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An Impossibility Result for Social Welfare Relations in Infinitely-Lived Societies

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AN IMPOSSIBILITY RESULT FOR SOCIAL WELFARE RELATIONS IN INFINITELY-LIVED SOCIETIES

MICHELE LOMBARDI AND ROBERTO VENEZIANI

Abstract. This paper extends the analysis of liberal principles in social choice recently proposed by Mariotti and Veneziani ([6]) to societies with an infinite number of agents. First, a novel characterisation of the inequalitarian leximax social welfare relation is provided based on the Individual Benefit Principle, which incorporates a liberal, non-interfering view of society. This result is surprising because the IBP has no obvious anti-egalitarian content. Second, it is shown that there exists no weakly complete social welfare relation that satisfies simultaneously the standard axioms of Finite Anonymity, Strong Pareto, and Weak Continuity, and a liberal principle of Non-Interference that generalises IBP.

JEL classification. D63 (Equity, Justice, Inequality, and Other Normative Criteria and Measurement); D70 (Analysis of Collective Decision-Making); Q01 (Sustainable development).

Keywords. Infinite utility streams, Individual Benefit Principle, leximax, Non-Interference, impossibility.

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1. Introduction

Liberal principles in philosophy and social choice tend to express some notion of individual autonomy or freedom. In a recent contribution, Mariotti and Veneziani ([6]) have proposed a new axiom - called the Harm Principle (HP) - suited for Social Welfare Orderings (SWOs), which is meant to capture a liberal view of non-interference. The basic content of HP can be illustrated as follows: consider two welfare allocations $u$ and $v$ such that $u$ is socially preferred to $v$, and two different welfare allocations $u'$ and $v'$ such that agent $i$ is worse off at these than at the corresponding starting allocations, the other agents are equally well off, and agent $i$ prefers $u'$ to $v'$. Whatever the origin of the decrease in agent $i$’s welfare, HP requires that society’s preference over $u'$ and $v'$ should agree with person $i$’s preferences: having already suffered a welfare loss in both allocations, and given that no other agent is affected, agent $i$ should not be punished in the SWO by changing social preferences against her.

Although HP incorporates no egalitarian content, Mariotti and Veneziani ([6]) have shown that, together with the standard axioms of Anonymity and Strong Pareto, it characterises the lexicin SWO in societies with a finite number of agents. Lombardi and Veneziani ([5]) have generalised this counterintuitive result by weakening HP and, based on the weak HP, they have provided novel characterisations of various SWOs related to Rawls’s difference principle, including the maximin and the ‘recursive maximin’ recently proposed by Roemer ([8],[9]). They have also used the weak HP to characterise the lexicin social welfare relation (SWR) as defined by Asheim and Tungodden ([1]) in economies with an infinite number of agents. The latter result is particularly relevant because the analysis of societies with an infinite number of agents is of focal interest, especially in the discussion of intergenerational justice, but impossibility results easily obtain, for there exists no SWO that satisfies Anonymity and Strong Pareto (see [4]).

This paper extends the analysis of liberal, non-interfering views in societies with an infinite number of agents in two main directions. First, the Individual Benefit Principle (IBP) - proposed by Mariotti and Veneziani ([7]) in economies with a finite number of agents - is analysed. The IBP also incorporates a liberal, noninterfering view of society and it can be taken
as the theoretical complement of HP, for it requires society not to switch social preferences when agent i’s welfare in both allocations u and v increases. Although it has no obvious inegalitarian content, we show that a weaker version of IBP suitable for SWRs in infinitely-lived societies, together with other standard axioms, provides a novel characterisation of the inegalitarian leximax SWR. This result generalises the characterisation of the leximax SWO in finite societies in ([7]).

Second, as noted by ([7]), HP and IBP can be taken as two parts of a single liberal view and a weaker version of the principle of Non-Interference ([7]) is proposed, which is suitable for SWRs in societies with an infinite number of agents. An interesting impossibility result for liberal approaches is derived, according to which there exists no weakly complete SWR that satisfies Finite Anonymity, Strong Pareto Optimality, Weak Preference Continuity, and Non-Interference.

2. The framework

Let $X \equiv \mathbb{R}^\mathbb{N}$ be the set of countably infinite utility streams, where $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of natural numbers. An element of $X$ is $1u = (u_1, u_2, ...)$ and, for $t \in \mathbb{N}$, $u_t$ is the utility level of a representative member of generation $t$. For $T \in \mathbb{N}$, $1u_T = (u_1, ..., u_T)$ denotes the $T$-head of $1u$ and $1u_{T+1} = (u_{T+1}, u_{T+2}, ...)$ denotes the $T$-tail of $1u$, so that $1u = (1u_{T:T+1})$. We write $const$ for the stream of constant level of well-being equal to $\epsilon \in \mathbb{R}$. A permutation $\pi$ is a bijective mapping of $\mathbb{N}$ on itself. A permutation $\pi$ of $\mathbb{N}$ is finite whenever there is $T \in \mathbb{N}$ such that $\pi(t) = t$ for all $t > T$. For any $1u \in X$ and any permutation $\pi$, let $\pi(1u) = (u_{\pi(t)})_{t \in \mathbb{N}}$ be a permutation of $1u$. For any $T \in \mathbb{N}$ and $1u \in X$, $1\bar{u}_T$ is a permutation of $1u_T$ such that the components are ranked in ascending order (i.e., $\bar{u}_1 \leq ... \leq \bar{u}_T$).

For any two utility paths $1u, 1v$, we write $1u \succeq 1v$ to mean $u_t \geq v_t$ for all $t \in \mathbb{N}$; $1u \succ 1v$ to mean $1u \succeq 1v$ and $1u \neq 1v$; and $1u \gg 1v$ to mean $u_t > v_t$ for all $t \in \mathbb{N}$.

Let $\succsim$ be a (binary) relation over $X$. For any $1u, 1v \in X$, we write $1u \succsim 1v$ for $(1u, 1v) \in \succsim$ and $1u \not\succsim 1v$ for $(1u, 1v) \not\in \succsim$; $\succsim$ stands for “at least as good as”. For any $1u, 1v \in X$, the asymmetric factor $\succ$ of $\succsim$ is defined by $1u \succ 1v$ if and only if $1u \succsim 1v$ and $1v \not\succsim 1u$, and the symmetric part $\sim$ of $\succsim$ is defined
by \( 1u \sim 1v \) if and only if \( 1u \succsim 1v \) and \( 1v \succsim 1u \). They stand, respectively, for “strictly better than” and “indifferent to”. A relation \( \succsim \) on \( X \) is said to be: reflexive if, for any \( 1u \in X \), \( 1u \sim 1u \); complete if, for any \( 1u, 1v \in X \), \( 1u \neq 1v \) implies \( 1u \succsim 1v \) or \( 1v \succsim 1u \); transitive if, for any \( 1u, 1v, 1w \in X \), \( 1u \succsim 1v \succsim 1w \) implies \( 1u \succsim 1w \). \( \succsim \) is a quasi-ordering if it is reflexive and transitive, while \( \succ \) is an ordering if it is a complete quasi-ordering. Let \( \succ \) and \( \succ' \) be relations on \( X \). \( \succ' \) is an extension of \( \succ \) if \( \succ \subseteq \succ' \) and \( \succ \subseteq \succ' \).

3. The Harm Principle and the Leximin SWR

The standard definition of the leximin SWR used in the literature to compare (countably) infinite utility streams is due to Asheim and Tungodden ([1]).

**Definition 3.1.** (Definition 2, [1], p. 224) For all \( 1u, 1v \in X \), \( 1u \sim^{LM} 1v \Leftrightarrow \exists \tilde{T} \geq 1 \) such that \( \forall \tilde{T} \geq \tilde{T} : \tilde{u}_\tilde{T} = \tilde{v}_\tilde{T} \), and \( 1u \succ^{LM} 1v \Leftrightarrow \exists \tilde{T} \geq 1 \) such that, \( \forall \tilde{T} \geq \tilde{T} \), \( \exists t \in \{1, \ldots, T\} \) with \( \tilde{u}_s = \tilde{v}_s \) (\( \forall 1 \leq s < t \)) and \( \tilde{u}_t > \tilde{v}_t \).

The characterisation of the leximin derived by ([5]) focuses on definition 3.1, and it is based on the following axioms.\(^1\)

**Finite Anonymity, FA:** \( \forall 1u \in X \) and \( \forall \) finite permutation \( \pi \) of \( \mathbb{N} \), \( \pi(1u) \sim 1u \).

**Strong Pareto Optimality, SPO:** \( \forall 1u, 1v \in X : 1u > 1v \Rightarrow 1u \succ 1v \).

**Weak Preference Continuity, WPC:** \( \forall 1u, 1v \in X : \exists \tilde{T} \geq 1 \) such that \( (1u_{\tilde{T}, \tilde{T} + 1} v) > 1v \forall \tilde{T} \geq \tilde{T} \Rightarrow 1u \succ 1v \).

**Weak Completeness, WC:** \( \forall 1u, 1v \in X, \exists \tilde{T} \geq 1 \pi(1u_{\tilde{T}, \tilde{T} + 1} v) \neq 1v \forall \) finite permutation \( \pi \) of \( \mathbb{N} \Rightarrow (1u_{\tilde{T}, \tilde{T} + 1} v) \succ 1v \) or \( 1v \succ (1u_{\tilde{T}, \tilde{T} + 1} v) \).

**Harm Principle, HP:** \( \forall 1u, 1v, 1u', 1v' \in X : \exists \tilde{T} \geq 1 \ 1u = (1u_{\tilde{T}, \tilde{T} + 1} v) \succ 1v, \) and \( 1u' \) and \( 1v' \) are such that, \( \exists i \leq T \),

\(^1\)Definition 3.1 is also known as the W-Leximin ([1], p.224). [5] also provide a characterisation of the S-Leximin ([1], p.224) and of the leximin SWR as defined by ([2]). Analogous impossibility results can be proved for the latter definitions.
implies $u'_i \succ v'_i$ whenever $u'_i > v'_i$.

**FA** and **SPO** are standard and need no further comment. **WPC** has been proposed by Asheim and Tungodden ([1], p. 223) and it represents a mainly technical, weak requirement to deal with infinite-dimensional vectors. **WC** states that a SWR should be able to compare vectors with the same tail: this seems an obviously desirable property, as it imposes a minimum requirement of completeness. Finally, **HP** formalises the Harm Principle in societies with an infinite number of agents. It is weaker than the version proposed by ([7]), because it does not require that $u'_i \succ v'_i$ and moreover it only holds for vectors with the same tail.² Lombardi and Veneziani ([5]) have proved the following Theorem.

**Theorem 3.2.** *(Theorem 3.5, [5], p. 12)* $\succ$ is an extension of $\succeq^{LM}$ if and only if $\succeq$ satisfies **FA**, **SPO**, **HP**, **WPC**, and **WC**.

As noted in [6] and [5], a characterisation of the leximin based on **HP** is surprising, because **HP** has no obvious egalitarian content, unlike the standard axiom of Hammond Equity (see, e.g., [3], and [1]). It is also quite surprising that, by a suitable change in the axiom incorporating a liberal view of non-interference, it is possible to characterise the strongly inegalitarian leximax SWR.

### 4. The Benefit Principle and the Leximax SWR

According to the leximax, that society is best which (lexicographically) maximises the welfare of its best-off members. In economies with an infinite number of agents, this intuition can be formalised as follows.

**Definition 4.1.** For all $u, v \in X$, $u \sim^{LX} v \iff \exists \bar{T} \geq 1$ such that $\forall T \geq \bar{T}$: $1\bar{u}_T = 1\bar{v}_T$, and $u \succ^{LX} v \iff \exists \bar{T} \geq 1$ such that, $\forall T \geq \bar{T}$, $\exists t \in \{1, ..., T\}$ with $\bar{u}_s = \bar{v}_s \quad (\forall t < s \leq T)$ and $\bar{u}_t > \bar{v}_t$.

²For a detailed discussion of the axioms, see ([5]).
In order to characterise the leximax SWR, the same axioms as for the leximin are used, except for \textbf{HP}, which is substituted with the Individual Benefit Principle. The \textbf{IBP} also captures a liberal requirement of noninterference and can be formalised as follows.

\textbf{Individual Benefit Principle, IBP:} \(\forall u, v, u', v' \in X : \exists T \geq 1
\)
\(1u = (u_{T,T+1}) \succ v \), and \(1u'\) and \(1v'\) are such that, \(\exists i \leq T \),
\[ u'_i > u_i \]
\[ v'_i > v_i \]
\[ u'_j = u_j \forall j \neq i \]
\[ v'_j = v_j \forall j \neq i \]
implies \(1u' \succ 1v'\) whenever \(u'_i > v'_i\).

In other words, consider two alternatives \(1u\) and \(1v\), whereby \(1u\) is socially preferred to \(1v\), and two different welfare allocations \(1u'\) and \(1v'\) such that agent \(i\) is better off at these than at the corresponding starting allocations, the other agents are equally well-off, and \(i\) prefers \(1u'\) to \(1v'\). \textbf{IBP} requires that society’s preference over \(1u'\) and \(1v'\) should agree with person \(i\)’s preferences: although \(i\)’s welfare has increased in both allocations, society should not ‘punish’ \(i\) by reversing social preferences. The moral intuition behind \textbf{IBP} is similar to the \textbf{HP}, and yet the next Theorem proves that the \textbf{IBP} leads to an extremely different result.

\textbf{Theorem 4.2.} \(\succ\) is an extension of \(\succ^{LX}\) if and only if \(\succ\) satisfies \textbf{FA, SPO, IBP, WPC, and WC}.

\textit{Proof.} \((\Rightarrow)\) Let \(\succ^{LX} \subseteq \succ\). It is easy to see that \(\succ\) meets \textbf{FA, SPO, WPC,} and \textbf{WC}. We show that \(\succ\) satisfies \textbf{IBP}. Take any \(1u, 1v, 1u', 1v' \in X\) such that \(1u = (u_{T,T+1}) \succ v \exists T \geq 1\), and \(1u', 1v'\) are such that, \(\exists i \leq T\), \(u'_i > u_i\), \(v'_i > v_i\), \(u'_j = u_j \forall j \neq i\), \(v'_j = v_j \forall j \neq i\). We show that \(1u' \succ 1v'\) whenever \(u'_i > v'_i\). As \(1u, 1v\) have the same tail, \(1u \succ^{LX} 1v\). Then, \(\exists T \geq 1\) such that, \(\forall T' \geq T\), \(\exists t \in \{1, ..., T'\}\) with \(\tilde{u}_s = \tilde{v}_s \forall t < s \leq T'\) and \(\tilde{u}_t > \tilde{v}_t\). Consider any \(T' \geq \hat{T}\). If \(\tilde{u}_{T'} > \tilde{v}_{T'}\), the result follows as \(\tilde{u}_{T'} \in \{u'_i, \tilde{u}_{T'}\}\) and \(\tilde{v}_{T'} \in \{v'_i, \tilde{v}_{T'}\}\). Therefore suppose \(\tilde{u}_{T'} = \tilde{v}_{T'}\). If \(\tilde{u}_t = \tilde{v}_t\) for all \(t \leq \ell \leq T'\), the result follows. Otherwise, let \(\tilde{v}_t \neq \tilde{v}'_t\) for some \(t \leq \ell \leq T'\). We distinguish
two cases.

Case 1. $\tilde{v}_t < v'_t < \tilde{v}_{t+1}$
Then, $\tilde{v}'_{t+1} > \tilde{v}'_t = v'_t > \tilde{v}_t$ and $\tilde{v}'_s = \tilde{v}_s$ for all $T' \geq s > t$. If $u'_i \in (v'_i, \tilde{u}_{t+1}]$, then $u'_i > v'_i = \tilde{u}'_i$. Otherwise, let $u'_i > \tilde{u}_{t+1}$. Thus, there exists $j \geq t + 1$ such that $u'_i = \tilde{u}'_j > \tilde{u}_j$. Let

$$m = \max \{t + 1 \leq j \leq T'\mid \tilde{u}'_j > \tilde{u}_j\}.$$ 

Since $\tilde{v}'_j = \tilde{u}_j = \tilde{u}_j \forall t < j \leq T'$ it follows that $\tilde{u}'_m > \tilde{v}'_m$. In both cases, there exists $t^* \leq T'$ such that $\tilde{u}'_s = \tilde{v}'_s \forall t^* < s \leq T'$ and $\tilde{u}'_{t^*} > \tilde{v}'_{t^*}$.

Case 2. $v'_1 \geq \tilde{v}_{T'}$

If $v'_1 \geq \tilde{v}_{T'}$, then $\tilde{u}'_{T'} > \tilde{v}_{T'}$ as $u'_1 > v'_1$. Otherwise, let $v'_1 < \tilde{v}_{T'}$. Let

$$\ell = \min \{t + 1 < j \leq T'\mid v'_j < \tilde{v}_j\}.$$ 

Then, $\tilde{v}_t > v'_t = \tilde{v}_{\ell-1} \geq \tilde{u}_{t-1} = \tilde{v}_{t-1}$. As $v'_t < u'_1$, it follows that $u'_1 > \tilde{u}_{t-1}$. If $u'_i \in (\tilde{u}_{t-1}, \tilde{u}_{\ell})$, then $\tilde{v}_{\ell-1} < u'_1 = \tilde{u}_{\ell-1}$. Otherwise, let $\tilde{u}_t < u'_1$. Then, there exists $\ell \leq m \leq T'$ such that $\tilde{v}'_m = \tilde{v}_m < \tilde{u}'_m = u'_1$ and if $m < T'$, $\tilde{u}'_s = \tilde{v}'_s \forall m < s \leq T'$. In both cases, there exists $t^* \leq T'$ such that $\tilde{u}'_s = \tilde{v}'_s \forall t^* < s \leq T'$ and $\tilde{u}'_{t^*} > \tilde{v}'_{t^*}$.

Since it holds for any $T' \geq \tilde{T}$, we have that $u'_1 \succ v'_1$ as $\succ^{LX} \subseteq \succ$.

($\Leftarrow$) Suppose that $\succ$ satisfies FA, SPO, IBP, WPC, and WC. We show that $\sim^{LX} \subseteq \sim$ and $\succ^{LX} \subseteq \succ$. Take any $u_1, v_1 \in X$. If $u_1 \sim^{LX} v_1$, then $T_1 + u_1 = T_1 + v_1 \forall T > \tilde{T}$, so FA implies $u_1 \sim v_1$.

Next, we show that $u_1 \succ v_1$ whenever $u_1 \succ^{LX} v_1$. Thus, suppose that $u_1 \succ^{LX} v_1$. Take any $T \geq \tilde{T}$ and consider the vector $w = (u_{T+1}, T+1)$. We want to show that $v_1 > w_1$. By FA and transitivity, we can consider $w_1 = (u_{T+1}, T+1)$ and $v_1 = (v_{T+1}, T+1)$. Suppose that $w_1 \succ v_1$. We distinguish two cases.

Case 1. $w_1 \succ v_1$
By SPO it follows that $\tilde{v}_l > \tilde{w}_l$, some $l < t \leq T$. Let

$$k = \max \{1 \leq l < t|\tilde{v}_l > \tilde{w}_l\}.$$ 

By FA, let $w_i = \tilde{w}_k$ and $v_i = \tilde{v}_k + g$ for some $0 < g \leq t - k$ with $\tilde{w}_k + g > \tilde{v}_k + g$. Let $d_1, d_2 > 0$, and consider vectors $w', v'$ formed from $w, v$ as follows: $\tilde{w}_k + g$ is raised to $\tilde{v}_k + g + d_1$ such that $\tilde{w}_k + g > \tilde{v}_k + g + d_1$; $\tilde{w}_k$ is raised to $\tilde{w}_k + d_2$ such that $\tilde{v}_k + g + d_1 > \tilde{w}_k + d_2 > \tilde{v}_k$; and all other entries of $v$ and $w$ are
unchanged. By FA, consider \( \tilde{w}' = (\tilde{w}'_{T:T+1} v) \) and \( \tilde{v}' = (\tilde{v}'_{T:T+1} v) \). By construction \( \tilde{w}'_j \leq \tilde{v}'_j \) for all \( T \geq j \geq k \), with \( \tilde{w}'_{k+g} > \tilde{v}'_{k+g} \) and \( \tilde{w}'_{k} > \tilde{v}'_{k} \). IBP implies \( \tilde{v}' \succ \tilde{w}' \), and by SPO \( d_1, d_2 \) can be chosen so that \( \tilde{v}' \succ \tilde{w}' \), without loss of generality. Consider two cases:

a) Suppose that \( \tilde{v}_k > \tilde{w}_k \), but \( \tilde{v}_l \geq \tilde{w}_l \) for all \( l < k \). It follows that \( \tilde{w}' > \tilde{v}' \), and so SPO implies that \( \tilde{w}' \succ \tilde{v}' \), a contradiction.

b) Suppose that \( \tilde{v}_l > \tilde{w}_l \) for some \( l < k \). Note that by construction \( \tilde{v}_l' = \tilde{v}_l \) and \( \tilde{w}_l' = \tilde{w}_l \) for all \( l < k \). Then, let

\[
k' = \max\{1 \leq l < k \mid \tilde{v}_l' > \tilde{w}_l' \}.
\]

The above argument can be applied to \( \tilde{w}'_l, \tilde{v}'_l \) to derive vectors \( \tilde{w}''_l, \tilde{v}''_l \) such that \( \tilde{w}''_j \geq \tilde{v}''_j \) for all \( j \geq k' \), whereas by IBP and SPO \( \tilde{w}'' \succ \tilde{v}'' \).

And so on. After a finite number of iterations \( q \), two vectors \( \tilde{w}^q, \tilde{v}^q \) can be derived such that, by IBP and SPO, \( \tilde{w}^q \succ \tilde{v}^q \) but, by SPO, \( \tilde{w}^q \succ \tilde{v}^q \), yielding the desired contradiction.

**Case 2.** \( \tilde{v} \sim \tilde{w} \)

By assumption, \( \tilde{v}_l < \tilde{w}_l \equiv \tilde{w}_l \). Therefore, define \( \tilde{w}'_l \) as follows: \( \tilde{w}'_l = \tilde{w}_l \forall \tau \in \mathbb{N}\setminus\{t\} \) and \( \tilde{w}'_l = \tilde{w}_l - \epsilon > \tilde{v}_l \), some \( \epsilon > 0 \). By SPO and transitivity, it follows that \( \tilde{v} \succ \tilde{w}' \) but \( \tilde{w}' \succ LX \tilde{v} \). Hence, the argument of **Case 1** above can be applied to \( \tilde{v} \) and \( \tilde{w}' \), yielding the desired contradiction.

As \( \tilde{v} \not\sim \tilde{w} \) WC implies \( \tilde{w} \succ \tilde{v} \). FA and transitivity imply that \( (1u_T, T+1v) \succ 1v \). Since this is true for any \( T \geq \tilde{T} \), WPC implies \( 1u \succ 1v \).

**Theorem 4.2** has an interesting theoretical implication. Consider the following axiom of Non-Interference, which incorporates the normative intuitions behind HP and IBP in a unified liberal framework, and generalises the principle of Non-Interference proposed by ([7]) to economies with an infinite number of agents.

**NON-INTERFERENCE**, NI: \( \forall 1u, 1v, 1u' \in X : \exists T \geq 1u = (1u_T, T+1v) \succ 1v, \) and \( 1u' \) and \( 1v' \) are such that, \( \exists i \leq T, \)

\[
(u'_i - u_i) (v'_i - v_i) > 0
\]

\[
u'_j = u_j \forall j \neq i
\]

\[
v'_j = v_j \forall j \neq i
\]
implies $1u' \succ 1v'$ whenever $u'_i > v'_i$.

As is well-known, there exists no SWO defined on an infinite bounded set of real vectors which satisfies Anonymity and SPO (see, [4]). Theorems 3.2 and 4.2 imply that there is no weakly complete SWR that satisfies FA, SPO, WPC, and NI.

**Theorem 4.3.** There exists no SWR on $X$ that satisfies FA, SPO, WPC, WC and NI.

**Proof.** By contradiction. Let $\nu, \mu \in \mathbb{R}$ with $\nu > \mu$ and consider vectors $1u, 1v \in X$ such that $1u = \text{con} \mu$ and $1v = (v_1, 2v)$, where $v_1 < \mu$ and $2v = \text{con} \nu$. By Theorem 3.2, $1u \gtrsim LM 1v$, so that $1u \succ 1v$, but by Theorem 4.2, $1v \gtrsim LX 1u$, so that $1v \succ 1u$, a contradiction. □

**Conclusions**

This paper analyses liberal axioms for SWRs in societies with an infinite number of agents. The leximax SWR is characterised by appealing to the Individual Benefit Principle, which incorporates a liberal, non-interfering view of society. This result is interesting per se, since it provides the first characterisation of the leximax in economies with an infinite number of agents, and because the IBP has no obvious anti-egalitarian content. It also has relevant implications for liberal approaches to social choice. For it allows us to show that there exists no weakly complete SWR that satisfies the standard axioms of Finite Anonymity, Strong Pareto, a weak requirement on continuity, and the liberal principle of Non-Interference.

**References**


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1. Appendix: Independence of axioms

Let \( \Pi \) be the set of all finite permutations.

For an example violating only \textbf{FA}, define \( \triangleright \) on \( X \) in the following way: \( \forall x, y \in X \)

1) \( 1x = 1y \Rightarrow 1x \sim 1y \)
2) \( 1x \neq 1y \) and \( 1x = \pi(1y) \Rightarrow \exists \pi \in \Pi : 1u \neq 1v \) and \( 1v \neq 1u \)
3) \( 1x \neq 1y \) and \( 1x \neq \pi(1y) \) \( \forall \pi \in \Pi : 1x \triangleright^{LX^*} 1y \Rightarrow 1x \triangleright 1y. \)

The SWR \( \triangleright \) on \( X \) is not an extension of the leximin \( \triangleright^{LX^*} \). The SWR \( \triangleright \) on \( X \) satisfies all properties except \textbf{FA}.

For an example violating only \textbf{SPO}, for all \( x, y \in X \), define \( \triangleright \) on \( X \) in the following way: \( 1x \sim 1y \). The SWR \( \triangleright \) on \( X \) is not an extension of the leximax \( \triangleright^{LX^*} \). Clearly, the SWR \( \triangleright \) on \( X \) satisfies all properties except \textbf{SPO}.

For an example violating only \textbf{WC}, for all \( x, y \in X \), define \( \triangleright \) on \( X \) in the following way: \( 1x \triangleright 1y \) if \( 1x > \pi(1y) \exists \pi \in \Pi; 1x \sim 1y \) if \( 1x = \pi(1y) \exists \pi \in \Pi \); and \( 1x \neq 1y \) and \( 1x \neq \pi(1y) \). Clearly, \( 1x \triangleright 1y \) and \( 1y \neq 1x \) if \( 1x \neq 1y \). \( 1y \neq \pi(1y) \) and \( 1x \neq \pi(1y) \) \( \forall \pi \in \Pi \). The SWR \( \triangleright \) on \( X \) is not an extension of the leximax \( \triangleright^{LX^*} \). Clearly, the SWR \( \triangleright \) on \( X \) satisfies all properties except \textbf{WC}.

For an example violating only \textbf{WPC}, define \( \triangleright \) on \( X \) in the following way: \( \forall x, y \in X \)

\[ \exists T \geq 1 \text{ s.t. } \forall x \neq 1y \text{ and } 1y \neq 1x \Rightarrow 1x \neq 1y \text{ and } 1y \neq 1x, \]

otherwise,

\[ 1x \triangleright^{LX^*} 1y \Rightarrow 1x \triangleright 1y. \]

\( \triangleright \) on \( X \) is a SWR. Fix \( \mu, \nu, \epsilon, \delta \in \mathbb{R} \), with \( \mu > \delta > \epsilon > \nu \). Let \( 1x = (\delta, \text{con}\epsilon) \) and \( 1y = (\mu, \text{con}\nu) \). Clearly, \( 1x, 1y \in X \), and \( (1yT, T+1x) \triangleright^{LX^*} 1x \forall T \geq 2, 1y \triangleright^{LX^*} 1x \), but \( 1x \neq 1y \) and \( 1y \neq 1x \). It follows that the SWR \( \triangleright \) on \( X \) is not an extension of the leximax SWR. The SWR \( \triangleright \) on \( X \) satisfies all properties except \textbf{WPC}.

For an example violating only \textbf{IBP*}, define \( \triangleright \) on \( X \) in the following way: \( \forall x, y \in X \)

\[ 1x \sim 1y \Leftrightarrow \exists \bar{T} \geq 1 \text{ s.t. } \forall T \geq \bar{T} : 1xT = 1yT, \]

and

\[ 1x \triangleright 1y \Leftrightarrow \exists \bar{T} \geq 1 \text{ s.t. } \forall T \geq \bar{T} : \exists \bar{t} \in \{1, \ldots, T\} \bar{u}_s = \bar{v}_s (\forall 1 \leq s < t) \text{ and } \bar{u}_t > \bar{v}_t. \]

\( \triangleright \) on \( X \) is a SWR (i.e., the w-leximin SWR). It follows that the SWR \( \triangleright \) on \( X \) is not an extension of the leximax SWR. The SWR \( \triangleright \) on \( X \) satisfies all properties except \textbf{IBP*}.

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