. The outcome of day 2 is σ_2. By construction,

- For every $i \in \bar{I}(k_1 - 2)$, $i \in I_A(\sigma_2)$.
- For every $i \in \underline{I}(k_1)$, $i \in I_B(\sigma_2)$.
- For every $i \in I(k_1 - 1)$, $i \in I_A(\sigma_2)$ if and only if $i \in I_B(\sigma_1)$.

Thus σ_2 can be written as

$$\sigma_2 = (\overbrace{\overbrace{A, \ldots, A}^{|\bar{I}(k_1-2)|}, \overbrace{A, \ldots\ldots, A}^{|I(k_1-1)\cap I_B(\sigma_1)|}}^{k_2}, \overbrace{B, \ldots\ldots, B}^{|I(k_1-1)\cap I_A(\sigma_1)|}, \overbrace{B, \ldots, B}^{|\underline{I}(k_1)|}).$$

Case 1. $k_2 > k_1$. Following $(\star)$, let $\sigma_3^i = br^i(\sigma_2)$ for every $i \in I$. We show that $k_3 = |\sigma_3|_A > k_2$. In σ_2, every $i \in I$ has at least k_1 others playing A. Therefore $i \in I_A(\sigma_3)$ for every $i \in \bar{I}(k_1 - 1)$. Hence $k_3 \geq k_2$. If $k_2 \geq n-1$, then $\sigma_3 = (A, \ldots, A)$. Thus we can assume $2 \leq k_2 \leq n-2$.

> Claim 1: If $I(k_1 - 1) \cap I_A(\sigma_1) = \varnothing$ then $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$.
>
> *Proof of Claim 1.* Assume that $I(k_1 - 1) \cap I_A(\sigma_1) = \varnothing$. It follows that $|\bar{I}(k_1 - 1)| = k_2$. If $\underline{I}(k_1) \cap \bar{I}(k_2 - 2) \neq \varnothing$, then $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$. If $\underline{I}(k_1) \cap \bar{I}(k_2 - 2) = \varnothing$, then $\bar{I}(k_2 - 2) = \bar{I}(k_1 - 1)$. Hence $|\bar{I}(k_2 - 2)| = k_2$. Thus (**G3**) implies that $I(k_2 - 1) \neq \varnothing$. Clearly, $I(k_2 - 1) \subset \underline{I}(k_1)$. Therefore $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$. ‖

It follows from Claim 1 that either $I(k_1 - 1) \cap I_A(\sigma_1) \neq \varnothing$ or $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$. Therefore, $k_3 > k_2$. Under the sample assignment $\sigma_{t+1}^i = br^i(\sigma_t)$ for every $i \in I$, we have shown that $k_2 > k_1$ implies $k_3 > k_2$. By induction, the play eventually reaches $(A, \ldots, A)$ if $k_2 > k_1$.

Case 2. $k_2 < k_1$. This case is entirely analogous to Case 1.

Case 3. $k_2 = k_1$ and $2 \leq k_2 \leq n-2$. In this case, σ_2 can be written as follows.

$$\sigma_2 = (\overbrace{\overbrace{A, \ldots, A}^{|\bar{I}(k_2-2)|}, \overbrace{A, \ldots\ldots, A}^{|I(k_2-1)\cap I_B(\sigma_1)|}}^{k_2}, \overbrace{B, \ldots\ldots, B}^{|I(k_2-1)\cap I_A(\sigma_1)|}, \overbrace{B, \ldots, B}^{|\underline{I}(k_2)|}).$$

If $I(k_2-1)=\varnothing$, then $|\bar{I}(k_2-2)|=k_2$. Thus (**G3**) implies $I(k_2-1)\neq\varnothing$.

Let $\sigma_3^i=br^i(\sigma_2)$. Then

$$\sigma_3=(\overbrace{\overbrace{A,\ldots,A}^{|\bar{I}(k_2-2)|},\ \overbrace{A,\ldots\ldots,A}^{|I(k_2-1)\cap I_A(\sigma_1)|}}^{k_3},\ \overbrace{B,\ldots\ldots,B}^{|I(k_2-1)\cap I_B(\sigma_1)|},\ \overbrace{B,\ldots,B}^{|\underline{I}(k_2)|}).$$

If $k_3\neq k_2$, then we can apply either Case 1 or Case 2. Thus we can assume that $k_3=k_2=k_1$. Then

$$|I(k_2-1)\cap I_A(\sigma_1)|=|I(k_2-1)\cap I_B(\sigma_1)|\geq 1. \tag{$\dagger$}$$

The inequality follows from $I(k_2-1)\neq\varnothing$. For every $i\in I(k_2-1)\cap I_A(\sigma_1)$, let $\sigma_4^i=br^i(\sigma_2)$. For everyone else, let $\sigma_4^i=br^i(\sigma_3)$. Then

$$\sigma_4=(\overbrace{\overbrace{A,\ldots,A}^{|\bar{I}(k_2-2)|},\ \overbrace{A,\ldots\ldots,A}^{|I(k_2-1)\cap I_A(\sigma_1)|},\ \overbrace{A,\ldots\ldots,A}^{|I(k_2-1)\cap I_B(\sigma_1)|}}^{k_4},\ \overbrace{B,\ldots,B}^{|\underline{I}(k_2)|}).$$

By ($\dagger$), $k_4>k_2$. Following ($\star$), let $\sigma_5^i=br^i(\sigma_4)$ for every $i\in I$. We show that $k_5=|\sigma_5|_A>k_4$. In σ_4, every $i\in I$ has at least k_2 others playing A. Therefore $i\in I_A(\sigma_5)$ for every $i\in\bar{I}(k_2-1)$. Hence $k_5\geq k_4$. If $k_4\geq n-1$, then $\sigma_5=(A,\ldots,A)$. Thus we can assume that $3\leq k_4\leq n-2$.

> Claim 2: If $\underline{I}(k_2)\cap\bar{I}(k_4-2)=\varnothing$ then $\underline{I}(k_2)\cap\bar{I}(k_4-1)\neq\varnothing$.
>
> *Proof of Claim 2.* Assume that $\underline{I}(k_2)\cap\bar{I}(k_4-2)=\varnothing$. It follows that $|\bar{I}(k_4-2)|=k_4$. Thus (**G3**) implies that $I(k_4-1)\neq\varnothing$. Since $k_4>k_2$, $I(k_4-1)\subset\underline{I}(k_2)$. Therefore $\underline{I}(k_2)\cap\bar{I}(k_4-1)\neq\varnothing$. $\|$

Noting that $\bar{I}(k_4-2)\subset\bar{I}(k_4-1)$, it follows from Claim 2 that $\underline{I}(k_2)\cap\bar{I}(k_4-1)\neq\varnothing$. Therefore $k_5>k_4$, which allows us to apply Case 1 for the rest of the play.

Case 4. $k_2=k_1=1$ or $k_2=k_1=n-1$. Consider the first case. Then σ_2 can be written as

$$\sigma_2=(A,\overbrace{B,\ldots\ldots,B}^{|I(0)\cap I_A(\sigma_1)|},\overbrace{B,\ldots,B}^{|\underline{I}(1)|}).$$

Letting $\sigma_3^1=br^1(\sigma_2)$ and $\sigma_3^i=br^i(\sigma_1)$ for every $i\neq 1$, we have $\sigma_3=(B,\ldots,B)$. The second case can be dealt with analogously. $\square$

Proof of Lemma 2. We give a sketch of the proof. Details can be found in Maruta and Okada (2007). Given a binary coordination game, consider the adaptive play in which $s\leq T/2$. Fix an initial state, which is an arbitrary sequence in Σ with length T. Since $s\leq T/2$, there is $\sigma_1\in\Sigma$ such that the play reaches the s-length concatenation of σ_1 in a finite number of steps with a positive probability. By Lemma 3, in the adaptive play with $(T, s) = (2, 1)$, there is a path starting from σ_1 that eventually leads to either $(A, \ldots, A)$ or $(B, \ldots, B)$. It suffices to replicate this path in the current setting. By providing players with appropriate samples, the replication is possible thanks to the assumption that $s \leq T/2$. □

A.2 Simple optimal solutions in the relevant program

The next result characterizes the conditions under which $(\mathbf{P^i_A})$ possesses a simple optimal solution.

Proposition 3. *Consider program $(\mathbf{P^i_A})$ for a player $i \in I$ in a binary coordination game. Denote by λ_1 and λ_2 the Lagrange multipliers for the best response constraint and the sample size constraint, respectively. The following conditions are equivalent:*

(1) *There is $k^* \in \arg\min_{\substack{k \\ z^i_k \neq 0}} \frac{k}{z^i_k}$ such that $b^i_{k^*+1} \geq a^i_{n-k^*}$.*

(2) *There is an optimal solution in which $\lambda_2 = 0$.*

(3) *The solution*

$$(x^*_1, \ldots, x^*_{n-1} : \lambda^*_1, \lambda^*_2) = \Big(\overbrace{0, \ldots, 0, \frac{s(a^i_n - b^i_1)}{z^i_{k^*}}}^{k^*}, 0, \ldots, 0 : \frac{k^*}{z^i_{k^*}}, 0\Big)$$

is optimal.

Proof. The result is a simple application of the duality theorem of linear programming. A complete proof is given in Maruta and Okada (2007). □

This result allows us to prove case (2) of Proposition 2. Consider a unanimity game. One can verify that the resistance $r(A, B)$ coincides with the minimum of the values of the optimal integer solutions of $(\mathbf{P^i_A})$. It suffices to compare optimal values of $(\mathbf{P^i_A})$ and that of $(\mathbf{P^i_B})$, the programs to evaluate $r(A, B)$ and $r(B, A)$, respectively.

Fix $i \in I$. In any unanimity game, $z^i_1 = \cdots = z^i_{n-2} = \alpha^i$. Thus $\arg\min(k/z^i_k) \subset \{1, n-1\}$. There are three cases to be distinguished:

Case 1. $\beta^i > (n-2)\alpha^i$. Then $\arg\min(k/z^i_k) = \{n-1\}$. Therefore, Proposition 3 implies that the optimal value of $(\mathbf{P^i_A})$ is $(n-1)s\alpha^i/(\alpha^i + \beta^i) < s$. If the sample size s is large enough so that $(n-1)\left\lceil s\alpha^i/(\alpha^i + \beta^i)\right\rceil < s$, then there is path from $\mathbf{A}$ to $\mathbf{B}$ that contain exactly $(n-1)\left\lceil s\alpha^i/(\alpha^i + \beta^i)\right\rceil$ mistakes, where $\lceil q \rceil$ is the smallest integer not less than q. On the other hand, again by Proposition 3, the optimal value of $(\mathbf{P^i_B})$ is s.

Case 2. $\alpha^i > (n-2)\beta^i$. Similarly, the optimal value of $(\mathbf{P^i_B})$ is $(n-1)s\beta^i/(\alpha^i+\beta^i) < s$ and that of $(\mathbf{P^i_A})$ is s.

Case 3. $\beta^i \leq (n-2)\alpha^i$ and $\alpha^i \leq (n-2)\beta^i$. The optimal values of the two programs are s.

Therefore:

$$r(A,B) = \min_{i \in I}\left\{(n-1)\left\lceil \frac{s\alpha^i}{\alpha^i+\beta^i}\right\rceil, s\right\} \quad \text{and} \quad r(B,A) = \min_{i \in I}\left\{(n-1)\left\lceil \frac{(n-1)s\beta^i}{\alpha^i+\beta^i}\right\rceil, s\right\},$$

from which it follows that, ignoring rounding, $r(B,A) \leq r(A,B)$ if and only if

$$\min_{i \in I} \frac{\beta^i}{\alpha^i+\beta^i} \leq \min_{i \in I} \frac{\alpha^i}{\alpha^i+\beta^i}.$$

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