Non-cooperative Bargaining and the Incomplete Information Core

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Abstract

We consider information transmission in the core of an exchange economy with incomplete information by non-cooperative bargaining theory. Reformulating the coalitional voting game by Serrano and Vohra [Information transmission in coalitional voting games, J. of Economic Theory (2007), 117-137] so that an informed agent proposes an allocation, we define a notion of the informational core. A coalition has an informational objection to the status-quo allocation if and only if there exists an equilibrium rejection in the coalitional voting game. We present a non-cooperative sequential bargaining game in which coalitional voting games are repeated, and prove that a refinement of a sequential equilibrium of the bargaining game necessarily yields an allocation in the informational core.

JEL classification: C71; C72; D51; D82
Key words: core, exchange economy, incomplete information, information transmission, non-cooperative bargaining

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1 Introduction

This paper considers the problem of resource allocations under uncertainty where agents have different information on a true state of an economy when they negotiate for a contract.[1] In the case of complete information where agents have no uncertainty about the state of an economy, the core has been the most fruitful solution concept to analyze coalitional bargaining on resource allocations. Roughly, the core is defined as the set of allocations to which no coalition of agents objects. A coalition is supposed to object to an allocation if all its members can improve upon their utility by re-allocating their endowments.

Wilson (1978) extends the notion of the core to an economy with incomplete information. Since his seminal paper, there has been a large volume of literature to explore an appropriate definition of the core for an economy with incomplete information. Forges et al. (2002) provide an excellent survey on the topic. In this paper, the core will be an *interim* concept in that agents evaluate allocations, given their private information.

A central issue in defining the core under incomplete information is that when a coalition attempts to object to some allocation, the objection itself may reveal members' private information and, as a result, agents do not agree to object to the allocation based on a new information. To deal with the issue of information leakage, Wilson employs two distinct approaches. The *coarse core* is based on the assumption that a coalition may object to an allocation if and only if it is commonly known by its members that they are better-off by objection. Under the requirement of common knowledge, an objection does not reveal agents' private information. The *fine core* is based on the assumption that a coalition may utilize unlimited communication among agents to make

[1] In this paper, we will employ the model of Wilson (1978) in which a true state of the economy is commonly known and verifiable when a contract is implemented. Therefore, incentive constraints are irrelevant to a feasible contract of allocations. In Section 5, we will discuss how our result can be extended to the case of unverifiable states where incentive constraints as well as physical constraints are imposed on a feasible allocation.

an objection.[2]

Recently, several authors refine the coarse core by allowing information transmission in the process of contracting, and relax the requirement of common knowledge in a coarse objection. Lee and Volij (2002) introduce the *coarse+ core* where a blocking coalition may include a subgroup of agents who are better-off by objecting against a status-quo allocation for every possible state of an economy. Even if some information may be leaked to such agents in negotiations, their willingness to objection is unchanged since the objection is a dominant action to them. Dutta and Vohra (2005) weaken a coarse objection and propose the notion of the *credible core* which is based on the idea that a coalition can object to a status-quo allocation over an event that can be credibly inferred from the act of objection itself. Serrano and Vohra (2007) present a non-cooperative support to the notion of a credible objection as a Bayesian equilibrium of a coalitional voting game.[3]

The purpose of this paper is to develop a non-cooperative approach to the core under incomplete information employed by Serrano and Vohra (2007). Our study is motivated by two points explained as follows. First, in Serrano and Vohra's (2007) model of a coalitional voting game, a proposal is made by an uninformed mediator, not by an informed agent. All members in a coalition vote simultaneously to discard a status-quo allocation in favor of a proposal. The proposal is accepted by unanimity. The model does not capture well an important aspect of a negotiation process that a proposal may transmit some private information of the proposer to responders. Since the members can coordinate their voting on any admissible event with help of the mediator's proposal in a Bayesian equilibrium, the credible core coincides with the fine

[2]In the fine core, communication is *direct* in the sense that information is transferred through messages. Communication changes agents' information structures which determine their permissible strategies. In contrast, this paper considers information transmission through observed actions, which may be regarded as *indirect* communication. For a recent study on a relationship between direct communication and the core, see Volij (2000).

[3]While Dutta and Vohra (2005) and Serrano and Vohra (2007) consider the credible core in the case of unverifiable states, the issue of endogenous information transmission is relevant even in the case of verifiable states as our analysis shows.

core in the case of verifiable states. Second, a coalitional voting game is not a whole process of negotiations in the sense that a status-quo allocation is exogenously given. The coalitional voting game approach is a preliminary step to consider a question how the incomplete information core can be supported as a non-cooperative equilibrium of some suitable bargaining model without a mediator.[4]

The results of this paper are summarized as follows. First, we reformulate Serrano and Vohra's (2007) coalitional voting game in the way that an informed agent may propose an alternative allocation against a status-quo allocation. If the proposal is made, all other members either accept or reject it sequentially. The proposal is agreed by unanimity. Based on the voting game, we introduce a new type of objection, called an *informational objection*, which prescribes that all members of a coalition will be better-off over a self-selection event that a proposer's private information is credibly transmitted to responders. We present a non-cooperative support to the informational objection in terms of a sequential equilibrium of the coalitional voting game. Second, we present a non-cooperative sequential bargaining model in which the coalitional voting games are repeated, and prove that an allocation belongs to the informational core of an economy if it is agreed (with probability one) in a stationary equilibrium of the bargaining game which satisfies (i) payoff-oriented response, (ii) self-selection, and (iii) no end-effect. The converse holds for a stronger notion of the informational core.

The remainder of the paper is organized as follows. Section 2 offers preliminaries. Section 3 introduces a coalitional voting game and gives a non-cooperative support to an informational objection. Section 4 presents a non-

[4]It should be noted that our non-cooperative bargaining approach is different from the competitive screening one of de Clippel (2007). de Clippel considers a competitive screening game in which at least two uninformed intermediaries offer simultaneously contracts to each agent, and the agent chooses one contract among those offered based on his private information. It is proved that the set of allocations supported by subgame perfect equilibria in the game coincides with a subset of the coarse core called the type-agent core. In the competitive screening game, each agent is only faced with a one-person choice problem. The problem of coalitional bargaining among agents underlying the core is out of consideration.

cooperative sequential bargaining game for an economy with incomplete information and provides its equilibrium analysis. Section 5 discusses the results.

2 Preliminaries

We consider an exchange economy with incomplete information. Let Ω be the set of possible states. We assume that Ω is finite. A subset E of Ω is called an *event*. $N = \{1, 2, \cdots, n\}$ is the set of players. A subset S of N is called a *coalition* of players. For each state $\omega \in \Omega$, the consumption set of player i is denoted by $X_i(\omega)$, which is a subset of the non-negative orthant R^l_+ of the l-dimensional Euclidean space. For simplicity, we assume that $X_i(\omega) = R^l_+$ for all $\omega \in \Omega$. The endowment $e_i(\omega)$ of player i when ω is a prevailing state is one element of R^l_+. We denote by $u_i : R^l_+ \times \Omega \to R$ player i's state-dependent von Neumann-Morgenstern utility function. Each player i receives utility $u_i(x, \omega)$ when he consumes a commodity bundle $x \in R^l_+$ at ω. We assume that $u_i(x, \omega) \geq u_i(0, \omega)$ for all $x \in R^l_+$ and all $\omega \in \Omega$. Let π, a probability distribution on Ω, denote the common *prior* of players. The probability judgement that a state ω prevails is denoted by $\pi(\omega)$. With no loss of generality, we assume that $\pi(\omega) > 0$ for all ω. The *posterior* belief $\pi_{|E}$ given an event E is defined by

$$\pi_{|E}(\omega) = \frac{\pi(\omega)}{\sum_{\omega' \in E} \pi(\omega')} \quad \text{for all } \omega \in E. \tag{2.1}$$

The information of player i is described by a field $\mathcal{F}_i$ of events which he can discern.[5] For an event E, $E \in \mathcal{F}_i$ means that player i knows whether the prevailing state is in the event E or in the complementary event E^c. $\mathcal{PF}_i$ denotes the finest partition of Ω contained in $\mathcal{F}_i$. $\mathcal{PF}_i(\omega)$ denotes the unique member of $\mathcal{PF}_i$ containing ω. We refer to $\mathcal{PF}_i(\omega)$ as player i's information

[5] A class $\mathcal{F}$ of events in Ω is called a field if (i) $\phi \in \mathcal{F}$, (ii) $A \in \mathcal{F}$ and $B \in \mathcal{F}$ imply $A \cup B \in \mathcal{F}$, and (iii) $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$.

at ω. An (exchange) economy with incomplete information is defined by $\mathcal{E} = (\Omega, \pi, \{u_i, e_i, \mathcal{F}_i\}_{i \in N})$.

For a coalition S, the field of events discernible to every player in S is given by the *coarse* field $\wedge_{i \in S} \mathcal{F}_i = \cap_{i \in S} \mathcal{F}_i$. The coarse field $\wedge_{i \in S} \mathcal{F}_i$ is the maximal field contained in all fields $\mathcal{F}_i$ $(i \in S)$. An event E is called *common knowledge within* S if all players in S can discern it, that is, $E \in \wedge_{i \in S} \mathcal{F}_i$. If E is a common knowledge within S, then E is described as a disjoint union of events in $\mathcal{P}\mathcal{F}_i$ for every $i \in S$. The *fine* field $\vee_{i \in S} \mathcal{F}_i$ is the minimal field of Ω containing all fields $\mathcal{F}_i$ $(i \in S)$. If all members in the coalition S can pool their private information, they could discern all events E in the fine field $\vee_{i \in S} \mathcal{F}_i$.

A *consumption bundle* for player i is a function $x_i : \Omega \to R^l_+$ that assigns a consumption vector $x_i(\omega) \in R^l_+$ to each state ω. For a consumption bundle x_i, we define the function $u_i(x_i) : \Omega \to R$ by $u_i(x_i)(\omega) = u_i(x_i(\omega), \omega)$. The conditional expected utility of player i for a consumption bundle x_i relative to $\mathcal{F}_i$ is an $\mathcal{F}_i$-measurable function $E(u_i(x_i)|\mathcal{F}_i) : \Omega \to R$, which is defined by

$$E(u_i(x_i)|\mathcal{F}_i)(\omega) = \sum_{\omega' \in I} \pi_{|I}(\omega') u_i(x_i)(\omega'), \quad I = \mathcal{P}\mathcal{F}_i(\omega) \tag{2.2}$$

for every $\omega \in \Omega$.

The trading process in an economy runs as follows: At date 0, a state $\omega \in \Omega$ is realized, and all players observe their private information $\mathcal{P}\mathcal{F}_i(\omega)$. At date 1, players negotiate to form a coalition S and to make a contract of consumption bundles $x = (x_i)_{i \in S}$ for its members. Several coalitions may form. During the negotiation, the members of S may reveal their private information through actions. At date 2, players may receive additional information with a new field $\mathcal{F}'_i \supset \mathcal{F}_i$ and the contract is implemented.

As in Wilson (1978), we consider the case that the true state becomes publicly known and is verifiable at the date of implementing the contract. In this case, $\mathcal{F}'_i = 2^\Omega$ (the set of all subsets of Ω) for all $i \in N$. Vohra (1999) extends the analysis of Wilson (1978) to the case that the true state is unverifiable at the date of contract implementation. We will discuss how our result can be extended to the case of unverifiable states in Section 5.

In order to define the core of an economy with incomplete information, we need to specify what each coalition can do for its members without cooperation of other players. When the true state is verifiable, a *feasible allocation* for coalition S is defined to be a collection $x = (x_i)_{i \in S}$ of consumption bundles for its members satisfying the physical constraint that $\sum_{i \in S} x_i(\omega) \leq \sum_{i \in S} e_i(\omega)$ for every $\omega \in \Omega$. The set of feasible allocations for coalition S is given by

$$A^S = \{x = (x_i)_{i \in S} : \Omega \to R_+^{ls} \mid \sum_{i \in S} x_i(\omega) \leq \sum_{i \in S} e_i(\omega) \text{ for all } \omega \in \Omega\} \tag{2.3}$$

where s is the cardinality of S. In what follows, a feasible allocation $x \in A^S$ for coalition S is simply called an *S-allocation*.

In the case of verifiable states, the set A^S of S-allocations makes sense since the true state is publicly known at the date of implementing contracts of allocations. On the other hand, when the true state is unverifiable, Vohra (1999) requires that a feasible S-allocation be incentive-compatible so that players are motivated to report true information to an enforcement agency.

Once the set of feasible S-allocations is determined, we can define a family of the core of an economy with incomplete information, depending on the extent of communication permitted within a coalition.

Definition 2.1. (Wilson 1978)

(1) A coalition S has a *coarse objection* to an N-allocation $x \in A^N$ if there exists a common knowledge event E within S and an S-allocation $y^S \in A^S$ such that

$$E(u_i(y_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega) \tag{2.4}$$

for all $i \in S$ and all $\omega \in E$.

(2) The *coarse core* is the set of all N-allocations to which no coalition has a

coarse objection.

(3) Let $C(S)$ be the set of all collections $(\mathcal{H}_i)_{i\in S}$ of fields such that $\mathcal{F}_i \subset \mathcal{H}_i \subset \vee_{i\in S}\mathcal{F}_i$. A coalition S has a *fine objection* to an N-allocation $x \in A^N$ if there exists some $(\mathcal{H}_i)_{i\in S} \in C(S)$ such that S has a coarse objection to x with respect to $(\mathcal{H}_i)_{i\in S}$.

(4) The *fine core* is the set of all N-allocations to which no coalition has a fine objection.

The coarse core is based on the idea that a coalition is permitted to object to the status-quo over an event if and only if the event is commonly known to all members of the coalition. One possible rationale for the requirement of common knowledge is that if not, then the act of objection itself may leak some members' private information and others may conclude that the proposed objection is not desirable. This problem of adverse selection does not occur if the objection yields no leakage of information. An objection does not reveal any information if and only if it is common knowledge within a coalition that all its members are better-off by the objection than the status-quo. In contrast to the coarse core, the fine core is based on the assumption that a coalition may utilize unlimited communication among agents to make an objection.

Recently, several authors pose some criticism to the traditional core concepts under incomplete information. Lee and Volij (2002) and Dutta and Vohra (2005) argue that the common knowledge restriction is too demanding in the coarse core. It may be the case that informational leakage does not alter members' preference to block the status-quo. The next example is due to Lee and Volij (2002).

Example 2.1. Consider a two-agent economy with two commodities. Let $\Omega = \{\omega_1, \omega_2\}$ be the set of states. A common prior π is given by $\pi(\omega_1) = \pi(\omega_2) = 1/2$. Agents have differential information: $\mathcal{PF}_1 = [\{\omega_1, \omega_2\}]$ and $\mathcal{PF}_2 = [\{\omega_1\}, \{\omega_2\}]$, and have identical state-independent utility functions $u(a, b) = \min(a, b)$. Table 2.1 shows the endowment $e = (e_1, e_2)$ and an allocation $y =$

(y_1, y_2).

Agents	$\mathcal{PF}_i$	endowment e		allocation y	
		w_1	w_2	w_1	w_2
1	$[\{w_1, w_2\}]$	$(2, 0)$	$(2, 0)$	$(3, 1)$	$(1, 1)$
2	$[\{w_1\}, \{w_2\}]$	$(1, 1)$	$(0, 2)$	$(0, 0)$	$(1, 1)$

Table 2.1: A two-agent economy with a coarse+ objection

The endowment e belongs to the coarse core since Ω is the only common knowledge event and agent 2 receives the highest utility at ω_1. Is the endowment e a reasonable outcome in this economy? Suppose that agent 2 proposes y at ω_2. Agent 1 prefers y to e, regardless of the state. Thus, agent 1 accepts agent 2's proposal y, and as a result, agent 2 will be better-off. Since agent 2 does prefer e to y at ω_1, his proposal y reveals credibly to agent 1 that the true state is ω_2. This information revelation, however, does not alter agent 1's preference for the allocation y.

Motivated by this example, Lee and Volij (2002) introduce the following refinement of the coarse core.

Definition 2.2. A coalition S has a *coarse+ objection* to an N-allocation $x \in A^N$ if there exist an S-allocation $y^S \in A^S$, a partition $\{A, P\}$ of S and a common knowledge event E within A such that

(1) $u_i(y_i^S)(\omega) > u_i(x_i)(\omega)$ for all $i \in P$ and all $\omega \in \Omega$,

(2) $E(u_i(y_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega)$ for all $i \in A$ and all $\omega \in E$.

The *coarse+ core* is the set of all N-allocations to which no coalition has a coarse+ objection.

In Example 2.1, agent 2 has a coarse+ objection y to the endowment e. The example shows that informational revealing does not contradict agents' objection, and thus we should weaken the notion of a coarse objection so that it can allow endogenous information transmission. It, however, seems to be unreasonable to assume that unlimited communication is possible within a coalition as the fine core does. The next example illustrates this point.

Example 2.2. Consider again an economy in Example 2.1. Table 2.2 shows a new endowment $e = (e_1, e_2)$ and an allocation $y = (y_1, y_2)$.

Agents	$\mathcal{PF}_i$	endowment e		allocation y	
		w_1	w_2	w_1	w_2
1	$[\{w_1, w_2\}]$	$(2, 0)$	$(2, 2)$	$(1, 1)$	$(1, 0)$
2	$[\{w_1\}, \{w_2\}]$	$(1, 3)$	$(1, 1)$	$(2, 2)$	$(2, 3)$

Table 2.2: A two-agent economy with a fine objection

In the example, agent 1 prefers e to y over the event $\{w_1, w_2\}$, whereas agent 2 prefers y to e, no matter of which a true state is, w_1 or w_2. Thus, neither e nor y is a coarse objection to the other. However, y is a fine objection to e over the event $\{w_1\}$. If Agent 2 could transmit his private information $\{w_1\}$ credibly to agent 1, then agent 1 accepts y. But, how can he do this? Since agent 2 prefers y to e on either event, the proposal y itself does not transmit any information to agent 1. Furthermore, agent 2 has an incentive to send a false information $\{w_1\}$ when a true state is w_2, and thus agent 1 can not trust agent 2's massage $\{w_1\}$.

Examples 1 and 2 pose a critical question: what kind of information agents can use credibly to organize an objection? To answer this question, we need a theory of endogenous information transmission in bargaining with incomplete information.

To consider this issue, we take the same viewpoint as Serrano and Vohra (2007, p.118). They argue that "the non-cooperative equilibrium theory is ideally suited to deal with the question of how much private information agents transmit to each other." Based on a Bayesian equilibrium of a coalitional voting game, Serrano and Vohra (2007) introduce the notion of a credible objection from which agents can infer each other's private information in a credible way. In the case of verifiable states where incentive constraints are irrelevant, the credible core turns out to be equal to the fine core. This result is due to the special rule of their voting game that an objection is proposed by an uninformed mediator, not by an informed agent. Since the mediator's proposal helps agents to coordinate their voting behavior over any admissible event, the fine objection is supported by agents' equilibrium behavior.

While the coalitional voting game approach is very useful to the study of the core with incomplete information, it is a preliminary step to a non-cooperative bargaining theory for the core. The next step is to develop a coalitional bargaining model without a mediator. By this reason, we will reformulate the voting game of Serrano and Vohra (2007) in the next section in the way that a privately informed agent proposes an objection against a status-quo allocation, and will consider how much information can be transmitted in the process of negotiations among agents.

3 The Informational Core

In this section, we consider a situation in which a coalition votes to make an objection to a status quo allocation. Let $x \in A^N$ be the status quo allocation, and let $y^S \in A^S$ be a feasible allocation for a coalition S, which is a candidate of an objection to x. A voting game for S has the following rule. First, a state $\omega \in \Omega$ is realized. Given his private information $\mathcal{PF}_i(\omega)$, one particular member $i \in S$ decides to propose $y^S \in A^S$ against the status quo x, or not. If not, the status quo x prevails. If y^S is proposed, then all other members in S either accept or reject it sequentially according to some fixed order. The order is irrelevant to our results. If all them accept y^S, then it is agreed. Otherwise, the status quo x prevails. This sequential voting game is denoted by $\Gamma^x(S, i, y^S)$.

The extensive form of $\Gamma^x(S, i, y^S)$ is given in Figure 3.1 when $S = \{1, 2\}$, $i = 1$, $\Omega = \{\omega_1, \omega_2\}$ and agents have differential information: $\mathcal{PF}_1 = [\{\omega_1\}, \{\omega_2\}]$ and $\mathcal{PF}_2 = [\{\omega_1, \omega_2\}]$.

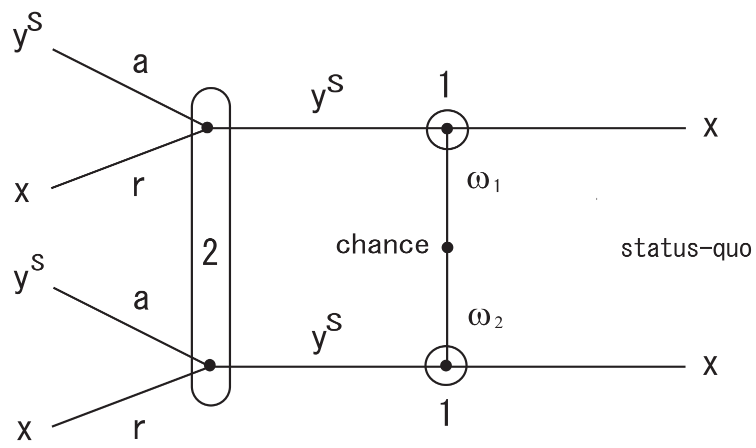

Figure 3.1. An extensive form of the voting game $\Gamma^x(S, 1, y^S)$

A (pure) strategy σ_i for proposer i in $\Gamma^x(S, i, y^S)$ is an $\mathcal{F}_i$-measurable function from Ω to $\{x, y^S\}$. Similarly, a strategy σ_j for responder j is an $\mathcal{F}_j$-measurable function from Ω to $\{accept, reject\}$. Notice that there is a natural one-to-one correspondence between each of player i's information set in the extensive form of $\Gamma^x(S, i, y^S)$ and an element of his information partition $\mathcal{PF}_i$ of the state space Ω.

The equilibrium concept that we employ for the voting game $\Gamma^x(S, i, y^S)$ is a sequential equilibrium (Kreps and Wilson 1982). A *sequential equilibrium* of $\Gamma^x(S, i, y^S)$ is a pair (σ, μ) where $\sigma = (\sigma_j)_{j \in S}$ is a strategy profile for members in coalition S and μ is a belief system which assigns to every information set I of every player in $\Gamma^x(S, i, y^S)$ his belief $\mu(I)$ on I, a probability distribution over the set of all nodes in I. Roughly, (σ, μ) is a sequential equilibrium of $\Gamma^x(S, i, y^S)$ if every player's strategy is a best response to all others' strategies at each of his information set under his belief about the state, where the belief system μ should be consistent with the strategy profile σ (and a slight deviation from it off equilibrium play) by the Bayes' rule. Since the notion of a sequential equilibrium is standard, we omit a precise definition of it.

Based on the voting game $\Gamma^x(S, i, y^S)$, we introduce a new type of an objection, called an informational objection, which takes into account the equilibrium revealing of the proposer's private information. The next example illustrates the idea of it.

Example 3.1. Consider again an economy in Example 2.2. Table 3.1 shows a new allocation $y = (y_1, y_2)$.

Agents	$\mathcal{PF}_i$	endowment e w_1	endowment e w_2	allocation y w_1	allocation y w_2
1	$[\{w_1, w_2\}]$	$(2,0)$	$(2,2)$	$(1,1)$	$(3,0)$
2	$[\{w_1\}, \{w_2\}]$	$(1,3)$	$(1,1)$	$(2,2)$	$(0,3)$

Table 3.1: A two-agent economy with an informational objection

As in Example 2.2, the coalition $\{1,2\}$ has a fine objection y to e over the event $\{w_1\}$. Unlike Example 2.2, agent 2 prefers y to e on the event $\{w_1\}$, not on the event $\{w_2\}$. If agent 1 knows this fact, he can rationally infer from the proposal y by agent 2 that a true state must be w_1. Thus, agent 2 can transmit his private information w_1 credibly to agent 1. In this case, we say that the coalition $\{1,2\}$ has an informational objection to the endowment e.

The idea of an informational objection is formalized as follows.

Definition 3.1. A coalition S has an *informational objection* to an N-allocation $x \in A^N$ if there exist an S-allocation $y^S \in A^S$, a member $i \in S$ and an event $E \in \mathcal{F}_i$ such that

(1) $E(u_i(y_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega)$ for all $\omega \in E$,

(2) $E(u_i(y_i^S)|\mathcal{F}_i)(\omega) \leq E(u_i(x_i)|\mathcal{F}_i)(\omega)$ for all $\omega \notin E$,

(3) $E(u_j(y_j^S)|\mathcal{PF}_j(\omega) \cap E) > E(u_j(x_j)|\mathcal{PF}_j(\omega) \cap E)$ for all $j \in S, j \neq i$ and all $\omega \in E$.

The *informational core* is the set of all N-allocations to which no coalition has an informational objection.

The notion of an informational objection can be explained as follows. Condition (1) means that proposer i prefers a proposal y^S to the status quo x over an event $E \in \mathcal{F}_i$. Condition (2) is that of self-selection, namely, it enables the proposer to reveal to responders credibly that the true state belongs to the event E. Condition (3) means that if the proposer offers y^S over the event E, then all responders prefer y^S to the status quo x on each of their possible information set $\mathcal{PF}_j(\omega)$ $(\omega \in E)$, inferring that the true state is in $\mathcal{PF}_j(\omega) \cap E$. Thus, if player i actually proposes y^S on E, then all responders accept it, given their updated beliefs. That is, the status-quo allocation x is objected over the event E.

Remark 3.1. Traditionally, the core is defined in terms of strict inequality as in Definition 3.1.(1) and (3). To be consistent with this tradition, we will strengthen the definition of a sequential equilibrium in the voting game $\Gamma^x(S, i, y^S)$ so that the proposer makes a proposal against the status-quo only if he is strictly better-off by doing so. The same thing should be applied to every responder's acceptance.

Dutta and Vohra (2005) first introduce the possibility of endogenous informational transmission into the concept of an objection under incomplete information. Their notion of the credible objection, however, is equivalent to a fine objection of Wilson (1978) when the state is verifiable and thus incentive constraints are irrelevant. An important difference between Dutta and Vohra's credible objection and our informational objection is that the latter allows only the informational transmission from a proposer to responders, while the former does informational sharing among all members in a coalition.

The following proposition shows a relationship among various concepts of objection.

Proposition 3.1. Let S be a coalition and $x \in A^N$.

(1) If S has a coarse objection to x, then S has a coarse+ objection to x.

(2) If S has a coarse+ objection to x, then S has an informational objection to x.

(3) If S has an informational objection to x, then S has a fine objection to x.

Proof. (1): trivial. (2): Suppose that a coalition S has a coarse+ objection to $x \in A^N$. By Definition 2.2, there exist an S-allocation $y^S \in A^S$, a partition $\{A, P\}$ of S and a common knowledge event E within A such that

$$u_i(y_i^S)(\omega) > u_i(x_i)(\omega) \quad \text{for all} \quad i \in P \quad \text{and all} \quad \omega \in \Omega, \tag{3.5}$$

$$E(u_i(y_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega) \quad \text{for all} \quad i \in A \quad \text{and all} \quad \omega \in E. \tag{3.6}$$

Define an S-allocation $z^S \in A^S$ such that $z^S(\omega) = y^S(\omega)$ if $\omega \in E$ and $z^S(\omega) = 0$ if $\omega \notin E$. We will show that S has an informational objection z^S to x. Select any member $i \in A$. By the construction of z^S and (3.6), z satisfies conditions (1) and (2) in Definition 3.1. Since E is a common knowledge event within A, it holds that $P\mathcal{F}_j(\omega) \cap E = P\mathcal{F}_j(\omega)$ for all $\omega \in E$ and all $j \in A, j \neq i$. Thus, (3.6) implies condition (3) in Definition 3.1 for all $j \in A, j \neq i$. Finally, it is clear that (3.5) implies condition (3) in Definition 3.1 for all $j \in P$, too. (3): Suppose that S has an informational objection to x, and thus that Definition 3.1 holds. Define the collection $(\mathcal{H}_i)_{i \in S}$ of fields as follows. It holds that $\mathcal{H}_i = \mathcal{F}_i$ for proposer i, and that all other $\mathcal{H}_j$ are the coarsest fields including $\mathcal{F}_i$ and the event E. By definition, $E \in \wedge_{i \in S} \mathcal{H}_i$. Conditions (1) and (3) in Definition 3.1 imply that S has a coarse objection y to x with respect to $(\mathcal{H}_i)_{i \in S}$. Q.E.D.

We are now in a position to justify an informational objection as a sequential equilibrium of the voting game $\Gamma^x(S, i, y^S)$. As in Serrano and Vohra

(2007), we will call a sequential equilibrium (σ, μ) of the voting game $\Gamma^x(S, i, y^S)$ an *equilibrium rejection* of the status quo x if proposer i proposes y^S with positive probability and all other members in S accept it at all possible states.

Theorem 3.2. A coalition S has an informational objection to $x \in A^N$ if and only if there exists an equilibrium rejection of x in the voting game $\Gamma^x(S, i, y^S)$ for some S-allocation $y^S \in A^S$.

Proof. Suppose that a coalition S has an informational objection to $x \in A^N$. By Definition 3.1, there exist an S-allocation $z^S \in A^S$, a member $i \in S$ and an event $E \in \mathcal{F}_i$ such that

$$E(u_i(z_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega) \text{ for all } \omega \in E \tag{3.7}$$

$$E(u_i(z_i^S)|\mathcal{F}_i)(\omega) \leq E(u_i(x_i)|\mathcal{F}_i)(\omega) \text{ for all } \omega \notin E \tag{3.8}$$

and

$$E(u_j(z_j^S)|\mathcal{P}\mathcal{F}_j(\omega) \cap E) > E(u_j(x_j)|\mathcal{P}\mathcal{F}_j(\omega) \cap E) \tag{3.9}$$

for all $\omega \in E$ and all $j \in S, j \neq i$. Define $y^S \in A^S$ by

$$y^S(\omega) = \begin{cases} z^S(\omega) & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E. \end{cases}$$

Construct a pair (σ, μ) of strategies and belief in $\Gamma^x(S, i, y^S)$ as follows. First, the strategy σ_i of proposer i is defined by

$$\sigma_i(\omega) = \begin{cases} y^S & \text{if } \omega \in E \\ x & \text{if } \omega \notin E \end{cases} \tag{3.10}$$

(proposer i proposes y^S only over E), and strategies σ_j of responders $j \in S$

are defined by

$$\sigma_j(\omega) = \begin{cases} \mathit{accept} & \text{if } \mathcal{PF}_j(\omega) \cap E \neq \emptyset \\ \mathit{reject} & \text{if otherwise.} \end{cases} \tag{3.11}$$

Secondly, to every responder j's information set $\mathcal{PF}_j(\omega)$, the belief system μ assigns the posterior belief $\pi_{|\mathcal{PF}_j(\omega)\cap E}$ if $\mathcal{PF}_j(\omega) \cap E \neq \emptyset$, and otherwise, the posterior belief $\pi_{|\mathcal{PF}_j(\omega)}$.

We will prove that (σ, μ) is a sequential equilibrium of $\Gamma^x(S, i, y^S)$. By construction, it is clear that the belief system μ is consistent with σ. We next examine the optimal response of each responder j on his every information set $\mathcal{PF}_j(\omega)$. When $\mathcal{PF}_j(\omega) \cap E \neq \emptyset$, he receives the conditional expected payoff $E(u_j(z_j^S)|\mathcal{PF}_j(\omega) \cap E)$, given the belief system μ and all other responders' strategies if he accepts the proposal, and receives $E(u_j(x_j)|\mathcal{PF}_j(\omega) \cap E)$ otherwise. Thus, by (3.9), it is optimal for him to accept the proposal. When $\mathcal{PF}_j(\omega) \cap E = \emptyset$, it clearly follows from the construction of y^S that it is optimal for j to reject the proposal. Given the responders' strategies, it can be shown by (3.7) that (3.10) prescribes the proposer i's (unique) optimal choice on E. Proposer i is indifferent between choosing y^S and x outside E since y^S is rejected. Thus, (σ, μ) is an equilibrium rejection of x in $\Gamma^x(S, i, y^S)$.

Conversely, suppose that there exists an equilibrium rejection (σ, μ) of $x \in A^N$ in the voting game $\Gamma^x(S, i, y^S)$ for some S-allocation $y^S \in A^S$. Let $E \in \mathcal{F}_i$ be the event on which proposer i chooses y^S in equilibrium, receiving his private information $\mathcal{PF}_i(\omega)$. By the definition of an equilibrium rejection, all responders $j \in S$ accept it for all states in E. Define $z^S \in A^S$ by

$$z^S(\omega) = \begin{cases} y^S(\omega) & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E, \end{cases}$$

Since (σ, μ) is an equilibrium rejection of x, the event E is non-empty. The equilibrium condition for proposer i implies that

$$E(u_i(z_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega)$$

for all $\omega \in E$ (See Remark 3.1). Notice that all responders accept z^S at every $\omega \in E$. By the definition of z^S, it is clear that, for all $\omega \notin E$

$$E(u_i(z_i^S)|\mathcal{F}_i)(\omega) = 0 \leq E(u_i(x_i)|\mathcal{F}_i)(\omega).$$

Since all responders j accept y^S on E in (σ, μ), it must hold that, for all $\omega \in E$

$$\sum_{\theta \in \mathcal{PF}_j(\omega) \cap E} \pi(\theta)\ u_i(y_i^S)(\theta) > \sum_{\theta \in \mathcal{PF}_j(\omega) \cap E} \pi(\theta)\ u_i(x_i)(\theta) \tag{3.12}$$

(See Remark 3.1). Note that proposer i never chooses y^S outside the event E. (3.12) is equivalent to

$$E(u_j(z_j^S)|\mathcal{PF}_j(\omega) \cap E) > E(u_j(x_j)|\mathcal{PF}_j(\omega) \cap E)$$

for all $\omega \in E$. Thus, S has an informational objection z^S to x. Q.E.D.

The theorem justifies the notion of an informational objection in terms of a sequential equilibrium of the voting game $\Gamma^x(S, i, y^S)$. In equilibrium, the proposer attempts to object to the status-quo allocation if and only if he can obtain a higher conditional expected payoff, given his private information. This action may transmit some of the proposer's private information to responders and, as a result, they update their prior belief. The objection is accepted whenever it is proposed.

4 The Non-cooperative Bargaining Game

The voting game introduced in the last section is not a whole bargaining game which is played in an economy with incomplete information. For example, even if an initial proposer selects the endowment as the status quo allocation, he can not enforce it on other agents, in general. The others can continue their negotiations for allocations. Furthermore, the voting game has a very restricted feature of the ultimatum bargaining. The game stops once the proposal is rejected. In a general bargaining situation, negotiations may continue after rejection.

In this section, we will present a non-cooperative coalitional bargaining game which agents play to reach a contract of allocations. We continue to assume that a true state becomes publicly known and verifiable when a contract of allocation is implemented. Any feasible allocation satisfying the physical constraint is implementable.

In the literature, several authors have presented non-cooperative bargaining models for the core in the case of complete information. See Moldovanu and Winter (1995), Okada (1992), Okada and Winter (2003) and Perry and Reny (1994) among others. In what follows, we will extend the bargaining model studied by Okada (1992) and Okada and Winter (2003) to the case of incomplete information.

The bargaining game consists of a sequence of proposals, responses and counter-proposals. Let $\alpha = (i_1, i_2, \cdots, i_n)$ be a predetermined order over the player set N. The order α determines an initial proposer and the order of responders. The game is played over possibly infinitely many periods $t = 1, 2, \cdots$. Let N_t ($\subset N$) be the set of "active" players in period t who have not bound to any contract. Let $N_1 = N$. The bargaining game in period t has the following steps.

(1) Given his information $\mathcal{PF}_i(\omega)$ on a state $\omega \in \Omega$, the first player $i \in N_t$ (according to the order α) proposes a pair (S, x^S) where $i \in S \subset N_t$ and

$x^S \in A^S$.

(2) All other members in S either accept or reject the proposal (S, x^S) sequentially according to α. If they all accept it, then the coalition S quits the game with the agreement of x^S. Then, the game goes to the next period with the new set of active players, $N_{t+1} = N_t - S$, and the same process as in period t is repeated.

(3) If the proposal (S, x^S) is rejected by any player j, then player j can make a counter-proposal (T, y^T) where $T \subset N_t$ and $y^T \in A^T$ (with a possible exception explained below). The same rule as (2) is applied.

There exists an upper bound K (> 1) of successive proposals made in each period t.[6] If no agreement is made up to K proposals, then the game goes to the next period $t+1$ with $N_{t+1} = N_t$. Thereafter, the same rule as in period t is applied. In particular, the same initial proposer as in period t is selected. The game stops if and only if there remain no active players. If the game does not stop, all players outside coalitions receive their endowments $e_i(\omega)$. No players discount future utility.

We denote by $\Gamma(\mathcal{E})$ this bargaining game for an economy $\mathcal{E}$ with incomplete information. Whenever he makes a choice, every player has perfect information on past moves of players (including himself). The bargaining game $\Gamma(\mathcal{E})$ is regarded as an extension of the Rubinstein's two-person alternating bargaining model to an n-person cooperative game with incomplete information. The game has the special property that the bargaining process re-starts completely after a predetermined number of proposals have been rejected. By this reason, we call $\Gamma(\mathcal{E})$ a *sequential coalitional bargaining game with re-starts.*

The following story may be helpful to interpret the bargaining game $\Gamma(\mathcal{E})$. A market opens every day. The trading process runs as follows. One player publicly announces a coalition and an allocation for it. If all members of

[6] If $K = 1$, the bargaining model is just the repetition of the ultimatum bargaining with the same proposer. There is no opportunity for responders to make counter-proposals.

the coalition accept the allocation, then the contract of it is made, and the market of the day is closed. The contract is not renegotiable. On the next day, the market re-opens and other players negotiate for trading under the same rule. Since negotiations take some time, there is an upper bound of proposals possible within one day. If there is no agreement at the end of the day, then the market will re-open on the next day, and the same player as today will start the trading process.

Let I_i^t be an information set of player i in the extensive form of $\Gamma(\mathcal{E})$ in period t. All nodes in I_i^t correspond to an identical sequence $z = (z_1, \cdots, z_{t-1}, z_t)$ of past actions since player i perfectly knows them. Here, z_k, $k = 1, \cdots, t$, denotes the sequence of actions in period k preceding I_i^t. z_t is empty when player i is a proposer in period t. Then, the information set I_i^t is uniquely represented by a pair $(\mathcal{PF}_i(\omega), z)$ of player i's private information and past actions. We call it the *history* of I_i^t. In the following, we will identify I_i^t with its history, and will write as $I_i^t = (\mathcal{PF}_i(\omega), z)$ whenever no confusion arises.

A *(pure) strategy* σ_i for player i in $\Gamma(\mathcal{E})$ is a function which assigns to each of his information set I_i^t a choice at I_i^t. As well as the voting game $\Gamma^x(S, i, y^S)$, we employ a sequential equilibrium as a non-cooperative solution concept for the bargaining game $\Gamma(\mathcal{E})$. A belief system μ is a function which assigns to every information set I_i in $\Gamma(\mathcal{E})$ a probability distribution $\mu(I_i)$ over the set of nodes in $I_i = (\mathcal{PF}_i(\omega), z)$. Since there is a natural one-to-one correspondence between the information set I_i and the information $\mathcal{PF}_i(\omega)$, we can regard $\mu(I_i)$ to be a probability distribution over the set $\mathcal{PF}_i(\omega)$.

It is well-known that there is a large multiplicity of sequential equilibria in a broad class of n-person sequential bargaining games including our bargaining game $\Gamma(\mathcal{E})$. By this reason, it is now standard in the literature of non-cooperative coalitional bargaining that the analysis is restricted to a stationary equilibrium (see Perry and Reny 1994, Chatterjee et al. 1993, Okada 1992, and Mordovanu and Winter 1995, for example).

Definition 4.1. A sequential equilibrium (σ, μ) of $\Gamma(\mathcal{E})$ is said to be *stationary* if every player's choice assigned by the strategy σ to his information set $I_i^t = (\mathcal{PF}_i(\omega), z)$ in each period t $(= 1, 2, \cdots)$ depends only on information $\mathcal{PF}_i(\omega)$, the set N_t of active players, and history z_t within period t.

A stationary equilibrium prescribes that every player's action does not depend on the whole history of actions. An important implication of it is that any player's bargaining behavior does not change even if agreements were rejected in past periods, as long as the same players are still active in negotiations and he has the same information about the prevailing state ω.

Besides the stationarity, we will consider three conditions on a sequential equilibrium of $\Gamma(\mathcal{E})$. The first condition is about a refinement of the belief system.

As the notion of an informational objection has already shown, it is important to consider what information players rationally infer from the actions of other players in $\Gamma(\mathcal{E})$. In general, proposers with different information may have different preferences over an allocation. Thus, the selection of an allocation may reveal proposers' private information to responders. It, however, should be noted that an allocation is a function from the state space Ω to the set of commodity bundles. Therefore, even if proposers with different information prefer different allocations, they can propose the same allocation rule (as a function) which assigns different allocations to different states. The constructed allocation rule never alters the conditional expected payoffs of proposers. By this reason, we can assume without any loss of generality that a proposer chooses an allocation rule independent of his private information, so that the proposer's choice does not convey his any information. This assumption is called the *principle of inscrutability* by Myerson (1983).

Myerson (1983) considered the ultimatum bargaining game in which an informed principal with the full bargaining power chooses and announces a coordination mechanism. Knowing the principal's choice, multiple subordinates select their reports and actions independently in the implementation game of the selected mechanism. By this structure, the issue of endogenous information transmission through subordinates' actions does not arise in his model. This is not the case in our bargaining game $\Gamma(\mathcal{E})$ in which responders' actions may reveal their private information. In this paper, to be compatible with the principle of inscrutability for the proposer, we restrict our analysis to an equilibrium in which responders' behavior do not reveal any private information on *equilibrium play.*[7]

By the principle of inscrutability, the belief system μ of every stationary equilibrium of $\Gamma(\mathcal{E})$ assigns to every information set $I_i = (\mathcal{PF}_i(\omega), z)$ on equilibrium play the posterior belief $\pi_{|\mathcal{PF}_i(\omega)}$ given information $\mathcal{PF}_i(\omega)$ by the Bayes' rule.

Next consider responders' belief at information sets off equilibrium play. We first remark that the voting game $\Gamma^x(S, i, y^S)$ in the last section is "embedded" into the bargaining game $\Gamma(\mathcal{E})$. Suppose that in an equilibrium (σ, μ) an allocation x is proposed and accepted. If the proposer deviates from the equilibrium by proposing an alternative S-allocation y^S, then other members of S are in the same position as in the voting game $\Gamma^x(S, i, y^S)$. A difference is that the game may continue after rejection unless it is the last proposal. Unlike Theorem 3.2, our aim is now to explain the status-quo allocation as an equilibrium behavior. In such an equilibrium, all responders' information sets in the voting game are off equilibrium play. Since the consistency of a sequential equilibrium never impose any restriction on the responders' belief off equilibrium play, we need a suitable refinement on their belief. The following example shows that a refinement of the belief system is critical for the analysis of $\Gamma^x(S, i, y^S)$.

Example 4.1. Consider a two-agent economy with one commodity and

[7]The definition of an equilibrium rejection in the last section satisfies this property since it requires that whenever a proposal is made, it is accepted by all responders.

two states. Let $\Omega = \{\omega_1, \omega_2\}$ be the set of states. A common prior μ is given by $\mu(\omega_1) = \mu(\omega_2) = 1/2$. Agents have differential information: $\mathcal{PF}_1 = [\{\omega_1\}, \{\omega_2\}]$ and $\mathcal{PF}_2 = [\{\omega_1, \omega_2\}]$. Two agents have identical state-independent strictly concave utility functions. Table 4.1 shows the endowment $e = (e_1, e_2)$ and an allocation $y = (y_1, y_2)$. Imagine a situation that there is a production possibility that the total resources may increase from 2 to 4 at ω_2 if two agents cooperate. Either agent prefers y to e at ω_2, but not at ω_1. It can be seen that e does not belong to the informational core since the coalition $\{1, 2\}$ has an informational objection y to e on the event $\{\omega_2\}$.

Agents	$\mathcal{PF}_i$	endowment e		allocation y	
		w_1	w_2	w_1	w_2
1	$[\{w_1\}, \{w_2\}]$	1	1	0	2
2	$[\{w_1, w_2\}]$	1	1	0	2

Table 4.1: An economy with two risk-averse agents

The voting game $\Gamma^e(\{1, 2\}, 1, y)$ has two sequential equilibria with different outcomes, (σ^1, μ^1) and (σ^2, μ^2). In the first equilibrium (σ^1, μ^1), agent 1 chooses the status-quo e, regardless of a state. Agent 2 rejects the proposal y under the prior belief. In this equilibrium, agent 2's information set is off equilibrium play, and his prior belief (in fact, any belief) is consistent with agent 1's strategy. Since agent 2 is risk-averse, his strategy is optimal under the prior. In the second equilibrium (σ^2, μ^2), agent 1 chooses the status-quo e at ω_1 and y at ω_2. Agent 2 accepts the proposal y. His equilibrium belief places the whole probability on ω_2.

We think that the first equilibrium is unreasonable on the ground that agent 2 rationally infers from the proposal y that a true state must be ω_2 according to the same logic as an informational objection.[8] To eliminate such

[8]While this is closely related to the idea of the intuitive criterion of Cho and Kreps (1987), it is not implied by the intuitive criterion since y is not "equilibrium dominated" for type ω_1 in their terminologies. At ω_1, the proposer's highest payoff from proposing y is equal to the equilibrium payoff 1.

an unreasonable equilibrium, we need to refine a sequential equilibrium based on the notion of an informational objection.

We introduce a refinement of a sequential equilibrium of $\Gamma^x(S, i, y^S)$ in which the belief system satisfies the self-selection condition of a proposer.

Definition 4.2. A sequential equilibrium (σ, μ) of $\Gamma^x(S, i, y^S)$ is said to satisfy *self-selection* if the belief system μ assigns to each of every responder j's ($j \in S, j \neq i$) information set I off equilibrium path his posterior $\pi_{|I^+}$ given the event I^+ where $I^+ = \{\omega \in I \mid E(u_i(y_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega)\}$. If I^+ is an empty set, then no restriction is imposed.

This definition means that, given a proposal and his private information I, every responder updates the prior belief π and infers that a true state must be in the event I^+ that the proposer prefers to object to the status quo. In other words, the objection makes the self-selection of the proposer possible in equilibrium.

We will impose two further conditions on a sequential equilibrium of $\Gamma(\mathcal{E})$.

Definition 4.3. A sequential equilibrium (σ, μ) of $\Gamma(\mathcal{E})$ is said to have *payoff-oriented response* if for every $U \subset N$ and every $i \in U$, there exists a function $a^{\mu}_{i,U} : \mathcal{PF}_i \to R$, such that when the set of active players is U, σ prescribes player i's response rule as follows: for any proposal (S, x), $i \in S \subset U$ and $x \in A^S$, player i accepts it, given information $I = \mathcal{PF}_i(\omega)$ if and only if $E_\mu(u_i(x_i)|\mathcal{F}_i)(\omega) \geq a^{\mu}_{i,U}(I)$ where $E_\mu(u_i(x_i)|\mathcal{F}_i)(\omega)$ is defined by

$$E_\mu(u_i(x_i)|\mathcal{F}_i)(\omega) = \sum_{\theta \in I} \mu(I)(\theta)\ u_i(x_i)(\theta). \tag{4.13}$$

In every sequential equilibrium of $\Gamma(\mathcal{E})$, every player employs a "cut-off" response rule in the sense that he accepts an allocation x, given information

$\mathcal{PF}_i(\omega)$, if and only if $E_\mu(u_i(x_i)|\mathcal{F}_i)(\omega) \geq a_i$ where a_i is some acceptance level. Generally, the acceptance level a_i may depend on the whole history of negotiations. For example, even if a player receives the same conditional expected utility by accepting the proposal, his response may be different, depending on the history. Definition 4.3 requires that every responder should be "payoff-oriented" so that he responds to a proposal in the same way as long as the set of active players and the belief system μ are identical. In particular, it is critical to our result that responders' acceptance levels do not depend on how many proposals are left within the present period in negotiations.

The final property of an equilibrium comes from the peculiar property of the bargaining game $\Gamma(\mathcal{E})$ with re-starts that there is an end of negotiations in each period when K successive proposals have been rejected. By this property, the following behavior may be possible in equilibrium. The initial proposer colludes with some player and they "waste" the opportunities of proposals just by repeating proposing and rejection between them until one of them becomes the last proposer within the period. In this case, the situation resembles to the ultimatum bargaining, and it may distort the bargaining outcome. To avoid this unreasonable equilibrium, we impose the following condition.

Definition 4.4. A sequential equilibrium (σ, μ) of $\Gamma(\mathcal{E})$ is said to have *no end-effect* if its equilibrium play satisfies the following property: if an agreement (S, x^S), $S \subset N$ and $x^S \in A^S$, is made by the K-th proposal in some period t, then every player $i \in S$ has the opportunity to make a decision on the equilibrium play before the agreement (S, x^S) in period t.

If a sequential equilibrium has no end-effect, the distortion of the last proposal due to an end-effect can be avoided since, if they want, all members in the contract could propose other allocations before the equilibrium agreement is reached.

We are now ready to prove the main theorem. In what follows, a stationary sequential equilibrium is simply referred to as an equilibrium.

Theorem 4.1. Let $\mathcal{E} = (\Omega, \pi, \{u_i, e_i, \mathcal{F}_i\}_{i\in N})$ be an economy with incomplete information. If a contract (N, x) is agreed (with probability one) in an equilibrium of the bargaining game $\Gamma(\mathcal{E})$ which satisfies (i) payoff-oriented response, (ii) self-selection, and (iii) no end-effect, then the N-allocation x belongs to the informational core of $\mathcal{E}$.

Proof. Suppose that a contract (N, x), $x \in A^N$, is agreed at all states in an equilibrium (σ, μ) of $\Gamma(\mathcal{E})$ which satisfies the three properties in the theorem, but that x does not belong to the informational core of $\mathcal{E}$. The proof is done in two steps.

Step 1. We will prove that when the set of active players is N and information $\mathcal{PF}_j(\omega)$ is privately revealed, every player j's acceptance level $a^{\mu}_{j,N}(\mathcal{PF}_j(\omega))$ in σ_j is equal to his conditional expected utility $E_\mu(u_j(x_j)|\mathcal{F}_j)(\omega)$ for x. Let player $i \in N, i \neq j$ be the last proposer in some period t. Suppose that player i proposes an allocation $y = (y_i, y_j) \in A^{\{i,j\}}$ to player j, on his information set $\mathcal{PF}_i(\omega)$ for any $\omega \in \Omega$. If player j accepts y, then he receives the conditional expected utility $E_\mu(u_j(y_j)|\mathcal{F}_j)(\omega)$. On the other hand, if player j rejects it, then negotiations go to the next period $t+1$. Since (σ, μ) is stationary, negotiations will result in the equilibrium allocation x. This means that player j receives the conditional expected utility $E_\mu(u_j(x_j)|\mathcal{F}_j)(\omega)$ by rejecting y. By these arguments, we can see that it is optimal for player j to accept proposal y if and only if

$$E_\mu(u_j(y_j)|\mathcal{F}_j)(\omega) \geq E_\mu(u_j(x_j)|\mathcal{F}_j)(\omega).$$

This implies that

$$a^{\mu}_{j,N}(\mathcal{PF}_j(\omega)) = E_\mu(u_j(x_j)|\mathcal{F}_j)(\omega) \quad \text{for every} \quad \omega \in \Omega. \tag{4.14}$$

Step 2. Since the equilibrium allocation x does not belong to the informational core, there exists some coalition S which has an informational objection to x, that is, there exist an S-allocation $y^S \in A^S$, a member $i \in S$ and an event $E \in \mathcal{F}_i$ such that

$$E(u_i(y_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega) \quad \text{for all} \quad \omega \in E \tag{4.15}$$

$$E(u_i(y_i^S)|\mathcal{F}_i)(\omega) \leq E(u_i(x_i)|\mathcal{F}_i)(\omega) \quad \text{for all} \quad \omega \notin E \tag{4.16}$$

and

$$E(u_j(y_j^S)|\mathcal{PF}_j(\omega) \cap E) > E(u_j(x_j)|\mathcal{PF}_j(\omega) \cap E) \tag{4.17}$$

for all $j \in S, j \neq i$ and all $\omega \in E$.

Consider first the case that the equilibrium contract (N, x) is agreed as the last proposal in the initial period.[9] Due to the no-end-effect property, player i has an opportunity to make a proposal on the equilibrium play before the last proposal (N, x). Suppose that player i deviates from the equilibrium play of (σ, μ) at one of his information sets $I_i \subset E$ and proposes the new contract (S, y^S). By the inscrutability principle, no players j in S receive any additional information except the fields $\mathcal{F}_j$ on the equilibrium play of (σ, μ) before I_i. In particular, player i has the posterior belief $\pi_{|I_i}$ on the information set I_i under the belief system μ. On the other hand, since (σ, μ) satisfies self-selection, it follows from (4.15) and (4.16) that all responders $j (\neq i)$ update their belief by the proposal y^S and infer that the true state is in the event E. Thus, the belief system μ assigns to their information sets $\mathcal{PF}_j(\omega)$ succeeding the new proposal (S, y^S) the posterior belief $\pi_{|\mathcal{PF}_j(\omega) \cap E}$ if $\mathcal{PF}_j(\omega) \cap E \neq \emptyset$.

By backward induction, we will show that all responders accept (S, y^S) at their information sets $\mathcal{PF}_j(\omega)$ for all $\omega \in E$. Assume that all responders except the last one accept it. Then, the last proposer j receives the

[9]The agreement is made in the initial period since the equilibrium is stationary. With no loss of generality, we assume that every possible equilibrium play has the same number of proposals before an agreement.

conditional expected utility $E(u_j(y_j^S)|\mathcal{PF}_j(\omega)\cap E)$ if he accepts it. Since $E(u_j(y_j^S)|\mathcal{PF}_j(\omega)\cap E)$ is greater than his acceptance level $a_{j,N}^{\mu}(\mathcal{PF}_j(\omega))$ by (4.17), the last proposer j accepts y^S by the payoff-oriented response condition of the equilibrium (σ,μ). By applying the same arguments to other responders backward, we can show that all responders accept (S,y^S). Since the proposal y^S is accepted on every play following the information set I_i, proposer i receives the conditional expected utility $E(u_i(y_i^S)|I_i)$ by proposing y^S on the information set I_i. Thus, by (4.15), proposer i would be better-off by proposing y^S on I_i than in equilibrium. This contradicts that (σ,μ) is an equilibrium.

Finally, consider the other case that the equilibrium contract (N,x) is agreed before the last proposal. In this case, every member in S has an opportunity to make a new proposal. Suppose that player i rejects (N,x) and proposes (S,y^S) on the information set $I_i\subset E$. By the same proof as in the first case, we can show that all responders accept it, and thus that player i would be better-off. The same contradiction as in the first case arises. Q.E.D.

The intuition for the theorem is as follows. When every player responds to any last proposal in each period, his rejection makes the game to re-start in the next period, and thereafter the equilibrium allocation x will be agreed in a stationary equilibrium. Thus, every responder receives his conditional expected utility for the equilibrium allocation x by rejecting the last proposal. By this fact, the optimal condition of response is that every responder's acceptance level is equal to his conditional expected utility for x. On the contrary to the theorem, suppose that the equilibrium allocation x does not belong to the informational core. Then, there exists some coalition S which has an informational objection to x. Specifically, there exists some particular member i of S and some S-allocation y^S such that, if i proposes y^S over the event where he credibly reveals his preference of y^S, all members in S would be better-off by accepting y^S than in x under their equilibrium belief. The no-end-effect property guarantees such an opportunity for player i to propose the new allocation y^S. By the inscrutability principle, player i has the posterior belief given information $\mathcal{F}_i$, and every responder's belief is updated according to the self-selection property. Thus, if player i deviates from the equilibrium and proposes y^S, y^S is accepted by all other members of S and player i will be better-off than in x. This contradicts that x is the equilibrium allocation.

To prove the inverse of Theorem 4.1, we need to weaken the notion of an informational objection as follows.

Definition 4.5. A coalition S has a *weakly informational objection* to an N-allocation $x \in A^N$ if there exist an S-allocation $y^S \in A^S$, a member $i \in S$ and an event $E \in \mathcal{F}_i$ such that

(1) $E(u_i(y_i)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega)$ for all $\omega \in E$,

(2) $E(u_i(y_i)|\mathcal{F}_i)(\omega) \leq E(u_i(x_i)|\mathcal{F}_i)(\omega)$ for all $\omega \notin E$,

(3) $E(u_j(y_j)|\mathcal{P}\mathcal{F}_j(\omega) \cap E) > E(u_j(x_j)|\mathcal{P}\mathcal{F}_j(\omega) \cap E)$ for some $\omega \in E$ and all $j \in S, j \neq i$.

The *strictly informational core* is the set of all N-allocations to which no coalition has a weakly informational objection.

The difference between the informational objection and the weakly informational objection lies in condition (3) about the responders' conditional expected utility. The weakly informational objection requires only that there exists *at least one* state in the self-selection event E for the proposer and all members of the coalition would be better-off by accepting the alternative proposal on the corresponding play, while the informational objection requires that the same thing happens for *every* state in the self-selection event E. Clearly, the strictly informational core is a subset of the informational core. Remark that the two sets coincide in an economy with complete information.

For a coalition S, a *sub-economy* of an economy $\mathcal{E} = (\Omega, \pi, \{u_i, e_i, \mathcal{F}_i\}_{i \in N})$ is defined as the economy in which the set of traders is restricted to S and all other elements are kept unchanged. Formally, a sub-economy $\mathcal{E}^S$ is defined by

$\mathcal{E}^S = (\Omega, \pi, \{u_i, e_i, \mathcal{F}_i\}_{i \in S})$.

Theorem 4.2. Let $\mathcal{E}$ be an economy with incomplete information. Assume that the strictly informational core of every sub-economy of $\mathcal{E}$ (including $\mathcal{E}$ itself) is non-empty. Then, for any N-allocation x in the strictly informational core of $\mathcal{E}$, there exists an equilibrium of the bargaining game $\Gamma(\mathcal{E})$ which satisfies (i) payoff-oriented response, (ii) self-selection, and (iii) no end-effect, and the N-allocation x is agreed (with probability one) in equilibrium.

Proof. By assumption, for every subset $S \subset N$ we can select an S-allocation x^S in the strictly informational core of the sub-economy $\mathcal{E}^S$ of $\mathcal{E}$. Select $x^N = x$ for $S = N$. Define the strategy σ_i of player i in the bargaining game $\Gamma(\mathcal{E})$ as follows. When the set of active players is S, player i, receiving every information $\mathcal{PF}_i(\omega)$,

(i) proposes (S, x^S), and

(ii) for any proposal (T, y^T) with $i \in T \subset S$, accepts it if and only if $(T, y^T) = (S, x^S)$, or $E_\mu(u_i(y_i^T)|\mathcal{F}_i)(\omega) > E_\mu(u_i(x_i^S)|\mathcal{F}_i)(\omega)$.

The belief system μ is constructed so that it is consistent with $\sigma = (\sigma_1, \cdots, \sigma_n)$ and moreover that, off equilibrium path, it assigns to each of every proposer i's information set I his posterior $\pi_{|I}$ given I and, when i proposes (T, y^T), it assigns to each of every responder j's information set J the posterior $\pi_{|J^+}$ given J^+ where $J^+ = \{\omega \in J \mid E(u_i(y_i^T)|\mathcal{F}_i)(\omega) > E(u_i(x_i^S)|\mathcal{F}_i)(\omega)\}$.

It can be easily seen that σ_i is a stationary strategy satisfying the three properties in the theorem. Further, when the strategy combination σ is employed, x is agreed immediately in the initial period.

We will prove that the pair (σ, μ) is a sequential equilibrium of $\Gamma(\mathcal{E})$. To do this, it suffices us to show that σ prescribes an optimal choice at each of every player's information set, given (σ, μ). Without loss of generality, we can assume that the set of active players is N. The same arguments can be applied to other cases.[10] It is clear that the response strategy (ii) is optimal at each of every player i's information set $I_i = (\mathcal{PF}_i(\omega), z)$, given (σ, μ).

Suppose that player i deviates from (i) at his any information set $\mathcal{PF}_i(\omega)$ and proposes another contract (S, y^S) such that $E(u_i(y^S)|\mathcal{F}_i)(\omega) > E(u_i(x)|\mathcal{F}_i)(\omega)$. Let

$$E = \{\omega \in \Omega \mid E(u_i(y_i^S)|\mathcal{F}_i)(\omega) > E(u_i(x_i)|\mathcal{F}_i)(\omega)\}.$$

Since x belongs to the strictly informational core, S does not have a weakly informational objection to x. Thus, for any $\omega \in E$ there exists some player $j \in S, j \neq i$, such that

$$E(u_j(y_j^S)|\mathcal{PF}_j(\omega) \cap E) \leq E(u_j(x_j)|\mathcal{PF}_j(\omega) \cap E).$$

Then, by (ii), player j rejects player i's proposal (S, y^S), on his information set $\mathcal{PF}_j(\omega)$, under the belief system μ. Since there exists some responder who rejects player i's proposal (S, y^S) on every play starting from the event E, player i's conditional expected utility remains to be the same as in equilibrium on each of his information sets $\mathcal{PF}_i(\omega) \subset E$. Thus, it is optimal for player i to propose (N, x). Q.E.D.

The intuition for the theorem is as follows. For every N-allocation x in the strictly informational core, we can construct an equilibrium of $\Gamma(\mathcal{E})$ such that every player proposes x independent of his private information and every responder accepts any non-equilibrium proposal y^S if and only if his conditional expected payoff for y^S is greater than that for x under his updated belief through endogenous information transmission. To support this strategy as an equilibrium, we need to strengthen the notion of an informational objection so that, when the player makes any non-equilibrium proposal in attempting

[10] Every subgame of $\Gamma(\mathcal{E})$ where the set of active players is S is not reached on the equilibrium play of (σ, μ). Since we can select the belief system μ such that no players in S receive no information except $\mathcal{F}_i$ from any history before the subgame, the same proof as in the case of N can be applied to the subgame played by S.

to increase his expected utility conditional to private information, there exists at least one responder to reject it for *every* possible state in the self-selection event. The notion of an informational objection guarantees only that there exists such a responder for *some* possible state in the self-selection event.

5 Discussion

In this section, we discuss our results in relation to two branches of literature, the incentive-compatible core in the case of unverifiable states, and non-cooperative coalitional bargaining models for the core with complete information.

As Vohra (1999) argues, a feasible allocation must be incentive compatible when a state of an economy is unverifiable when a contract is implemented. That is, it should be optimal for every member of a coalition to report his private information (type) truthfully when all others do so. Given the new set of feasible allocations with incentive constraints, our model of a coalitional voting game can be applied to the case of unverifiable states without any difficulty. More generally, by the revelation principle, agents can propose any indirect (communication) mechanism followed by votes. If the mechanism is accepted, all members in the coalition are asked to report their messages. Then, the mechanism assigns a net-trade according to reported messages. In a game of the mechanism, agents may report strategically, receiving a new information revealed by others' voting.

Dutta and Vohra (2005) define the credible core where incentive compatibility is required over an event which is reasonably believed through information transmission. Serrano and Vohra (2007) provide a non-cooperative support to the credible core. The credible core is reduced to the Wilson's fine core allowing unlimited communication in the case of verifiable states. In Serrano and Vohra's voting game, a proposal of an uninformed mediator makes it possible the coordination of all members' voting behavior on every admissible event as the fine core presumes. In this paper, we have shown that the issue of endogenous information transmission is relevant even in the case of verifiable states since a proposal by an informed agent may reveal his private information to other agents only in a credible way.

We now turn to discuss the non-cooperative coalitional bargaining model in this paper. In the literature, several non-cooperative coalitional bargaining models for the core have been presented in the case of complete information. A natural extension of the Rubinstein's two-person alternating-offers model to the n-person coalitional bargaining situation seems to be a sequential bargaining model without re-starts. The protocol of the model was first studied by Selten (1981). In the model, an initial player selected by some predetermined order proposes a coalitional allocation and the first rejector becomes the next proposer. This process is repeated until all players join coalitions. However, every stationary subgame perfect equilibrium payoff allocation of the model does not belong to the core as the next example shows.[11]

Consider a coalitional game (N, v) with transferable utility where $N = \{1, 2, 3\}$ and the characteristic function v satisfies: $v(\{1, 2, 3\}) = v(\{1, 2\}) = 3$, $v(\{1, 3\}) = 2$, and $v(\{2, 3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. The core is the set of all payoff profiles $(x_1, x_2, 0)$ where $2 \leq x_1 \leq 3$. It can be seen that the following stationary strategy profile is a subgame perfect equilibrium of the sequential bargaining game explained above. Player 1 proposes $(\{1, 2\}, (1, 2))$ and accepts any proposal if he is offered at least 1. Player 2 proposes $(\{1, 2\}, (1, 2))$ and accepts any proposal if he is offered at least 2. Player 3 proposes $(\{1, 3\}, (1, 1))$ and accepts any proposal if he is offered at least 1. If either player 1 or player 2 is an initial proposer, then the allocation $(1, 2, 0)$ is realized. If player 3 is an initial proposer, then the allocation $(1, 0, 1)$ is realized. Neither allocation belongs to the core.

This example illustrates that the cooperative concept of domination underlying the core is not straightforwardly justified by non-cooperative game

[11] See Moldovanu and Winter (1995).

theory. Although the equilibrium allocation $(1,2,0)$ proposed by player 1 is dominated by a payoff allocation $(1.5,0.5)$ of the coalition of players 1 and 3, it is rejected by player 3 since he can receive a higher payoff 1 than 0.5 by doing so in equilibrium. What is critical to a responder is a comparison between a payoff offered to him and the continuation payoff, a payoff which he expects to receive by rejection. As the example above shows, the current payoff on a table and the continuation payoff may be different if the equilibrium allocation is sensitive to an order of proposers. Notice that this problem does not arise in the coalitional voting game since the status-quo allocation prevails by the rule of the game if any proposal is rejected.

In the literature, several different approaches have been introduced to avoid the sensitivity of an equilibrium to an order of proposers in negotiations. Our approach, the possibility of re-starts combined by the payoff-oriented response rule, is just one of them. Some of other approaches include the order-independent equilibrium (Moldovanu and Winter 1995)[12], a continuous-time model (Perry and Reny 1994) and the competition to make offers (Evans 1997). Notice that all these approaches are invented to prove the result (corresponding to Theorem 4.1) that every stationary subgame perfect equilibrium allocation is included in the core. We believe that Theorem 4.1 can also be proved by other approaches if they are suitably extended to the case of incomplete information. In our view, no approach is superior to others. They reflect a diversity of bargaining situations in real life. The fact that the core can be justified by various non-cooperative bargaining models supports it as a cooperative solution which may be applicable to broad situations.

[12] Recently, Horniaček (2008) shows that any non core-allocation can be eliminated from the set of stationary subgame perfect equilibrium allocations of Moldovanu and Winter's (1995) model if one allows players' preference for coalitions as well as payoffs.

6 Concluding Remarks

In this paper, we have introduced a notion of the informational core in an exchange economy with incomplete information based on the idea that an objection by a coalition should take into account endogenous information transmission from an informed proposer to other members. We have proved that a refinement of a sequential equilibrium of a non-cooperative bargaining model eliminates any non-core-allocation. This paper is a first step towards a non-cooperative bargaining foundation for the core under incomplete information. Many questions remain open. In particular, the existence of and the characterization of the informational core is left to future work.

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