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A Note on Aumann's Core Equivalence Theorem without Monotonicity

by

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Abstract

In an exchange economy with a continuum of traders, we establish the equivalence theorem on the core and the set of competitive allocations without assuming monotonicity of traders' preferences. Under weak assumptions we provide two alternative core equivalence theorems. The first one is for irreducible economies under Debreu's assumption on quasi-equilibria. The second one is an extension of Aumann's theorem under weaker assumptions than monotonicity.

Keywords: core; equivalence; monotonicity; quasi-equilibrium; irreducibility

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1 Introduction

In his seminal paper, Aumann (1964) established the equivalence between the core and the set of competitive equilibrium allocations in an economy with a continuum of traders. His core equivalence theorem is very general in that he assumed a set of very weak conditions on traders' preferences. In fact, in his proof, neither irreflexivity nor transitivity is assumed on traders' preference relations. Our purpose is to revisit his equivalence theorem under somewhat different or general assumptions.

In very general economies, Hildenbrand (1968, 1974) showed that any core allocation is a quasi-equilibrium with relaxing Aumann's monotonicity assumption of preferences to local non-satiation. In our Theorem 1, by applying one of claims of Hildenbrand (1982), we first restate his theorem of Hildenbrand (1968, 1974).

While Aumann's equivalence theorem is extended in our Theorem 3, it is easier to provide Theorem 3 by focusing on the relationship between competitive equilibria and quasi-equilibria. Since any quasi-equilibrium is a competitive equilibrium if each trader's income is positive, and since any equilibrium allocation belongs to the core, it suffices for establishing the equivalence theorem to show the positivity of each trader's income in quasi-equilibria. Taking this point into consideration, we will consider alternative assumptions by which, in quasi-equilibria, the positivity of each trader's income is ensured.

The simplest condition ensuring the positive income of each trader is the positivity of each trader's initial endowment of commodities. Other several conditions are known to ensure the positivity of each trader's income in quasi-equilibria. One of the most general conditions is the irreducibility assumption that was initiated by McKenzie (1959). A weaker version of the irreducibility assumption was introduced by Debreu (1962) to prove an existence theorem of competitive equilibria in finite economies. In our Theorem 2, under Debreu's assumption, we show that any quasi-equilibrium becomes a competitive equilibrium and thus establish the equivalence between the core and the set of equilibrium allocations. The result is consistent with the equivalence theorem of Yamazaki (1978) where a primitive condition on initial income distributions is assumed.

In the economies under Debreu's assumption, each trader's income in a quasi-equilibrium is positive and each trader participates in trade. On the other hand, in the economy that Aumann (1964) considered, there might exist non-negligible individuals who have no endowments and cannot participate in trade. To extend Aumann's equivalence theorem, we would like to look for some conditions which are weaker than both the monotonicity assumption and Debreu's assumption. To include Aumann's economies, we will assume a somewhat weaker condition than monotonicity, which is called the potential desirability of commodities defined by Hara (2006) and introduce a weaker form of Debreu's assumption which reflects the potential desirability of commodities. Under the assumptions we will establish the second equivalence theorem which is an extension of Aumann's theorem. Thus, it should be noted that the monotonicity assumption is dispensable to Aumann's equivalence theorem but it is just required that for each commodity there is a non-null coalition of some traders for whom the commodity is desirable.[1] In addition, since our proof is a simple modification of Aumann's proof, it is meant that the technique of his proof is very general and useful.

In what follows, in section 2 we present a model of exchange economy with a continuum of traders and state a well-known proposition that any equilibrium allocation is a core allocation. In section 3, under a set of weaker assumptions on preference relations, we prove that any core allocation is a quasi-equilibrium allocation. In section 4, the first core equivalence theorem is obtained under Debreu's assumption which is weaker than irreducibility. In addition, under an assumption which is related to the desirability of commodities, the second equivalence theorem is established. Section 5 is devoted to concluding remarks, in which we refer to the related results of Hildenbrand (1968, 1974).

2 The Model

There are n-types of commodities being traded in the economy. A *commodity bundle* is a point in the non-negative orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$ and the *consumption set* of each trader is $\mathbb{R}^n_+$. Let the set of traders be the closed unit interval $T = [0,1]$. The space of traders is an atomless measure space $(T, \mathscr{T}, \lambda)$ where $\mathscr{T}$ is σ-algebra of Borel subsets of $T = [0,1]$ and λ is the Lebesgue measure with $\lambda(T) = 1$.

Following Aumann (1964), an *assignment* is a function $\boldsymbol{f} : T \to \mathbb{R}^n_+$ such that $\boldsymbol{f}(t)$ is a commodity bundle assigned to each trader $t \in T$ and each component of the assignment is Lebesgue integrable over T. Let $\boldsymbol{e} : T \to \mathbb{R}^n_+$ be a fixed assignment in which $\boldsymbol{e}(t)$ denotes an *initial endowment* of each trader $t \in T$. The sum of initial endowments $\boldsymbol{e}$ is defined by $\int_{t \in T} \boldsymbol{e}(t) d\lambda$. For simplicity, we omit the symbols of t and $d\lambda$ in the integral, and so $\int_{t \in T} \boldsymbol{e}(t) d\lambda$ is denoted by $\int_T \boldsymbol{e}$. An *allocation* is an assignment $\boldsymbol{f} : T \to \mathbb{R}^n_+$ with $\int_T \boldsymbol{f} = \int_T \boldsymbol{e}$.

[1] Hildenbrand (1982) mentioned that the monotonicity is not essential for Aumann's proof.

Let $\succ_t$ be the *preference relation* of each trader $t \in T$ defined on the consumption set $\mathbb{R}^n_+$, i.e., $\succ_t \subset \mathbb{R}^n_+ \times \mathbb{R}^n_+$ which satisfies the following property.

ASSUMPTION 1 (Measurability): For a pair of any two assignments $(\boldsymbol{f}, \boldsymbol{g})$, the set $\{t \mid \boldsymbol{f}(t) \succ_t \boldsymbol{g}(t)\}$ is Lebesgue measurable in T.

Note that in this section the measurability on preference relations is just assumed. This assumption is a purely mathematical assumption with no economic interpretation.

Next we state other notations and definitions. A *coalition* of traders is a Lebesgue measurable subset of T. A coalition S with $\lambda(S) > 0$ can *improved upon* an allocation $\boldsymbol{f} : T \to \mathbb{R}^n_+$ if there is an assignment $\boldsymbol{g} : T \to \mathbb{R}^n_+$ such that $\boldsymbol{g}(t) \succ_t \boldsymbol{f}(t)$ for a.e. $t \in S$, and $\int_S \boldsymbol{g} = \int_S \boldsymbol{e}$. The *core* is the set of all allocations that no non-null coalition can improve upon. A *competitive equilibrium* is a pair of a *price vector* $p \in \mathbb{R}^n$ with $p \neq 0$ and an *allocation* $\boldsymbol{f}$ such that for a.e. $t \in T$, $\boldsymbol{f}(t)$ is a *maximal* element with respect to $\succ_t$ in t's *budget set* $\{x \in \mathbb{R}^n_+ \mid p \cdot x \leq p \cdot \boldsymbol{e}(t)\}$. An *equilibrium allocation* is the allocation $\boldsymbol{f}$ of the competitive equilibrium, and an *equilibrium price vector* is the price vector p of the competitive equilibrium. In the standard way it is shown that any equilibrium allocation is in the core. We state the following well-known fact without proof.

PROPOSITION 1: *Under Assumption 1, any equilibrium allocation belongs to the core.*

3 The Core and Quasi-Equilibrium Allocations

We define the *quasi-equilibrium* as follows.[2] A *quasi-equilibrium* is a pair of a *price vector* $p^* \in \mathbb{R}^n$ with $p^* \neq 0$ and an *allocation* $\boldsymbol{f}^*$ such that for a.e. $t \in T$, $\boldsymbol{f}^*(t)$ is maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p^* \cdot x \leq p^* \cdot \boldsymbol{e}(t)\}$ and/or $p^* \cdot \boldsymbol{f}^*(t) = p^* \cdot \boldsymbol{e}(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$. A *quasi-equilibrium allocation* is the allocation $\boldsymbol{f}^*$ of the quasi-equilibrium, and a *quasi-equilibrium price vector* is the price vector p^* of the quasi-equilibrium.

Next we assume that for a.e. $t \in T$ preference relation $\succ_t$ satisfies the two assumptions below.

ASSUMPTION 2 (Local non-satiation): For a.e. $t \in T$, for any $x \in \mathbb{R}^n_+$ and any $\varepsilon > 0$, there exists a point $y \in \mathbb{R}^n_+$ with $\|y - x\| \leq \varepsilon$ such that $y \succ_t x$.

[2]The notion of quasi-equilibrium was first defined by Debreu (1962).

Assumption 3 (Continuity): For a.e. $t \in T$, for any $x \in \mathbb{R}^n_+$, the upper contour set $\{y \in \mathbb{R}^n_+ \mid y \succ_t x\}$ is open in $\mathbb{R}^n_+$.

Note that we do not necessarily assume that the preference relation of almost every trader satisfies reflexivity, completeness, transitivity, or convexity etc, which are usually assumed in the existence theorems of competitive equilibria. In his paper, Aumann (1964) assumed the following condition of monotonicity of preferences and proved the core equivalence theorem.

Monotonicity: For a.e. $t \in T$, if $y \geq x$ and $y \neq x$ then $y \succ_t x$.

Monotonicity implies local non-satiation, but the converse does not hold. In this paper, instead of monotonicity, we assume the local non-satiation of preference relations.

In order to obtain the core equivalence theorem without Monotonicity, first we will restate the theorem of Hildenbrand (1968, 1974) that any core allocation is a quasi-equilibrium allocation. The proof of the following theorem proceeds in the following way. We first define some notations, secondly get a useful lemma by applying one of claims of Hildenbrand (1982),[3] and finally prove the theorem by using a separating hyperplane theorem.

Theorem 1: *Under Assumptions 1, 2, and 3, any core allocation is a quasi-equilibrium allocation.*

Proof: Let an allocation $\boldsymbol{f} : T \to \mathbb{R}^n_+$ be in the core. Define

$$\boldsymbol{P}(t) = \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}(t)\} \text{ and } \boldsymbol{F}(t) = \mathrm{int}\boldsymbol{P}(t) - \boldsymbol{e}(t).$$

By Assumptions 2 and 3, $\boldsymbol{F}(t)$ is a non-empty open set. By Assumption 1, for each $x \in \mathbb{R}^n$ we can define a measurable set $\boldsymbol{F}^{-1}(x)$ by

$$\boldsymbol{F}^{-1}(x) = \{t \in T \mid x \in \boldsymbol{F}(t)\}.$$

Define

$$N = \{r \in \mathbb{R}^n \mid r : \text{rational points},^4 \quad \lambda(\boldsymbol{F}^{-1}(r)) = 0\}.$$

[3]This lemma is obtained by a simple modification of Lemma 4.1 in Aumann (1964).

[4]A rational point in $\mathbb{R}^n$ is a vector whose all components are rational.

Since N is a denumerable set, we have $\lambda(\bigcup_{r\in N}\boldsymbol{F}^{-1}(r)) = 0$. Let us define a measurable set U by

$$U = T \setminus \bigcup_{r\in N}\boldsymbol{F}^{-1}(r),$$

which has full measure, i.e., $\lambda(U) = 1$. If a rational point $r \in \mathbb{R}^n$ belongs to $\boldsymbol{F}(t)$ for some $t \in U$, then the set $\boldsymbol{F}^{-1}(r)$ is of positive measure. This property is used in the proof of the following lemma.

Let us denote the convex hull of $\bigcup_{t\in U}\boldsymbol{F}(t)$ by Δ, i.e.,.

$$\Delta = \text{co}\bigcup_{t\in U}\boldsymbol{F}(t)$$

LEMMA 1: *Δ is non-empty and $0 \notin \Delta$.*[5]

PROOF: The non-emptiness follows from that of $\boldsymbol{F}(t)$ for a.e. $t \in T$. Assume, on the contrary, that $0 \in \Delta$. By the definition of Δ, there are some finitely many points in $\bigcup_{t\in U}\boldsymbol{F}(t)$ such that the origin is expressed as a convex combination of those points, i.e., there are some traders $t_i \in U \;\; (i = 1, 2, \ldots, k)$, $x_i \in \boldsymbol{F}(t_i)$, and $\beta_i > 0$ such that $\sum_{i=1}^{k}\beta_i = 1$ and $\sum_{i=1}^{k}\beta_i x_i = 0$.

Let us choose a rational number α_i sufficiently close to β_i for each $i = 2, 3, \ldots, k$ except 1 and define a number α_1 as follows.

$$\alpha_1 = 1 - \sum_{i=2}^{k}\alpha_i.$$

Since α_i is a rational number sufficiently close to β_i for each $i = 2, 3, \ldots, k$, α_1 is also a rational number close to β_1 and $\alpha_1 > 0$.

Since $\boldsymbol{F}(t_i)$ is an open set for each $i = 2, \ldots, k$, there exist some rational points $r_2, r_3, \ldots, r_k$ such that each point r_i is sufficiently close to x_i and $r_i \in \boldsymbol{F}(t_i)$ for each $i = 2, 3, \ldots, k$. Define a point $r_1 \in \mathbb{R}^n$ as follows.

$$r_1 = -\frac{1}{\alpha_1}\sum_{i=2}^{k}\alpha_i r_i.$$

Since r_1 is a rational point close to x_1 and $\boldsymbol{F}(t_1)$ is an open set, we can assume that $r_1 \in \boldsymbol{F}(t_1)$. Thus, the following property is satisfied.

$$r_i \in \boldsymbol{F}(t_i) \;\; (i = 1, 2, \ldots, k) \;\text{ and }\; \sum_{i=1}^{k}\alpha_i r_i = 0. \tag{1}$$

[5] This lemma is obtained by a simple modification of Lemma 4.1 in Aumann (1964).

Since $t_i \in U$, by definition of U, $t_i \notin \bigcup_{r \in N} \boldsymbol{F}^{-1}(r)$. Therefore, for any $r \in N$, $t_i \notin \boldsymbol{F}^{-1}(r)$, i.e., $r \notin \boldsymbol{F}(t_i)$. Thus, since $r_i \in \boldsymbol{F}(t_i)$, $r_i \notin N$, which implies that $\lambda(\boldsymbol{F}^{-1}(r_i)) > 0$. Hence, there exist some sets $S_1, S_2, \ldots, S_k$ and a positive number δ such that

$$S_i \subset \boldsymbol{F}^{-1}(r_i), \;\; \lambda(S_i) = \delta\alpha_i \;\; (i = 1, 2, \ldots, k), \;\; \text{and} \;\; S_i \cap S_j = \emptyset \;\; (i \neq j).$$

Define a coalition S and an assignment $\boldsymbol{g} : T \to \mathbb{R}^n_+$ as follows.

$$\begin{aligned} S &= \bigcup_{i=1}^{k} S_i \\ \boldsymbol{g}(t) &= \begin{cases} r_i + \boldsymbol{e}(t) & \text{if } t \in S_i \\ \boldsymbol{e}(t) & \text{if } t \notin S. \end{cases} \end{aligned}$$

If t belongs to S, then there exists some i such that $t \in S_i \subset \boldsymbol{F}^{-1}(r_i)$ for whom

$$\boldsymbol{g}(t) = r_i + \boldsymbol{e}(t) \succ_t \boldsymbol{f}(t).$$

In addition, since S is the union of disjoint sets $S_1, S_2, \ldots, S_k$, we have

$$\begin{aligned} \int_S \boldsymbol{g} &= \sum_{i=1}^{k} r_i \lambda(S_i) + \int_S \boldsymbol{e} \\ &= \delta \sum_{i=1}^{k} \alpha_i r_i + \int_S \boldsymbol{e} \\ &= \int_S \boldsymbol{e} \end{aligned}$$

where the last equality follows from (1). This means that non-null coalition S can improve upon allocation $\boldsymbol{f}$. Thus, $\boldsymbol{f}$ is not a core allocation, which is a contradiction. ■

By a separating hyperplane theorem, Lemma 1 implies that there exists a price vector $p \in \mathbb{R}^n$ with $p \neq 0$ such that $p \cdot x \geq 0$ for any $x \in \Delta$. Therefore, by definition of Δ, for each $t \in U$,

$$p \cdot y \geq p \cdot \boldsymbol{e}(t) \;\; \text{for any} \;\; y \in \text{int}\boldsymbol{P}(t)$$

If $y \in \boldsymbol{P}(t)$, then, by continuity, there exists $y' \in \text{int}\boldsymbol{P}(t)$ sufficiently close to y such that $y' \succ_t \boldsymbol{f}(t)$, which implies that $p \cdot y' \geq p \cdot \boldsymbol{e}(t)$. Since we can pick y' arbitrarily close to y, by letting y' go to y, we have $p \cdot y \geq p \cdot \boldsymbol{e}(t)$. Hence, we conclude that for each $t \in U$,

$$y \succ_t \boldsymbol{f}(t) \;\; \Rightarrow \;\; p \cdot y \geq p \cdot \boldsymbol{e}(t). \tag{2}$$

By local non-satiation of preference relations, for any small $\varepsilon > 0$ there exists $y \in \mathbb{R}^n_+$ with $\|y - \boldsymbol{f}(t)\| \leq \varepsilon$ such that $y \succ_t \boldsymbol{f}(t)$. From (2), it follows that $p \cdot y \geq p \cdot \boldsymbol{e}(t)$. By letting y go to $\boldsymbol{f}(t)$, $p \cdot \boldsymbol{f}(t) \geq p \cdot \boldsymbol{e}(t)$. If $\lambda(\{t \in T \mid p \cdot \boldsymbol{f}(t) > p \cdot \boldsymbol{e}(t)\}) > 0$, then, by integration of the inequality, we have $p \cdot \int_T \boldsymbol{f} > p \cdot \int_T \boldsymbol{e}$. This is a contradiction to the premise that $\boldsymbol{f}$ is an allocation. Therefore, for each $t \in U$,

$$p \cdot \boldsymbol{f}(t) = p \cdot \boldsymbol{e}(t). \tag{3}$$

If $p \cdot \boldsymbol{e}(t) > \inf\,\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ and $y \succ_t \boldsymbol{f}(t)$, we can show that $p \cdot y > p \cdot \boldsymbol{e}(t)$. Indeed, assume on the contrary that $p \cdot y = p \cdot \boldsymbol{e}(t)$. By continuity, there exists y' sufficiently close to y such that $y' \succ_t \boldsymbol{f}(t)$ and $p \cdot y' < p \cdot \boldsymbol{e}(t)$, a contradiction to (2). Thus, $(p, \boldsymbol{f})$ is a quasi-equilibrium. This completes the proof of Theorem 1. ∎

By $\mathscr{W}$, $\mathscr{C}$, and $\mathscr{Q}$, we denote respectively the set of equilibrium allocations, the core, and the set of quasi-equilibrium allocations. From Proposition 1 and Theorem 1, it follows that $\mathscr{W} \subset \mathscr{C} \subset \mathscr{Q}$ under Assumptions 1, 2, and 3. Therefore, if $\mathscr{W} \supset \mathscr{Q}$, then $\mathscr{W} = \mathscr{C}$, i.e., the core coincides with the set of equilibrium allocations. One of the assumptions under which $\mathscr{W} \supset \mathscr{Q}$ holds is the following.

Positivity of initial endowments: $\boldsymbol{e}(t) \gg 0$ for a.e. $t \in T$.[6]

In fact, $\boldsymbol{e}(t) \gg 0$ implies that $p \cdot \boldsymbol{e}(t) > \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for any $p \in \mathbb{R}^n$ with $p \neq 0$. Therefore, by the definition of quasi-equilibrium, any quasi-equilibrium is a competitive equilibrium. Thus, we have the following as a corollary of Theorem 1.

COROLLARY 1: *In addition to Assumptions 1, 2, and 3, under the assumption of Positivity of initial endowments, the core coincides with the set of equilibrium allocations, i.e.,* $\mathscr{W} = \mathscr{C}$.

[6] For x and y in $\mathbb{R}^n$, $x \gg y$ means that $x^i > y^i$ for all coordinate i.

4 Irreducible Economies

The assumption that every trader has initially a positive amount of every commodity is too strong. In what follows, we assume that the amount of each commodity is positive in the whole economy.

ASSUMPTION 4: $\int_T \boldsymbol{e} \gg 0$.

As we have seen in the previous section, we can prove the equivalence between the core and the set of equilibrium allocations by showing that any quasi-equilibrium is a competitive equilibrium. By the definition of quasi-equilibrium, any quasi-equilibrium is a competitive equilibrium if all traders' incomes in the quasi-equilibrium are positive. More accurately, a quasi-equilibrium $(p^*, \boldsymbol{f}^*)$ is a competitive equilibrium if $p^* \cdot \boldsymbol{e}(t) > \inf \{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for a.e. $t \in T$. The following is a well-known condition to ensure the positivity of traders' incomes.

Irreducibility: An economy is *irreducible* if for any allocation $\boldsymbol{f} : T \to \mathbb{R}^n_+$ and any measurable partition (S, S') of T with $0 < \lambda(S) < 1$, there is an assignment $\boldsymbol{g} : T \to \mathbb{R}^n_+$ such that

$$\int_{S'} (\boldsymbol{e} - \boldsymbol{g}) + \int_S \boldsymbol{f} \in \int_S \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}(t)\}.^7$$

This primitive condition on economies originated with McKenzie (1959). Irreducibility expresses the property that the initial endowments in any coalition are desirable for every trader in its complementary coalition.

Let us assume the following for any quasi-equilibrium.

ASSUMPTION 5: In a quasi-equilibrium $(p^*, \boldsymbol{f}^*)$, if $p^* \cdot \boldsymbol{f}^*(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders,[8] then it occurs for almost every trader. (Equivalently, if $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders, then it occurs for almost every trader.)

Assumption 5 is used by Debreu (1962) in order to guarantee the existence of competitive equilibria in a private ownership economy with finite traders.

[7] The integral $\int_S \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}(t)\}$ denotes a set defined by

$$\{\int_S \boldsymbol{h} \mid \boldsymbol{h} : T \to \mathbb{R}^n_+, \ \ \boldsymbol{h}(t) \succ_t \boldsymbol{f}(t) \ \text{ a.e. } t \in T\}.$$

[8] "Some traders" means that the set of such traders has a positive measure.

It is easy to show that under Assumption 2 of local non-satiation, Irreducibility implies Assumption 5. Indeed, let $(p^*, \boldsymbol{f}^*)$ be a quasi-equilibrium, S be a measurable subset of T and S' be its complement defined by

$$\begin{aligned} S &= \{t \in T \mid p^* \cdot \boldsymbol{e}(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}\}, \\ \text{and} \quad S' &= \{t \in T \mid p^* \cdot \boldsymbol{e}(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}\}. \end{aligned}$$

We show that $\lambda(S) > 0$ implies $\lambda(S') = 0$. Assume on the contrary that $\lambda(S') > 0$. Then, by Irreducibility, there is an assignment $\boldsymbol{g} : T \to \mathbb{R}^n_+$ such that

$$\int_{S'}(\boldsymbol{e} - \boldsymbol{g}) + \int_S \boldsymbol{f}^* \in \int_S \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}^*(t)\}.$$

By the definition of quasi-equilibrium, for each $t \in S$, $\boldsymbol{f}^*$ is maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p^* \cdot x \leq p^* \cdot \boldsymbol{e}(t)\}$. Therefore,

$$p^* \cdot \int_S \boldsymbol{f}^* \leq p^* \cdot \int_S \boldsymbol{e} < p^* \cdot \int_S \{x \in \mathbb{R}^n_+ \mid x \succ_t \boldsymbol{f}^*(t)\}.$$

Hence, we have $0 < p^* \cdot \int_{S'}(\boldsymbol{e} - \boldsymbol{g})$. On the other hand, by definition of S', for each $t \in S'$, $p^* \cdot \boldsymbol{e}(t) \leq p^* \cdot \boldsymbol{g}(t)$, and $p^* \cdot \int_{S'}(\boldsymbol{e} - \boldsymbol{g}) \leq 0$, a contradiction. Now, if $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders, then $\lambda(S) > 0$ because $p^* \cdot \boldsymbol{e}(t) \geq p^* \cdot \boldsymbol{f}^*(t)$. Therefore, $\lambda(S') = 0$. In addition, by local non-satiation we can show that $S' = \{t \in T \mid p^* \cdot \boldsymbol{f}^*(t) = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}\}$. Hence, $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for every trader. This proves that, under Assumption 2, Irreducibility implies Assumption 5.

By using Assumption 5 instead of Monotonicity, we show that the core coincides with the set of equilibrium allocations which is equivalent to the set of quasi-equilibrium allocations.

THEOREM 2: *Under Assumptions 1, 2, 3, 4, and 5, the core coincides with the set of equilibrium allocations, i.e.,* $\mathscr{W} = \mathscr{C}$.

PROOF: By Proposition 1, the set of equilibrium allocations is a subset of the core. To prove the converse, let $\boldsymbol{f} : T \to \mathbb{R}^n_+$ be a core allocation. Then, by Theorem 1, there exists a price vector $p \neq 0$ such that $(p, \boldsymbol{f})$ is a quasi-equilibrium.
CASE 1: If vector p has some negative components, then $\inf\{p \cdot x \mid x \in \mathbb{R}^n_+\} = -\infty$. Therefore, $p \cdot \boldsymbol{f}(t) > -\infty = \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for all $t \in T$.
CASE 2: If vector p has no negative component, then $\inf\{p \cdot x \mid x \in \mathbb{R}^n_+\} = 0$. Also, by Assumption 4, $p \cdot \int_T \boldsymbol{f} = p \cdot \int_T \boldsymbol{e} > 0$.

Therefore, $p \cdot \boldsymbol{f}(t) > 0 = \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for some $t \in T$. Thus, by Assumption 5, $p \cdot \boldsymbol{f}(t) > \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for all $t \in T$. Hence, in any case, by the definition of quasi-equilibrium, $\boldsymbol{f}(t)$ is maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p \cdot x \leq p \cdot \boldsymbol{e}(t)\}$ for each $t \in T$, i.e., $\boldsymbol{f}$ is an equilibrium allocation. ■

Since Irreducibility implies Assumption 5, we have the following corollary of Theorem 2.

COROLLARY 2: *In addition to Assumptions 1, 2, 3, and 4, under the assumption of Irreducibility, the core coincides with the set of equilibrium allocations, i.e.,* $\mathscr{W} = \mathscr{C}$.

The following is an example of economies which do not satisfy Irreducibility, but Assumption 5.

EXAMPLE 1: Let $n = 2$ and (S^0, S^1) be a measurable partition of T such that $\lambda(S^0) = \lambda(S^1) = 0.5$. For each trader t, initial endowment $\boldsymbol{e}(t)$ and utility function U_t which corresponds to preference relation $\succ_t$ are defined by the following:

$$\boldsymbol{e}(t) = \begin{cases} (1,3) & \text{for } t \in S^0 \\ (1,0) & \text{for } t \in S^1 \end{cases} \quad \text{and} \quad U_t(x_1,x_2) = \begin{cases} \min\{x_1,x_2\} & \text{for } t \in S^0 \\ x_1 & \text{for } t \in S^1 \end{cases}$$

A pair $(p^*, \boldsymbol{f}^*)$ of a price vector and an allocation defined by

$$p^* = (1,0) \quad \text{and} \quad \boldsymbol{f}^*(t) = \begin{cases} (1,3-y) & \text{for } t \in S^0 \\ (1,y) & \text{for } t \in S^1 \end{cases} \qquad (\text{where } 0 \leq y \leq 2)$$

is a competitive equilibrium as well as a quasi-equilibrium.

In the economy of Example 1, every trader has a positive income in quasi-equilibrium $(p^*, \boldsymbol{f}^*)$ for any y with $0 \leq y \leq 2$, and therefore Assumption 5 is satisfied. However, Irreducibility is not satisfied. In fact, let $y = 2$ and, in the definition of Irreducibility, put $\boldsymbol{f} = \boldsymbol{f}^*$, $S = S^0$, and $S' = S^1$. Then, to make traders in S^0 better off, both commodities are needed, while traders in S^1 have only commodity 1.

Next, we would like to consider an assumption which is weaker than Monotonicity and to introduce the concept of *potential desirability of commodities* defined by Hara (2006). The intuition of the potential desirability is that for any commodity there exists a group of traders with influential power for whom the commodity is desirable. Note that the potential desirability does not require that any small amount of each commodity is desirable for a group, but just that some constant amount of the commodity are desirable for the group. Let us define the potential desirability and introduce an assumption of potential desirability as follows.

DEFINITION: Commodity m is *potentially desirable* for a coalition $S \subset T$ with respect to a constant number $\alpha > 0$ if $x + \alpha \mathbf{1}_m \succ_t x$ for all $x \in \mathbb{R}^n_+$ and for all $t \in S$.[9]

ASSUMPTION 6: For each commodity $m = 1, 2, \ldots, n$, there exists a coalition $S^m \subset T$ with $\lambda(S^m) > 0$ and a number $\alpha^m > 0$ such that commodity m is potentially desirable for coalition S^m with respect to α^m.

This assumption says that for each commodity there are some traders for whom the commodity is desirable, but the commodity is not necessarily desirable for all traders. It also says that a particular positive amount of the commodity is desirable for the traders.

Moreover, we add the following assumption which reflects both properties of Assumptions 5 and 6.

ASSUMPTION 7: In a quasi-equilibrium $(p^*, \boldsymbol{f}^*)$, if $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ occurs for some traders, then it occurs for some traders in S^m for all $m = 1, 2, \ldots, n$.[10]

Clearly, Monotonicity implies Assumption 6, whereas the converse is not true. Note that it is possible for a trader to be included in more than one coalition of $S^1, S^2, \ldots, S^n$. Under Monotonicity, every trader belongs to S^m for all $m = 1, 2, \ldots, n$. Thus, Monotonicity implies Assumption 7.

Now let us consider economies in which Assumption 7 is satisfied while Assumption 5 is not. The following is an example of such economies.

EXAMPLE 2: Let $n = 2$ and (S^0, S^1, S^2) be a measurable partition of T such that $\lambda(S^1) = \lambda(S^2) > 0$. For each trader t, the initial endowment $\boldsymbol{e}(t)$ and the preference relation $\succ_t$ which is depicted by a utility function U_t are defined by the following:

$$\boldsymbol{e}(t) = \begin{cases} (0,0) & \text{for } t \in S^0 \\ (0,1) & \text{for } t \in S^1, \\ (1,0) & \text{for } t \in S^2 \end{cases} \qquad U_t(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{for } t \in S^0 \\ x_1 & \text{for } t \in S^1 \\ x_2 & \text{for } t \in S^2 \end{cases}$$

Here, commodity 1 is desirable for traders in S^1, commodity 2 is desirable for

[9] By $\mathbf{1}_m$ we denote a vector whose m-th coordinate is 1 and whose other coordinates are 0.

[10] S^m is the non-null coalition defined in Assumption 6 for each $m = 1, 2, \ldots, n$.

those in S^2, both are desirable for those in S^0. A pair $(p^*, \boldsymbol{f}^*)$ of a price vector and an allocation defined by

$$p^* = (1,1) \quad \text{and} \quad \boldsymbol{f}^*(t) = \begin{cases} (0,0) & \text{for } t \in S^0 \\ (1,0) & \text{for } t \in S^1 \\ (0,1) & \text{for } t \in S^2 \end{cases}$$

is a competitive equilibrium as well as a quasi-equilibrium that is unique.

In the above example, Assumption 6 is satisfied. Moreover, Assumption 7 is satisfied for quasi-equilibrium $(p^*, \boldsymbol{f}^*)$, while Assumption 5 isn't if $\lambda(S^0) > 0$.

From now on, in order to obtain a core equivalence theorem for economies which satisfy Assumptions 1, 2, 3, 4, 6 and 7, we show the following lemmas.

Lemma 2: *Under Assumption 6, for any quasi-equilibrium* $(p^*, \boldsymbol{f}^*)$, $p^* \geq 0$ *with* $p^* \neq 0$.

Proof: Suppose that $p^*_m < 0$ for some m. Then, by Assumption 6, for all $t \in S^m$,

$$\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m \succ_t \boldsymbol{f}^*(t) \text{ and } p^* \cdot (\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m) < p^* \cdot \boldsymbol{f}^*(t) \leq p^* \cdot \boldsymbol{e}(t),$$

i.e., $\boldsymbol{f}^*(t)$ is not maximal with respect to $\succ_t$ in t's budget set $\{x \in \mathbb{R}^n_+ \mid p^* \cdot x \leq p^* \cdot \boldsymbol{e}(t)\}$.

On the other hand, $p^* \cdot \boldsymbol{f}^*(t) > -\infty = \inf\{p^* \cdot x \mid x \in \mathbb{R}^n_+\}$ for all $t \in T$. Therefore, by the definition of quasi-equilibrium, $\boldsymbol{f}^*(t)$ is maximal with respect to $\succ_t$ in t's budget set for all $t \in T$, a contradiction. ■

Lemma 3: *Under Assumptions 4, 6, and 7, for any quasi-equilibrium* $(p^*, \boldsymbol{f}^*)$, $p^* \gg 0$.

Proof: Since $p^* \geq 0$ by Lemma 2, $\inf\{p \cdot x \mid x \in \mathbb{R}^n_+\} = 0$. Also, by Assumption 4, $p^* \cdot \int_T \boldsymbol{f}^* = p^* \cdot \int_T \boldsymbol{e} > 0$. Therefore, $p^* \cdot \boldsymbol{f}^*(t) > 0 = \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for some $t \in T$. Thus, by Assumption 7, for each $m = 1, 2, \ldots, n$, $p^* \cdot \boldsymbol{f}^*(t) > \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$ for some $t \in S^m$. Therefore, by the definition of quasi-equilibrium, for each $m = 1, 2, \ldots, n$, $\boldsymbol{f}^*(t)$ is maximal with respect to $\succ_t$ in t's budget set for some $t \in S^m$.

Now, suppose $p^*_m = 0$ for some m. Then, by Assumption 6, for all $t \in S^m$,

$$\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m \succ_t \boldsymbol{f}^*(t) \text{ and } p^* \cdot (\boldsymbol{f}^*(t) + \alpha^m \mathbf{1}_m) = p^* \cdot \boldsymbol{f}^*(t) \leq p^* \cdot \boldsymbol{e}(t),$$

i.e., $\boldsymbol{f}^*(t)$ is not maximal with respect to $\succ_t$ in t's budget set, a contradiction. ■

By using Lemmas 2 and 3, we can get the following theorem.

THEOREM 3: *Under Assumptions 1, 2, 3, 4, 6, and 7, the core coincides with the set of equilibrium allocation, i.e.,* $\mathscr{W} = \mathscr{C}$. *Moreover, every equilibrium price vector is strictly positive.*

PROOF: By Theorem 1, any core allocation is a quasi-equilibrium allocation. By Lemma 3, the price vector associated with any quasi-equilibrium is strictly positive. Let $\boldsymbol{f} : T \to \mathbb{R}^n_+$ be a core allocation. Then, there exists $p \gg 0$ such that $(p, \boldsymbol{f})$ is a quasi-equilibrium. Note that $\inf\{p \cdot x \mid x \in \mathbb{R}^n_+\} = 0$.
CASE 1: For $t \in T$ with $p \cdot \boldsymbol{e}(t) > 0$, by the definition of quasi-equilibrium, $\boldsymbol{f}(t)$ is maximal with respect to $\succ_t$ in t's budget set, since $p \cdot \boldsymbol{e}(t) > \inf\{p \cdot x \mid x \in \mathbb{R}^n_+\}$.
CASE 2: For $t \in T$ with $p \cdot \boldsymbol{e}(t) = 0$, clearly $\boldsymbol{e}(t) = \boldsymbol{f}(t) = 0$. Suppose that $\boldsymbol{f}(t)$ is not maximal with respect to $\succ_t$ in t's budget set. Then, $0 \succ_t \boldsymbol{f}(t)$, since t's budget set contains only the origin 0 of $\mathbb{R}^n_+$. If the set S of traders t for whom this happens has positive measure, then S can improve upon $\boldsymbol{f}$ via $\boldsymbol{e}$, contradicting that $\boldsymbol{f}$ is a core allocation. Thus, the set S is null and can be ignored.

This proves that $(p, \boldsymbol{f})$ is a competitive equilibrium where $p \gg 0$. ■

Note that Assumptions 6 and 7 are weaker than Monotonicity. This fact means that the assumption of Theorem 3 is weaker than that of Aumann's theorem, i.e., Theorem 3 is an extension of Aumann's equivalence theorem.

Since Example 2 satisfies all the assumptions in Theorem 3, we can apply the theorem to the example and conclude that $\boldsymbol{f}^*$ is a unique core allocation. Under the assumptions of Theorem 3, the core might be smaller than the set of quasi-equilibrium allocations. However, the example suggests the following corollary.

COROLLARY 3: *Let the preference relation of almost every trader be irreflexive. Under Assumptions 1, 2, 3, 4, 6, and 7, the following three sets of allocations: (i) the core, (ii) the set of equilibrium allocations, (iii) the set of quasi-equilibrium allocations are equivalent, i.e.,* $\mathscr{W} = \mathscr{C} = \mathscr{Q}$.

PROOF: The argument in the proof of Theorem 3 can be applied not only to any core allocation, but also to any quasi-equilibrium allocation, since the assertion of Case 2 is obviously true under the irreflexivity assumption. ■

EXAMPLE 3: Everything is the same as Example 2 except for the preference relations of traders in S^0. For $t \in S^0$, let $\succ_t$ be a relation such that $\mathbf{0} \succ_t x$ for all $x \in \mathbb{R}^2_+$. Namely, traders in S^0 prefer nothing but the origin $\mathbf{0}$ of $\mathbb{R}^2_+$. Note that their preference relations are not irreflexive since $\mathbf{0} \succ_t \mathbf{0}$.

In Example 3, if $\lambda(S^0) > 0$, the pair $(p^*, \boldsymbol{f}^*)$ defined in Example 2 is a unique quasi-equilibrium, while it is not a competitive equilibrium. Since Assumption 7 is satisfied for $(p^*, \boldsymbol{f}^*)$, Theorem 3 holds. However, since $\boldsymbol{f}^*$ is not a core allocation, in this case both the core and the set of equilibrium allocations are empty. On the other hand, if $\lambda(S^0) = 0$, this is a case of Corollary 3, and $\boldsymbol{f}^*$ is a core allocation as well as an equilibrium allocation.

With respect to the non-emptiness of the core, it is shown by Hara (2006) that, under Assumptions 1, 2, 3, and 6,[11] there is a competitive equilibrium in exchange economies with the following additional assumptions:

(i) Positivity of initial endowments, i.e., $\boldsymbol{e}(t) \gg 0$ for almost every $t \in T$,

(ii) Preference-indifference relation of each trader is complete and transitive.

By Hara's existence theorem, we can simply guarantee the non-emptiness of the core.

5 Conclusion

As we have shown, the monotonicity of preferences is not essential for Aumann's equivalence theorem. On the other hand, general equivalence theorems have been established by Hildenbrand (1968, 1974). In his paper (1968) he presented a coalition production economy and proved an equivalence theorem in a very general measure-theoretic framework by using powerful mathematical theorems such as Liapunov's Theorem and the Measurable Selection Theorem. Especially, he allowed consumption sets to vary with traders and proved a theorem that any core allocation is a quasi-equilibrium. The theorem is more general than Theorem 1 in this paper, since the coalition production economy includes our exchange economy as a special case. However, in asserting that the equivalence theorem holds, he assumed the monotonicity of preference relations. Furthermore, in his book (1974) he proved an equivalence theorem in an atomless exchange economy by assuming some regular conditions on preference relations such as irreflexivity, transitivity, and monotonicity.[12] Thus, his equivalence theorems are not more

[11]With respect to the potential desirability, in his paper, Hara (2006) just assumes that for each commodity m a number α^m depends on each trader $t \in S^m$.

[12]In Problem 9 of Hildenbrand (1974, p.143), he claimed that it is possible to prove an equivalence theorem for an irreducible exchange economy by using an analogous argument to Theorem

general than Theorems 2 and 3 in this paper, which hold even for some reducible economies without Monotonicity such as those in Examples 1 and 2. This is the first significance of this note. As the second significance of this note, we should note that our proof of the equivalence theorem is very elementary by virtue of both Aumann's technique and Hildenbrand's one, and we do not require a general approach of measure theory.

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1 in Hildenbrand (1974, p.133), but it seems that the monotonicity assumption is indispensable in the argument. However, there is no problem with using the method of proof in Hildenbrand (1968).