Heterogeneous Impatience in a Continuous-Time Model

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Abstract

In a continuous-time economy with complete markets, we study how the heterogeneity in the individual consumers' risk tolerance and impatience affects the representative consumer's risk tolerance and impatience. We derive some formulas, which indicate that the representative consumer's impatience decrease over time, and whether his risk tolerance increases or decreases over time depends on the sign of some weighted covariance between the individual consumers' cautiousness (derivative of risk tolerance with respect to own consumptions) and impatience. These results are then used to show that the short rate tends to decrease over time and the market price of risk is volatile in some special cases of heterogeneous economies.

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1 Introduction

In this paper, we consider a dynamic economic model of continuous-time consisting of multiple consumers. Our purpose is to assess the impacts of heterogeneity of consumers' impatience (often measured as discount rates) and risk attitudes (often measured as absolute risk aversion or its reciprocal, absolute risk tolerance) on equilibrium asset pricing. It is well known that if the asset markets are complete, we can define a representative consumer as the value function of the problem of maximizing a weighted sum of individual consumers' utilities and that we can use his marginal utilities, evaluated at the aggregate consumption process, as the state-price deflator to price all assets at any point in time. The task of investigating the impacts of heterogeneous impatience and risk attitudes on asset prices, therefore, boils down to the task of investigating the impacts of heterogeneous impatience and risk attitudes on the representative consumer's utility function. This is what we shall do in this paper.

We aim at establishing general properties of the representative consumer's utility function arising from heterogeneity. By being general, what we mean here is that the results ought to be independent of the number of (types of) consumers in the economy, the functional forms of their utility (or felicity) functions, and the stochastic characteristics of the consumption processes.

We also aim at establishing some formulas describing how the heterogeneity affects the representative consumer's utility function. To see why obtaining such a formula, as opposed to a merely qualitative prediction, is useful, let us compare one of our results (Theorem 3 and its corollary, Corollary 3) with an existing one (Proposition 5 of Gollier and Zeckhauser (2005)). Proposition 5 of Gollier and Zeckhauser (2005) showed that if all consumers have constant but unequal impatience (so that the discount factor is a negative exponential function of time) and exhibit decreasing absolute risk aversion, then the representative consumer's impatience will decrease over time. On the other hand, Theorem 3 and Corollary 3 of this paper give formulas that relate the individual consumers' impatience and cautiousness (the derivative of absolute risk tolerance) to the derivative of the representative consumer's discount rate with respect to time. One can easily see from our formulas that as long as there are many consumers, the representative consumer's impatience would decrease over time even if the assumption of decreasing absolute risk aversion is violated, to some extent, by a small number of consumers. But the qualitative prediction by Gollier and Zeckhauser, as it stands, does not allow us to judge whether the prediction would still be true when the deviations are small. While formulas allow us to see that some conclusions are robust to small deviations from the assumptions, qualitative predictions do not. This is why we aim at obtaining formulas rather than just qualitative predictions.

Throughout the paper, we assume that all consumers' utility functions are time-additive. Although this is a fairly common assumption in finance and macroeconomics, it does exclude some utility functions, such as recursive utilities and utilities of habit formation. By excluding them, we do not mean that they are unimportant or uninteresting. Rather, our intention is to make full use of the existing analytical techniques on the impacts of heterogeneous risk attitudes, as presented in Hara, Huang, and Kuzmics (2007), under the assumption of expected utilities and equal impatience, to the analysis of heterogeneous impatience.

This paper is most closely related to Gollier and Zeckhauser (2005). They investigated how the representative consumer's impatience is affected by the heterogeneity of the individual consumers' impatience. They also showed that if there are infinitely many consumers and their impatience are exponentially distributed, then the representative consumer may exhibit hyperbolic discounting. But their model is a deterministic one and, as discussed above, their results tend to be qualitative. In a discrete-time model under uncertainty, Malamud and Trubowitz (2006) showed that the way in which the representative consumer's risk aversion changes over time is determined by a weighted covariance of the individual consumers' impatience and cautiousness. In Sections 3 and 4, we use a similar approach to obtain more general results with simpler proofs. The methodology of this paper closely follows that of Hara, Huang, and Kuzmics (2007). Hara (2006) applied the techniques of Hara, Huang, and Kuzmics (2007) to a continuous-time setup under the assumption that all consumers have the discount rate. This paper can thus be considered as an extension of that paper to the case of unequal impatience. Hara (2008) elaborated on the result in Gollier and Zeckhauser (2005) on hyperbolic discounting by showing that the representative consumer's discount factor is a power function of some completely monotone function if and only if it can be derived from some economy populated by consumers with constant and equal relative risk aversion.

An important message of the results in this paper is that if the representative consumer is truly representative, in the sense that his utility function is derived from a group of heterogeneous consumers, then his utility function is quite likely to exhibit decreasing discount rates and unlikely to be multiplicatively separable between time and aggregate consumption levels. This fact cast serious doubts on the plausibility of the prevalent use of constant discount rates and multiplicatively separable utility functions for the representative consumer in the representative-consumer models of dynamic macroeconomics.

In Section 5, we uses these results to see how the short-rate process and the market-price-of-risk process in a heterogeneous economy are different from those in a homogeneous economy, by assuming that the aggregate consumption process is a geometric Brownian motion and each individual consumer's subjective discount rate and coefficient of relative risk aversion are constant. If the subjective discount rates and the coefficients of relative risk aversion are equal across all consumers, then the short-rate process and the market-price-of-risk process are deterministic and constant. We show that otherwise, they can be stochastic. Moreover, while the short-rate process tends to be decreasing over time, whether the market-price-of-risk process tends to be increasing or decreasing over time depends on the correlation between the individual consumers' discount rates and coefficients of relative risk aversion.

This paper is organized as follows: In Section 2, we lay out the setup of this paper and explain basic concepts for our analysis. In Section 3, we identify the implications of heterogeneous impatience and risk attitudes on the representative consumer's risk attitudes. In Section 4, we identify the implications on his impatience. In Section 5, we investigate the short-rate process and the market-price-of-risk process. In Section 6, we summarize our results and suggest two directions of future research.

2 Setup and Existing Results

The setup of this paper is as follows. The economy is subject to uncertainty, which is represented by a probability measure space $(\Omega, \mathscr{F}, P)$. The time span is $[0, T]$ with $0 < T < \infty$, which is of finite length, although the analysis in Sections 3 and 4 would be valid even when the time span were $[0, \infty)$. The gradual information revelation is represented by a filtration $(\mathscr{F}_t)_{t \in [0,T]}$. There is only one type of good on each time and state.

The economy consists of I consumers. Each consumer i has a possibly time-dependent felicity function $u_i : \boldsymbol{R}_{++} \times [0, T] \to \boldsymbol{R}$, which is at least twice continuously differentiable,[1] and satisfies $\partial u_i(x^i, t)/\partial x^i > 0 > \partial^2 u_i(x^i, t)/\partial \left(x^i\right)^2$ for every $(x^i, t) \in \boldsymbol{R}_{++} \times [0, T]$ and the Inada condition, that is, for every $t \in [0, T]$, $\partial u_i(x^i, t)/\partial x^i \to 0$ as $x^i \to \infty$, and $\partial u_i(x^i, t)/\partial x^i \to \infty$ as $x^i \to 0$. His utility function U_i over stochastic consumption processes are then defined by taking time additivity and state-independent expected utility:

$$U_i(c^i) = E\left(\int_0^T u_i(c^i_t, t)\,\mathrm{d}t\right),$$

where $c^i = \left(c^i_t\right)_{t \in [0,T]}$ is an adapted process taking values in $\boldsymbol{R}_{++}$.[2]

The key parameters of the felicity function u_i (and thus of the utility function U_i) are *risk tolerance* and *impatience*. The risk tolerance $s_i : \boldsymbol{R}_{++} \times [0, T] \to \boldsymbol{R}_{++}$ is defined by

$$s_i(x^i, t) = -\frac{\partial u_i(x^i, t)/\partial x^i}{\partial^2 u_i(x^i, t)/\partial (x^i)^2}.$$

This is nothing but the reciprocal of the Arrow-Pratt measure of absolute risk aversion. In this dynamic setup, this also measures tolerance to intertemporal consumption fluctuations. The partial derivative with respect to x_i, $\partial s_i(x_i, t)/\partial x_i$, is called the *cautiousness*. The impatience, or *discount rate*, $r_i : \boldsymbol{R}_{++} \times [0, T] \to \boldsymbol{R}$ is defined by

$$r_i(x^i, t) = -\frac{\partial^2 u_i(x^i, t)/\partial x^i \partial t}{\partial u_i(x^i, t)/\partial x^i}.$$

An important class of utility functions is one of multiplicatively separable utility functions. A utility function u_i is *multiplicatively separable* if there are two functions $v_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$ and $d_i : \boldsymbol{R}_{+} \to \boldsymbol{R}_{++}$ such that $u_i(x^i, t) = d_i(t) v_i(x^i)$ for every $(x^i, t) \in \boldsymbol{R}_{++} \times [0, T]$. Then $s_i(x^i, t) = -v_i'(x^i)/v_i''(x^i)$ and $r_i(x^i, t) = -d_i'(t)/d_i(t)$. We thus write $s_i(x_i)$ for $s_i(x^i, t)$ and $r_i(t)$

[1] The degree of continuous differentiability needed in each of the subsequent results will be made clear in its proof.

[2] To be exact, we need to impose some additional restrictions on c^i to make the integral well defined (finite). As such restrictions are irrelevant to the subsequent analysis, we shall not explicitly state them.

for $r_i(x^i, t)$ in this case. Then

$$\frac{d_i(t)}{d_i(0)} = \exp\left(-\int_0^t r_i(\tau)\,\mathrm{d}\tau\right)$$

for every t. If, in addition, there exists a $\beta^i > 0$ such that $d_i(t) = \exp(-\beta^i t)$, then $r_i(t) = \beta^i$ for every $t \in [0, T]$. This is the case of exponential discounting.

To find a Pareto efficient allocation of a given aggregate consumption process $c = (c_t)_{t\in[0,T]}$ and its supporting (decentralizing) state-price deflator, it is sufficient to choose positive numbers $\lambda_1, \dots, \lambda_I$ and consider the following maximization problem:

$$\begin{aligned} \max_{(c^1,\dots,c^I)} \quad & \sum_i \lambda_i U_i(c^i) \\ \text{subject to} \quad & \sum_i c^i = c. \end{aligned} \tag{1}$$

Since the utility functions U_i are additive with respect to both time and states and the probabilistic belief P is common across consumers, it can be rewritten as

$$\sum_i \lambda_i U_i(c^i) = E\left(\int_0^T \sum_i \lambda_i u_i(c_t^i, t)\,dt\right).$$

Hence, to solve the maximization problem (1), it suffices to solve

$$\begin{aligned} \max_{(x_1,\dots,x_I)\in \boldsymbol{R}^I_{++}} \quad & \sum_i \lambda_i u_i(x^i, t) \\ \text{subject to} \quad & \sum_i x^i = x. \end{aligned} \tag{2}$$

for each pair of a realized aggregate consumption level $x \in \boldsymbol{R}_{++}$ and time $t \in [0, T]$. It can be easily proved that under the stated conditions, there is a unique solution, which we denote by $(f_1(x, t), \dots, f_I(x, t))$. It can also be shown that for each f_i is continuously differentiable in both variables. We can define the value function of this problem $u : \boldsymbol{R}_{++} \times [0, T] \to \boldsymbol{R}$ by

$$u(x, t) = \sum_i \lambda_i u_i\left(f_i(x, t), t\right).$$

This is the felicity function of the *representative consumer*. It need not be multiplicatively separable between time t and the consumption level x when all individual consumers discounts future utilities exponentially but at differing rates. The representative consumer's utility function is

$$U(c) = E\left(\int_0^T u(c_t, t)\,\mathrm{d}t\right).$$

Just as for an individual consumer's utility function, we define risk tolerance and impatience as follows:

$$s(x, t) = -\frac{\partial u(x, t)/\partial x}{\partial^2 u(x, t)/\partial x^2} \text{ and } r(x, t) = -\frac{\partial^2 u(x, t)/\partial x \partial t}{\partial u(x, t)/\partial x}.$$

The cautiousness is defined as the partial derivative $\partial s(x,t)/\partial x$ with respect to the aggregate consumption level x.

The representative consumer is, of course, not an "actual" consumer, who would trade on financial markets. Rather, he is a theoretical construct, who can be used to identify asset prices. Specifically, if u is the representative consumer's felicity function and $c = (c_t)_{t\in[0,T]}$ is the aggregate consumption process, then his marginal utility process evaluated at the aggregate consumption, $(\partial u(c_t,t)/\partial x)_{t\in[0,T]}$, is the state price process. This means that the price at time t of an asset with dividend rate process $\delta = (\delta_t)_{t\in[0,T]}$[3] is equal to the discounted sum of its future dividends:

$$E_t\left(\int_t^T \frac{\partial u(c_\tau,\tau)/\partial x}{\partial u(c_t,t)/\partial x}\delta_\tau \,\mathrm{d}\tau\right).$$

Moreover, suppose that the aggregate consumption process is defined by a stochastic differential equation

$$\mathrm{d}c_t = a(c_t,t)\,\mathrm{d}t + b(c_t,t)\,\mathrm{d}B_t,$$

where $a : \boldsymbol{R}_{++}\times[0,T]\to\boldsymbol{R}$ and $b : \boldsymbol{R}_{++}\times[0,T]\to\boldsymbol{R}$. Then, as we will see in Section 5, the short-rate process, which keeps track of interest rates for risk-free lending and borrowing for infinitesimally short intervals of time, and the market-price-of-risk process, which keeps track of the expected rates of return, in excess of the short rates, on risky assets to be earned by accepting a unit increase in the standard deviation of the rate of return, can be represented in terms of the representative consumer's risk tolerance s and impatience r, and the drift term a and the diffusion term b of the aggregate consumption process.

Although we analyze the Pareto efficient allocations, if the asset markets are complete, then our analysis is applicable to the equilibrium allocations and asset prices. This is because the first welfare theorem holds in complete markets, so that the equilibrium allocations are Pareto efficient and the equilibrium asset prices are given by the marginal utility process. Since the $u_i(\cdot,t)$ are concave, the second welfare theorem also holds, so that every Pareto efficient allocation is an equilibrium allocation for some distribution of initial endowments. Hence an analysis of Pareto efficient allocations is also an analysis of equilibrium allocations.

The solution to the maximization problem (1) is a Pareto efficient allocation. When it is an equilibrium allocation, the individual consumers' wealth shares, evaluated by the equilibrium prices, are positively related to the utility weights λ_i in (1). All the properties we shall explore in the subsequent analysis are valid regardless of the choice of utility weights. Hence, these properties are also valid for the equilibrium allocations regardless of wealth distributions.

To develop our analysis, we now list up some of the results that have been in the existing

[3]This means that the cumulative dividend process $D = (D_t)_{t\in[0,T]}$ of this asset is given by $D_0 = 0$ and $\mathrm{d}D_t = \delta_t\,\mathrm{d}t$, that is, $D_t = \int_0^t \delta_\tau\,\mathrm{d}\tau$.

literature and hold for every $(x,t) \in \boldsymbol{R}_{++} \times [0,T]$. By Theorems 4 and 5 of Wilson (1968),

$$s(x,t) = \sum_i s_i(f_i(x,t),t), \tag{3}$$

$$\frac{\partial f_i}{\partial x}(x,t) = \frac{s_i(f_i(x,t),t)}{s(x,t)}. \tag{4}$$

By differentiating both sides of (3) with respect to x and applying (4), we obtain

$$\frac{\partial s}{\partial x}(x,t) = \sum_i \frac{\partial f_i}{\partial x}(x,t)\frac{\partial s_i}{\partial x^i}\left(f_i(x,t),t\right) = \sum_i \frac{s_i\left(f_i(x,t),t\right)}{s(x,t)}\frac{\partial s_i}{\partial x^i}\left(f_i(x,t),t\right) \tag{5}$$

This shows that the representative consumer's cautiousness is the weighted average of the individual consumers' counterparts, where the weights are proportional to their absolute risk tolerance.

By equality (10) and Proposition 3 of Gollier and Zeckhauser (2005),

$$r(x,t) = \sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)} r_i(f_i(x,t),t) \tag{6}$$

$$\frac{\partial f_i}{\partial t}(x,t) = s_i(f_i(x,t),t)\left(r(x,t) - r_i(f_i(x,t),t)\right). \tag{7}$$

(6) means that the representative consumer's impatience is the weighted average of the individual consumers' counterparts where the weights are proportional to their absolute risk tolerance. (7) means that if an individual consumer is more patient than the representative consumer, the former's consumption level would grow over time were the aggregate consumption level to be constant, and the growth rate is proportional to his absolute risk tolerance.

By applying Theorem 4 of Hara, Huang, and Kuzmics (2006) to the $u_i(\cdot,t)$ for a fixed t and using (4), we can obtain

$$\frac{\partial^2 s}{\partial x^2}(x,t) = \sum_i \left(\frac{s_i\left(f_i(x,t),t\right)}{s(x,t)}\right)^2 \frac{\partial^2 s_i}{\partial \left(x^i\right)^2}\left(f_i(x,t),t\right) \tag{8}$$

$$+ \frac{1}{s(x,t)} \sum_i \frac{s_i(f_i(x,t))}{s(x,t)} \left(\frac{\partial s_i}{\partial x^i}(f_i(x,t),t) - \frac{\partial s}{\partial x}(x,t)\right)^2. \tag{9}$$

To understand this formula, note first that by (4), the first term on the right-hand side can be written as

$$\sum_i \frac{s_i\left(f_i(x,t),t\right)}{s(x,t)} \left(\frac{\partial^2 s_i}{\partial \left(x^i\right)^2}\left(f_i(x,t),t\right)\frac{\partial f_i}{\partial x}(x,t)\right)$$

Here, the term $\left(\partial^2 s_i\left(f_i(x,t),t\right)/\partial\left(x^i\right)^2\right)\left(\partial f_i(x,t)/\partial x\right)$ is the change in the cautiousness of consumer i arising from the increase in his consumption level, which is, in turn, caused by an increase in the aggregate consumption level. Thus the first term is the weighted average of these individual effects. It represents the direct effect on the representative consumer's cautiousness by an increase in aggregate consumption. By (3), the second term on the right-hand side is the weighted variance of the individual consumers' cautiousness, divided by the representative consumer's absolute risk aversion. It represents the indirect effect on the representative consumer's cautiousness by an increase in aggregate consumptions arising from the heterogeneity in the individual consumers' cautiousness. This formula, therefore, shows that the heterogeneity in the individual consumers' cautiousness increases the representative consumer's cautiousness, thereby making his risk tolerance, as a function of aggregate consumption levels, more convex. The formulas we seek to obtain in this paper are of this nature, which decompose the effect of a change in aggregate consumptions or a passage of time on the change in the representative consumer's risk tolerance or impatience into the direct and indirect effects, the latter of which arises from the heterogeneity in the individual consumers' counterparts.

3 Representative Consumer's Risk Tolerance

The first result of this paper is concerned with how the representative consumer's risk tolerance varies over time. In the special case of exponential discounting, an essentially identical formula was already given in the proof of Theorem 3.3 of Malamud and Trubowitz (2006).

Theorem 1 *For every* $(x,t) \in \boldsymbol{R}_{++} \times [0,T]$,

$$\begin{aligned}\frac{\partial s}{\partial t}(x,t) = &\sum_i \frac{\partial s_i}{\partial t}\left(f_i(x,t),t\right) \\ &- s(x,t)\sum_i \frac{s_i(f_i(x,t))}{s(x,t)}\left(\frac{\partial s_i}{\partial x^i}(f_i(x,t),t) - \frac{\partial s}{\partial x}(x,t)\right)\left(r_i(f_i(x,t)) - r(x,t)\right). \quad (10)\end{aligned}$$

This theorem tells us that the rate, per unit of time, of changes in the representative consumer's risk tolerance can be decomposed into two terms. The first term on the right-hand side is easy to grasp. As shown by (3), the representative consumer's risk tolerance is the sum of the individual consumers' counterparts. Thus the first term represents the direct effect on risk tolerance by time. It is equal to zero when all individual consumers' felicity functions u_i are multiplicatively separable.

By (5) and (6), the second term of (10) is equal to the weighted covariance, multiplied by the representative consumer's risk tolerance, between the individual consumers' cautiousness and impatience, where the weights are proportional to the individual consumers' risk tolerance. Since the second term would be zero if all consumers' cautiousness or impatience are equal to one another, it captures the tendency of changes in the representative consumer's impatience that arise from the heterogeneity in the individual consumers' cautiousness and impatience.

Combining the first and second terms of the right-hand side of (10), Theorem 1 states that the rate of changes, per unit of time, in the representative consumer's risk tolerance is the sum of the individual counterparts subtracted by the weighted covariance, multiplied by his own risk tolerance, between the individual consumers' cautiousness and impatience.

Proof of Theorem 1 Differentiate both sides of (3) with respect to t, then we obtain

$$\begin{aligned}\frac{\partial s}{\partial t}(x,t) &= \sum_i \left(\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)\frac{\partial f_i}{\partial t}(x,t) + \frac{\partial s_i}{\partial t}(f_i(x,t),t)\right) \\ &= \sum_i \frac{\partial s_i}{\partial t}(f_i(x,t),t) + \sum_i \frac{\partial s_i}{\partial x^i}(f_i(x,t),t) s_i(f_i(x,t),t)\left(r(x,t) - r_i(f_i(x,t),t)\right), \end{aligned} \tag{11}$$

where the last equality follows from (7). Since

$$\sum_i s_i(f_i(x,t),t)\left(r(x,t) - r_i(f_i(x,t),t)\right) = 0,$$

the second term of (11) can be written as

$$\sum_i \left(\frac{\partial s_i}{\partial x^i}(f_i(x,t),t) - \frac{\partial s}{\partial x}(x,t)\right) s_i(f_i(x,t),t)\left(r(x,t) - r_i(f_i(x,t),t)\right).$$

This is equal to the second term of the right-hand side of (10). ///

Theorem 1 has a couple of implications. The first one is a generalization of Theorem 3.3 of Malamud and Trubowitz (2006) to the case of multiplicatively separable utility functions.

Corollary 1 *Suppose that u_i is multiplicatively separable for every i. Then*

$$\frac{\partial s(x,t)/\partial t}{s(x,t)} = -\sum_i \frac{s_i(f_i(x))}{s(x,t)}\left(s_i'(f_i(x,t)) - \frac{\partial s}{\partial x}(x,t)\right)(r_i(t) - r(x,t))$$

for every $(x,t) \in \boldsymbol{R}_{++} \times [0,T]$. Moreover,

1. *If $(s_1'(f_1(x,t)),\dots,s_I'(f_I(x,t)))$ and $(r_1(t),\dots,r_I(t))$ are comonotone (that is, $\left(s_i'(f_i(x,t)) - s_j'(f_j(x,t))\right)(r_i(t) - r_j(t)) \geq 0$ for every pair of two consumers i and j), then $\partial s(x,t)/\partial t \leq 0$. This weak inequality holds as an equality if and only if $s_1'(f_1(x,t)) = \cdots = s_I'(f_I(x,t))$ or $r_1(t) = \cdots = r_I(t)$.*

2. *If $(s_1'(f_1(x,t)),\dots,s_I'(f_I(x,t)))$ and $(r_1(t),\dots,r_I(t))$ are anti-comonotone (that is, $\left(s_i'(f_i(x,t)) - s_j'(f_j(x,t))\right)(r_i(t) - r_j(t)) \leq 0$ for every pair of two consumers i and j), then $\partial s(x,t)/\partial t \geq 0$. This weak inequality holds as an equality if and only if $s_1'(f_1(x,t)) = \cdots = s_I'(f_I(x,t))$ or $r_1(t) = \cdots = r_I(t)$.*

An important case of this corollary is Corollary 3.4 of Malamud and Trubowitz, which deals with constant relative risk aversion and exponential discounting. In this case, $s_i'(x^i)$ is equal to the reciprocal of constant relative risk aversion and $r(t)$ is equal to the constant impatience, so that the validity of the assumption of (anti-)comonotonicity can be checked without reference to the choice of the consumption levels x^i or time t.

4 Representative Consumer's Impatience

In this section, we turn our attention to the representative consumer's impatience. The first result of this section is concerned with how the representative consumer's impatience is affected by aggregate consumption levels.

Theorem 2 *For every* $(x,t) \in \boldsymbol{R}_{++} \times [0,T]$,

$$\frac{\partial r}{\partial x}(x,t) = \sum_i \left(\frac{s_i\left(f_i(x,t),t\right)}{s(x,t)} \right)^2 \frac{\partial r_i}{\partial x^i}\left(f_i(x,t),t\right) + \frac{1}{s(x,t)} \sum_i \frac{s_i(f_i(x,t))}{s(x,t)} \left(\frac{\partial s_i}{\partial x^i}(f_i(x,t),t) - \frac{\partial s}{\partial x}(x,t) \right) \left(r_i(f_i(x,t)) - r(x,t)\right). \quad (12)$$

Just as Theorem 1, this theorem tells us that the change in the representative consumer's impatience can be decomposed into two terms. The first term on the right-hand side is easy to grasp. By (4), the first term can be rewritten as

$$\sum_i \frac{s_i\left(f_i(x,t),t\right)}{s(x,t)} \left(\frac{\partial r_i}{\partial x^i}\left(f_i(x,t),t\right) \frac{\partial f_i}{\partial x}(x,t) \right)$$

Here, the term $\left(\partial r_i\left(f_i(x,t),t\right)/\partial x^i\right)\left(\partial f_i(x,t)/\partial x\right)$ is the change in the impatience of consumer i arising from the increase in his consumption level, which, in turn, caused by an increase in the aggregate consumption level. Thus the first term is the weighted average of these individual effects. It represents the direct effect on the representative consumer's impatience by the change in aggregate consumption. It is equal to zero when all individual consumers' felicity functions u_i are multiplicatively separable.

The second term is equal to the weighted covariance, divided by the representative consumer's risk tolerance, between the individual consumers' cautiousness and impatience, where the weights are proportional to the individual consumers' risk tolerance. Since the second term would be zero if all consumers' cautiousness or impatience are equal to one another, it captures the tendency of changes in the representative consumer's impatience that arise from the heterogeneity in the individual consumers' cautiousness and impatience. Theorem 2 states that the change in the representative consumer's impatience is the sum of the individual counterparts, added by the weighted covariance, divided by his own risk tolerance, between the individual consumers' cautiousness and impatience.

Proof of Theorem 2 By (6),

$$s(x,t)r(x,t) = \sum_i s_i(f_i(x,t),t)r_i(f_i(x,t),t). \quad (13)$$

Differentiate both sides of (13) with respect to x, then we obtain

$$\begin{aligned}
&\frac{\partial s}{\partial x}(x,t)r(x,t)+s(x,t)\frac{\partial r}{\partial x}(x,t)\\
=&\sum_i\left(\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)\frac{\partial f_i}{\partial x}(x,t)r_i\left(f_i(x,t),t\right)+s_i\left(f_i(x,t),t\right)\frac{\partial r_i}{\partial x^i}(f_i(x,t),t)\frac{\partial f_i}{\partial x}(x,t)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
r(x,t)=&\frac{1}{s(x,t)}\sum_i s_i\left(f_i(x,t),t\right)\frac{\partial r_i}{\partial x^i}(f_i(x,t),t)\frac{\partial f_i}{\partial x}(x,t)\\
&+\frac{1}{s(x,t)}\left(\sum_i\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)\frac{\partial f_i}{\partial x}(x,t)r_i\left(f_i(x,t),t\right)-\frac{\partial s}{\partial x}(x,t)r(x,t)\right)\\
=&\sum_i\left(\frac{s_i\left(f_i(x,t),t\right)}{s(x,t)}\right)^2\frac{\partial r_i}{\partial x^i}(f_i(x,t),t)\\
&+\frac{1}{s(x,t)}\left(\sum_i\frac{s_i\left(f_i(x,t),t\right)}{s(x,t)}\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)r_i\left(f_i(x,t),t\right)-\frac{\partial s}{\partial x}(x,t)r(x,t)\right),
\end{aligned}$$

where the last equality follows from (4). By (5) and (6),

$$\begin{aligned}
&\sum_i\frac{s_i\left(f_i(x,t),t\right)}{s(x,t)}\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)r_i\left(f_i(x,t),t\right)-\frac{\partial s}{\partial x}(x,t)r(x,t)\\
=&\sum_i\frac{s_i(f_i(x,t))}{s(x,t)}\left(\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)-\frac{\partial s}{\partial x}(x,t)\right)\left(r_i(f_i(x,t))-r(x,t)\right).
\end{aligned}$$

The proof is thus completed. ///

In the case of multiplicatively separable utility functions, we have the following corollary.

Corollary 2 *Suppose that u_i is multiplicatively separable for every i. Then*

$$\frac{\partial r}{\partial x}(x,t)=\frac{1}{s(x,t)}\sum_i\frac{s_i(f_i(x))}{s(x,t)}\left(s_i'(f_i(x,t))-\frac{\partial s}{\partial x}(x,t)\right)\left(r_i(t)-r(x,t)\right)$$

for every $(x,t)\in\boldsymbol{R}_{++}\times[0,T]$. Moreover,

1. *If $(s_1'(f_1(x,t)),\dots,s_I'(f_I(x,t)))$ and $(r_1(t),\dots,r_I(t))$ are comonotone (that is, $\left(s_i'(f_i(x,t))-s_j'(f_j(x,t))\right)(r_i(t)-r_j(t))\geq 0$ for every pair of two consumers i and j), then $\partial r(x,t)/\partial x\geq 0$. This weak inequality holds as an equality if and only if $s_1'(f_1(x,t))=\cdots=s_I'(f_I(x,t))$ or $r_1(t)=\cdots=r_I(t)$.*

2. *If $(s_1'(f_1(x,t)),\dots,s_I'(f_I(x,t)))$ and $(r_1(t),\dots,r_I(t))$ are anti-comonotone (that is, $\left(s_i'(f_i(x,t))-s_j'(f_j(x,t))\right)(r_i(t)-r_j(t))\leq 0$ for every pair of two consumers i and j), then $\partial r(x,t)/\partial x\leq 0$. This weak inequality holds as an equality if and only if $s_1'(f_1(x,t))=\cdots=s_I'(f_I(x,t))$ or $r_1(t)=\cdots=r_I(t)$.*

We next give a formula for $\partial r(x,t)/\partial t$, which shows how the representative consumer's impatience varies over time.

Theorem 3 *For every* $(x,t) \in \boldsymbol{R}_{++} \times [0,T]$,

$$
\begin{aligned}
\frac{\partial r}{\partial t}(x,t) = &\sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)} \frac{\partial r_i}{\partial t}(f_i(x,t),t) \\
&+ \sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)} (r_i(f_i(x,t),t) - r(x,t))^2 \frac{\partial s_i}{\partial x^i}(f_i(x,t),t) \\
&+ \sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)} (r_i(f_i(x,t),t) - r(x,t)) \\
&\times \left(\frac{\frac{\partial s_i}{\partial t}(f_i(x,t),t)}{s_i(f_i(x,t),t)} - \frac{\frac{\partial s}{\partial t}(x,t)}{s(x,t)} + \frac{\partial r_i}{\partial x^i}(f_i(x,t),t) s_i(f_i(x,t),t) \right). \qquad (14)
\end{aligned}
$$

This theorem tells us that the rate, per unit of time, of changes in the representative consumer's impatience can be decomposed into three terms. The first term is easy to grasp. As shown by (6), the representative consumer's impatience is equal to the weighted average of the individual consumers' counterparts, where the weights are proportional to their risk tolerance. Thus the first term represents the direct effect, by time, on the representative consumer's impatience, while the weights are hypothetically fixed. It is equal to zero if all the consumers' felicity functions u_i are of exponential discounting.

The third term represents the change in the representative consumer's impatience caused by the impact on the individual consumers' risk tolerance by time, and also by the impact on their impatience by consumption levels. It is equal to zero if all consumers' felicity functions are multiplicatively separable.

The second term is most interesting. It represents the impact on the representative consumer's impatience when the individual consumers have differing impatience. As mentioned above, the representative consumer's impatience is equal to the weighted average of the individual consumers' counterparts, and the weights are proportional to their risk tolerance. If their impatience are different, then the risk-sharing rules f_i would depend on time t; that is, the partial derivative $\partial f_i(x,t)/\partial t$ would be different from zero. Unless the cautiousness, $\partial s_i(f_i(x,t),t)/\partial x_i$, is zero (which would be the case if u_i exhibited constant absolute, rather than relative, risk aversion), the change in consumption levels has an impact on the individual consumers' risk tolerance, and thus on the representative consumer's impatience, which is the weighted average of the individual consumers' impatience, with the weights given by their risk tolerance. The second term, therefore, captures the change in the representative consumer's impatience arising from the heterogeneity in the individual consumers' impatience.

Proof of Theorem 3 By (6) and differentiation for a product,

$$
\frac{\partial r}{\partial t}(x,t) = \sum_i \frac{d}{dt}\left(\frac{s_i(f_i(x,t),t)}{s(x,t)}\right) r_i(f_i(x,t),t) + \sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)} \frac{d}{dt}\left(r_i(f_i(x,t),t)\right). \qquad (15)
$$

By (7),

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{s_i(f_i(x,t),t)}{s(x,t)}\right)\\
=&\frac{\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)}{s(x,t)}\frac{\partial f_i}{\partial t}(x,t)+\frac{\frac{\partial s_i}{\partial t}(f_i(x,t),t)}{s(x,t)}-\frac{s_i(f_i(x,t),t)}{(s(x,t))^2}\frac{\partial s}{\partial t}(x,t)\\
=&\frac{s_i(f_i(x,t),t)}{s(x,t)}\left(\frac{\frac{\partial s_i}{\partial t}(f_i(x,t),t)}{s_i(f_i(x,t),t)}-\frac{\frac{\partial s}{\partial t}(x,t)}{s(x,t)}+\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)\left(r_i(f_i(x,t),t)-r(x,t)\right)\right).
\end{aligned}
$$

By (3),

$$
\sum_i \frac{d}{dt}\left(\frac{s_i(f_i(x,t),t)}{s(x,t)}\right)=0.
$$

Thus,

$$
\begin{aligned}
&\sum_i \frac{d}{dt}\left(\frac{s_i(f_i(x,t),t)}{s(x,t)}\right)r_i\left(f_i(x,t),t\right)\\
=&\sum_i \frac{d}{dt}\left(\frac{s_i(f_i(x,t),t)}{s(x,t)}\right)\left(r_i\left(f_i(x,t),t\right)-r(x,t)\right)\\
=&\sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)}\frac{\partial s_i}{\partial x^i}(f_i(x,t),t)\left(r_i\left(f_i(x,t),t\right)-r(x,t)\right)^2\\
&+\sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)}\left(\frac{\frac{\partial s_i}{\partial t}(f_i(x,t),t)}{s_i(f_i(x,t),t)}-\frac{\frac{\partial s}{\partial t}(x,t)}{s(x,t)}\right)\left(r_i\left(f_i(x,t),t\right)-r(x,t)\right)
\end{aligned}
\tag{16}
$$

Again by (7),

$$
\begin{aligned}
&\frac{d}{dt}\left(r_i\left(f_i(x,t),t\right)\right)\\
=&\frac{\partial r_i}{\partial x^i}\left(f_i(x,t),t\right)\frac{\partial f_i}{\partial t}(x,t)+\frac{\partial r_i}{\partial t}\left(f_i(x,t),t\right)\\
=&\frac{\partial r_i}{\partial x^i}\left(f_i(x,t),t\right)s_i\left(f_i(x,t),t\right)\left(r_i\left(f_i(x,t),t\right)-r(x,t)\right)+\frac{\partial r_i}{\partial t}\left(f_i(x,t),t\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)}\frac{d}{dt}\left(r_i\left(f_i(x,t),t\right)\right)\\
=&\sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)}\frac{\partial r_i}{\partial t}\left(f_i(x,t),t\right)\\
&+\sum_i \frac{s_i(f_i(x,t),t)}{s(x,t)}\frac{\partial r_i}{\partial x^i}\left(f_i(x,t),t\right)s_i\left(f_i(x,t),t\right)\left(r_i\left(f_i(x,t),t\right)-r(x,t)\right).
\end{aligned}
\tag{17}
$$

Thus, by (15), (16), and (17), we obtain (14). ///

The right-hand side of (14) in Theorem 3 can be much simplified if we concentrate on the case of multiplicatively separable felicity functions. The following corollary follows from Theorem 3 and (6)

Corollary 3 *Suppose that u_i is multiplicatively separable for every i. Then*

$$\frac{\partial r}{\partial t}(x,t) = \sum_i \frac{s_i(f_i(x,t))}{s(x,t)} r_i'(t) - \sum_i \frac{s_i(f_i(x,t))}{s(x,t)} \left(r_i(t) - r(x,t)\right)^2 s_i'(f_i(x,t))$$

for every $(x,t) \in \boldsymbol{R}_{++} \times [0,T]$. Moreover,

1. *If $r'(t) \leq 0$ and $s_i'(f_i(x,t)) \geq 0$ for every i, then $\partial r(x,t)/\partial t \leq 0$. This weak inequality holds as an equality if and only if $r_1'(t) = \cdots = r_I'(t) = 0$ and, in addition, either $r_1(t) = \cdots = r_I(t)$ or $s_1'(f_1(x,t)) = \cdots = s_I'(f_I(x,t)) = 0$.*

2. *If $r'(t) \geq 0$ and $s_i'(f_i(x,t)) \leq 0$ for every i, then $\partial r(x,t)/\partial t \geq 0$. This weak inequality holds as an equality if and only if $r_1'(t) = \cdots = r_I'(t) = 0$ and, in addition, either $r_1(t) = \cdots = r_I(t)$ or $s_1'(f_1(x,t)) = \cdots = s_I'(f_I(x,t)) = 0$.*

An important case of this corollary is where all individual consumers have constant impatience. Then $r_i'(t) = 0$ for every i and t, and hence

$$\frac{\partial r}{\partial t}(x,t) = -\sum_i \frac{s_i(f_i(x,t))}{s(x,t)} \left(r_i(t) - r(x,t)\right)^2 s_i'(f_i(x,t)) \tag{18}$$

This means that the representative consumer's impatience decreases (increases) over time whenever the individual consumers have constant but unequal impatience, and their risk tolerance are increasing (decreasing) functions of their own consumptions. This is exactly the claim of Proposition 5 of Gollier and Zeckhauser (2005). Notice, however, that (18) implies, in addition, that even if some individual consumers do not have increasing (decreasing) risk tolerance, the representative consumer's impatience may well be decreasing (increasing), if most individual consumers have increasing (decreasing) risk tolerance.

5 Short Rates and Market Price of Risk

In this section, we apply the formulas obtained in the preceding sections to show how the short-rate process and the market-price-of-risk process depend on the heterogeneity of risk attitudes and impatience. We shall do so in three steps. First, we review well known results on the relationship between the state price deflator, the equivalent martingale measure, the short-rate process, and the market-price-of-risk process, taking the state-price deflator as a primitive datum. Second, assuming that the aggregate consumption process is given by a stochastic differential equation, we derive some general formulas regarding the short rates and market price of risk when the state price deflator is the representative consumer's marginal utility process evaluated at the aggregate consumption process. Third, in some special cases, we show

how the short rates and market price of risk in a heterogeneous economy differ from their counterparts in the standard representative-consumer model.

5.1 State-price deflator and equivalent martingale measure

Duffie (2001, Chapter 6) is a standard reference on the following materials. Let $B = (B_t)_{t\in[0,T]}$ be a one-dimensional Brownian motion and $(\mathscr{F}_t)_{t\in[0,T]}$ be the standard filtration generated by B. Let $\pi = (\pi_t)_{t\in[0,T]}$ be a strictly positive Ito process, referred to as the *state-price deflator*, satisfying

$$d\pi_t = \mu_t^\pi \, dt + \sigma_t^\pi \, dB_t,$$

where $\mu^\pi = (\mu_t^\pi)_{t\in[0,T]}$ and $\sigma^\pi = (\sigma_t^\pi)_{t\in[0,T]}$ are adapted processes. Define the *short-rate process* $\rho = (\rho_t)_{t\in[0,T]}$ by $\rho_t = -\mu_t^\pi/\pi_t$, and the *market-price-of-risk process* $\eta = (\eta_t)_{t\in[0,T]}$ by $\eta_t = -\sigma_t^\pi/\pi_t$.[4] Then define an Ito processes $\zeta = (\zeta_t)_{t\in[0,T]}$ by $\zeta_0 = 1$ and $d\zeta_t/\zeta_t = -\rho_t \, dt$; and define another Ito process $\xi = (\xi_t)_{t\in[0,T]}$ by $\xi_0 = 1$ and $d\xi_t/\xi_t = -\eta_t \, dB_t$. Then,

$$\begin{aligned}
\zeta_t &= \exp\left(-\int_0^t \rho_\tau \, d\tau\right), \\
\xi_t &= \exp\left(-\int_0^t \eta_\tau \, dB_\tau - \frac{1}{2}\int_0^t \eta_\tau^2 \, d\tau\right), \\
\frac{\pi_t}{\pi_0} &= \zeta_t \xi_t.
\end{aligned}$$

In particular, ξ is a martingale.

To see what the short-rate process ρ represents, suppose that the price $\Lambda_{t,\tau}$ at time t of the (risk-free) discount bond that pays one unit at time $\tau > t$ is determined by the state-price deflator π via

$$E_t\left(\frac{\pi_\tau}{\pi_t}\right).$$

Since $E_t\left((\zeta_\tau - \zeta_t)(\xi_\tau - \xi_t)\right) = 0$ and since ξ is a martingale,

$$\Lambda_{t,\tau} = \exp\left(-\int_t^\tau \rho_s \, ds\right).$$

This relation tells us that the short-rate process represents the continuously compounded risk-free interest rates for infinitesimally short periods of time.

Next, to see what the market-price-of-risk process η represents, suppose that the price process $S = (S_t)_{t\in[0,T]}$ of an asset with the dividend rate process $\delta = (\delta_t)_{t\in[0,T]}$ is determined by the state-price deflator π via

$$S_t = E_t\left(\int_t^T \frac{\pi_\tau}{\pi_t}\delta_\tau \, d\tau\right).$$

[4] For the following argument to be correct, it is necessary that these processes satisfy some sorts of integrability conditions, such as Novikov's condition. But we shall not explicitly state such conditions, as imposing them does not affect the formulas we will obtain at the end of our analysis.

This is equivalent to saying that $S_T = 0$ and the *deflated gain process* $G = (G_t)_{t\in[0,T]}$ defined by

$$G_t = \int_0^t \pi_\tau \delta_\tau \, \mathrm{d}\tau + \pi_t S_t$$

is a martingale. Moreover, since

$$S_t = \frac{1}{\pi_t}\left(E_t\left(\int_0^T \pi_\tau \delta_\tau \, \mathrm{d}\tau\right) - \int_0^t \pi_\tau \delta_\tau \, \mathrm{d}\tau\right),$$

S is an Ito process by the martingale representation theorem and Ito's lemma. We, thus, write

$$\mathrm{d}S_t = \mu_t^S \, \mathrm{d}t + \sigma_t^S \, \mathrm{d}B_t,$$

where $\mu^S = (\mu_t^S)_{t\in[0,T]}$ and $\sigma^S = (\sigma_t^S)_{t\in[0,T]}$ are adapted processes. Since S is an Ito process, so is G; and since G is a martingale, its drift term is zero. Hence

$$\eta_t = \frac{\mu_t^S/S_t + \delta_t/S_t - \rho_t}{\sigma_t^S/S_t}.$$

This justifies the term "market price of risk": it is the expected rate of return, in excess of the short rate, on risky assets to be earned by accepting a unit increase in the standard deviation of the rate of return. Note that this equality holds regardless of the choice of the dividend rate process δ.

To get another meaning of the market-price-of-risk process η, note that ξ is a strictly positive martingale with mean one, and hence there is a probability measure Q equivalent to P for which ξ is the density process, that is,

$$E_t\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right) = \xi_t.$$

Define an Ito process $B^Q = (B_t^Q)_{t\in[0,T]}$ by $B_0^Q = 0$ and $\mathrm{d}B_t^Q = \mathrm{d}B_t + \eta_t \, \mathrm{d}t$, that is, $B_t^Q = B_t + \int_0^t \eta_\tau \, \mathrm{d}\tau$. By Girsanov's theorem, B^Q is a standard Brownian motion under Q. Define the *discounted gain process* $H = (H_t)_{t\in[0,T]}$ by

$$H_t = \int_0^t \exp\left(-\int_0^\tau \rho_s \, \mathrm{d}s\right)\delta_\tau \, \mathrm{d}\tau + \exp\left(-\int_0^t \rho_\tau \, \mathrm{d}\tau\right) S_t.$$

By Ito's lemma, this is an Ito process under P and hence under Q. It also follows from the lemma that its drift term under Q is equal to zero. Thus it is a martingale under Q. For this reason, Q is called the *equivalent martingale measure*. Since $S_T = 0$, H is a martingale under Q (if and) only if

$$S_t = E^Q\left(\int_t^T \exp\left(-\int_t^\tau \rho_s \, \mathrm{d}s\right)\delta_\tau \, \mathrm{d}\tau\right).$$

That is, the asset price is equal to the expected discounted sum of future dividends under the equivalent martingale measure Q. The market-price-of-risk process η can therefore be considered as the process of adjustments in the drift term needed to turn the asset price into the expected discounted sum of future dividends.

5.2 State prices as marginal utilities

Suppose that the aggregate consumption process $c = (c_t)_{t\in[0,T]}$ is a solution to the stochastic differential equation

$$\mathrm{d}c_t = a(c_t, t)\,\mathrm{d}t + b(c_t, t)\,\mathrm{d}B_t,$$

where $a : \boldsymbol{R}_{++} \times [0,T] \to \boldsymbol{R}$ and $b : \boldsymbol{R}_{++} \times [0,T] \to \boldsymbol{R}$ and these are twice continuously differentiable. Also let $u : \boldsymbol{R}_{++} \times [0,T] \to \boldsymbol{R}$ be the representative consumer's felicity function, derived as in Section 2. Then the state-price deflator π is defined by

$$\pi_t = \frac{\partial u}{\partial x}(c_t, t).$$

Define $g : \boldsymbol{R}_{++} \times [0,T] \to \boldsymbol{R}$ by

$$g(x,t) = r(x,t) + \frac{a(x,t)}{s(x,t)} - \frac{1}{2}\left(\frac{b(x,t)}{s(x,t)}\right)^2 \left(1 + \frac{\partial s}{\partial x}(x,t)\right).$$

By applying Ito's lemma as in Hara (2006, Proposition 3) to $\partial u/\partial x$ and c, we obtain

$$\rho_t = g(c_t, t).$$

Thus, g represents the short rates as a (deterministic) function of time and aggregate consumption levels. Similarly, define $h : \boldsymbol{R}_{++} \times [0,T] \to \boldsymbol{R}$ by

$$h(x,t) = \frac{b(x,t)}{s(x,t)}, \tag{19}$$

then

$$\eta_t = h(c_t, t). \tag{20}$$

Thus, h represents the market price of risk as a (deterministic) function of time and aggregate consumption levels.

5.3 Heterogeneity in risk attitudes and impatience

We now analyze how the short rate process ρ and the market-price-of-risk process η are affected by the heterogeneity in risk attitudes and impatience through the two functions g and h. To simplify the subsequent analysis, we assume that the felicity function u_i of the individual consumer i is given by

$$u_i(x^i, t) = \exp(-\beta^i t)\frac{(x^i)^{1-\gamma^i} - 1}{1-\gamma^i},$$

where $\beta^i \in \boldsymbol{R}_{++}$ and $\gamma^i \in \boldsymbol{R}_{++}$. By convention, when $\gamma^i = 1$, we mean $u_i(x^i, t) = \exp(-\beta^i t)\ln x^i$. Here β^i is the constant subjective discount rate and γ^i is the constant coefficient of relative risk aversion. Thus $r_i(x^i,t) = \beta^i$ and $s_i(x^i,t) = x^i/\gamma^i$ for every $(x^i,t) \in \boldsymbol{R}_{++} \times [0,T]$. We also assume that $a(x,t) = \mu x$ and $b(x,t) = \sigma x$ for every $x \in \boldsymbol{R}_{++}$, where $\mu \in \boldsymbol{R}$ and $\sigma \in \boldsymbol{R}$. Here μ is the expected instantaneous growth rate of aggregate consumptions and σ^2 is the variance of the instantaneous growth rate of aggregate consumptions. Then, c is a geometric Brownian motion satisfying

$$\frac{c_t}{c_0} = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

Let's see what the short-rate process $\rho = (g(c_t,t)_t)_{t\in[0,T]}$ and the market-price-of-risk process $\eta = (h(c_t,t)_t)_{t\in[0,T]}$ are like. First, as the benchmark case of an homogeneous economy, suppose that $\beta^1 = \cdots = \beta^I$ and $\gamma^1 = \cdots = \gamma^I$. Then (5) and (6) imply that

$$u(x,t) = \exp(-\beta t)\frac{(x)^{1-\gamma} - 1}{1-\gamma},$$

where $\beta = \beta^i$ and $\gamma = \gamma^i$ for every i.[5] That is, the representative consumer has the same risk attitudes and impatience as the individual consumers. Then

$$\rho_t = \beta + \mu\gamma - \frac{\sigma^2}{2}\gamma(1+\gamma)$$

and

$$\eta_t = \sigma\gamma.$$

Thus the short-rate process and the market-price-of-risk processes are deterministic and constant, just as in the Black-Scholes model, where the stock price process is assumed to be a geometric Brownian motion.

Second, consider the case where the γ^i's are all equal but the β^i's are not. This is the case where the individual consumers have the same risk attitudes but different impatience, studied in details by Hara (2008). In this case, by (5) and Corollary 1, the representative consumer's utility function is multiplicatively separable between time t and aggregate consumption levels x, and has the constant coefficient of relative risk aversion equal to the individual consumers' counterparts. We can, thus, write

$$u(x,t) = d(t)\frac{(x)^{1-\gamma} - 1}{1-\gamma},$$

where $\gamma = \gamma^i$ for every i and $d : [0,T] \to \boldsymbol{R}_{++}$. Write $r(t) = -d'(t)/d(t)$, then $r'(t) < 0$ by (18). Since

$$\rho_t = r(t) + \mu\gamma - \frac{\sigma^2}{2}\gamma(\gamma+1),$$

the short-rate process is deterministic and strictly decreasing. On the other hand, since $\eta_t = \sigma\gamma$, the market-price-of-risk process is deterministic and constant.

Third, consider the case where the β^i's are all equal but the γ^i's are not. This is the case

[5] To be exact, we should say that $u(x,t)$ is a scalar multiple of $\exp(-\beta t)((x)^{1-\gamma} - 1)/(1-\gamma)$. This remark also applies to the subsequent expressions of $u(x,t)$.

where the individual consumers have the same impatience but different risk attitudes, studied in details by Hara (2006). By (6) and Corollary 1, then, the representative consumer's utility function is multiplicatively separable between time t and aggregate consumption levels x, and has the constant discount rate equal to the individual consumers' counterparts. We can, thus, write

$$u(x,t) = \exp(-\beta t)u(x),$$

where $\beta = \beta^i$ for every i and $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$. Since $\partial^2 s_i(x^i,t)/\partial(x^i)^2 = 0$ for every i and (x^i,t), (8) implies that $\partial^2 s(x,t)/\partial(x)^2 > 0$. This implies that $q'(x) < 0$, where $q(x) = -u''(x)x/u'(x)$. Moreover, according to (18) of Hara (2006),

$$\rho_t = \beta + \mu q(c_t) - \frac{\sigma^2}{2} q(c_t)\left(q(c_t) + 1 - \frac{q'(c_t)c_t}{q(c_t)}\right). \tag{21}$$

Thus the short-rate process is stochastic but time-invariant, that is, ρ_t depends on c_t but not directly on t. On the other hand, since $\eta_t = \sigma q(c_t)$, the market-price-of-risk process is stochastic but η_t is a strictly decreasing deterministic function of c_t. This means that the market price of risk is higher the lower the aggregate consumption level c_t; and the market-price-of-risk process is time-invariant like the short-rate process.

Finally, consider the case where neither the β^i nor the γ^i are all equal. By (18), $\partial r(x,t)/\partial t < 0$. Write

$$q(x,t) = -\frac{\dfrac{\partial^2 u}{\partial x^2}(x,t)x}{\dfrac{\partial u}{\partial x}(x,t)}.$$

Then $\partial q(x,t)/\partial x < 0$ for every (x,t) because $\partial^2 s(x,t)/\partial x^2 < 0$ for every (x,t). Just as we did for (21), we can show that

$$\rho_t = r(c_t,t) + \mu q(c_t,t) - \frac{\sigma^2}{2} q(c_t,t)\left(q(c_t,t) + 1 - \frac{\dfrac{\partial q}{\partial x}(c_t,t)c_t}{q(c_t,t)}\right).$$

This shows that the short-rate process ρ is stochastic and not time-invariant, but it is difficult to qualitatively determine whether the short rates depends monotonically on time t or aggregate consumption levels c_t. As for the market price of risk,

$$\eta_t = \sigma q(c_t,t).$$

Since $\partial q(x,t)/\partial x < 0$, the market price of risk is higher the lower the aggregate consumption level c_t. By Corollary 1, if the β^i and γ^i are comonotone, then $\partial q(x,t)/\partial t < 0$, while if they are anti-comonotone, then $\partial q(x,t)/\partial t > 0$. Thus, if, for any pair of two consumers, the more risk-averse consumer is less patient, then the market price of risk tends to decrease over time; and if the more risk-averse consumer is more patient, then the market price of risk tends to increase over time. More generally, as Corollary 1 suggests, if the subjective discount rates β^i

and the coefficients γ^i of relative risk aversion are positively correlated, then the market price of risk tends to decrease over time; and if they are negatively correlated, then it tends to increase over time. Jagannathan, McGrattan, and Scherbina (2000) (and the references therein) found that the equity premium declined significantly in the U.S. during the last three decades of the twentieth century.[6] This is consistent, in our model, with the case where the subjective discount rates β^i and the coefficients γ^i of relative risk aversion are positively correlated.[7]

The analysis of this section can be summarized as follows. Using the (state,time)-by-(state,time) (that is, (x,t)-by-(x,t)) formulas on the representative consumer's risk attitudes and impatience obtained in the preceding sections, we investigated what the short rates and the market price of risk are like when the aggregate consumption process is a geometric Brownian motion and each individual consumer's subjective discount rate and coefficient of relative risk aversion are constant. The analysis relies on stochastic calculus, especially Ito's lemma. We found that the short rates tend to decline over time and the market price of risk tends to be lower the higher the aggregate consumption levels. The change in the market price of risk over time depends on the correlation between the individual consumers' risk attitudes and impatience: it decreases over time if more risk-averse consumers are likely to be less patient, while it increases over time if more risk-averse consumers are likely to be more patient.

6 Conclusion

We have investigated implications of heterogeneous impatience in an economy populated by multiple consumers who have time-separable utility functions. We have found some formulas showing how the representative consumer's risk attitudes and impatience will change over time. We have applied these results to derive some properties of the short-rate process and the market-price-of-risk process in an economy of heterogeneous consumers.

There are some issued yet to be explored in this setting. First, while we concentrated on the case of time-additive utility functions, we should look into whether there is any coherence of our analysis with recursive or stochastic differential utility functions.[8] These utility functions are in general not time-additive, but they are still tractable and useful for many applications. It would therefore be reasonable to try to extend our analysis to these utility functions. Second, we have not investigated whether the heterogeneity of impatience would give rise to most commonly used properties in the literature on the term structure of interest rates. One of such properties is mean reversion, so that the drift term of the short-rate process is negative when the short

[6]Note, however, that when calculating the equity premium, they used the yield to maturity of the U.S. Treasury Bonds and the yield of stock market indices, while our short rates are instantaneous risk-free interest rates and our market price of risk is the instantaneous expected rates of return, per unit standard deviation, on risky assets in excess of short rates.

[7]Alternatively, as (19) and (20) indicate, the decline in the equity premium could be attributed to the decline in the volatility of the consumption growth rates, although the volatility is assumed to be constantly equal to σ in our model. Lettau, Ludvigson, and Wachter (2008, Section 1) gathered some empirical evidences of the declining consumption volatility.

[8]I am grateful to Tomoyuki Nakajima for suggesting this line of research.

rate exceeds some threshold, while it is positive when the short rate goes below the threshold. It will quite important to identify under what conditions of heterogeneity of impatience the short-rate process would be mean-reverting.[9]

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[9] I am grateful to Masaaki Kijima for suggesting this line of research.

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